Part 2: Nonnormality Issues

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Bidiagonal Matrix

$$\mathbf{A} = \begin{bmatrix} -1 & b & & & & \\ & -1 & b & & & \\ & & -1 & b & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & b \\ & & & & & & -1 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

Eigenvalues equal to -1

Nonnormal when $b \neq 0$

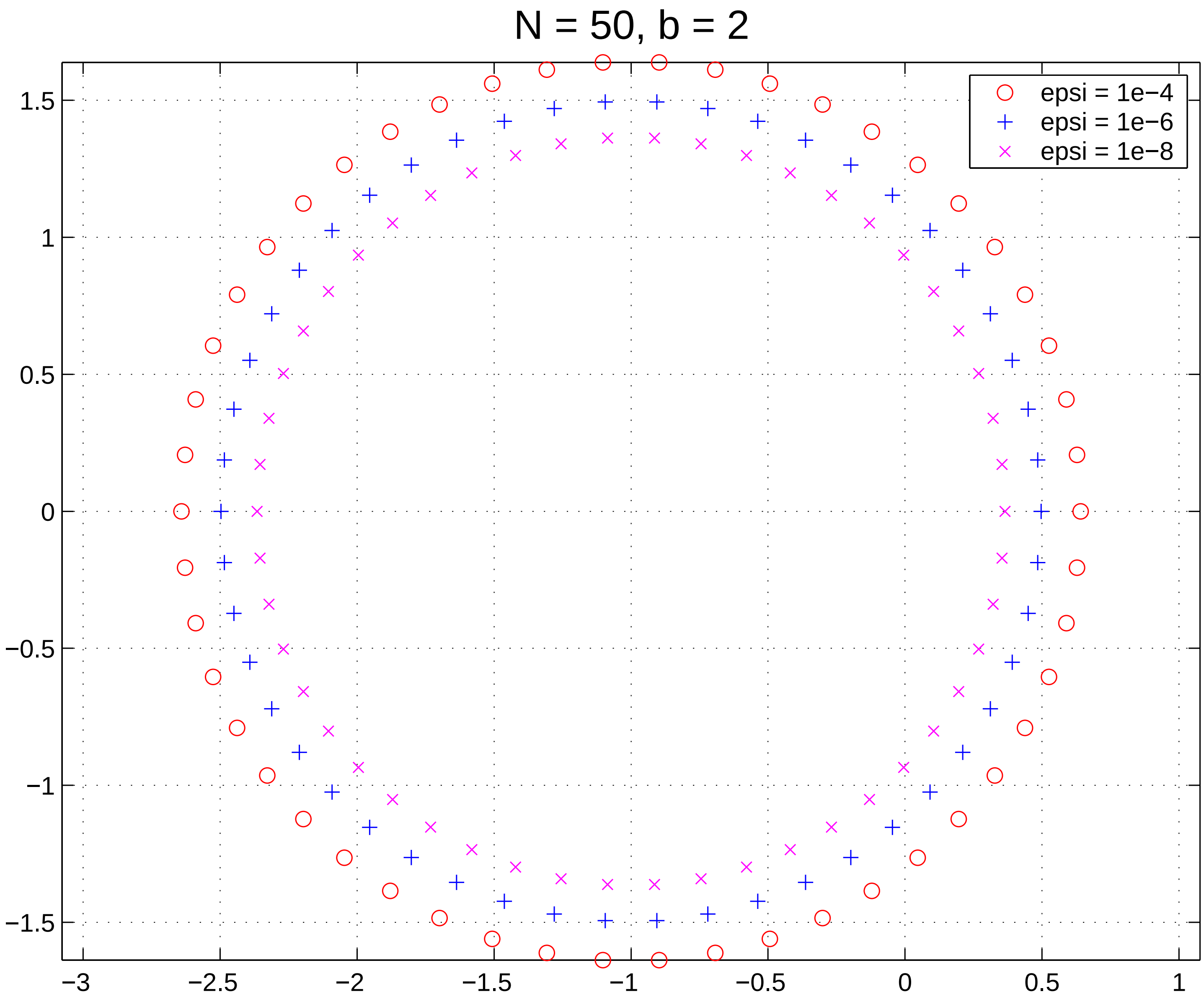
Perturbed Matrix

$$\mathbf{A}_\epsilon = \begin{bmatrix} -1 & b & & & & \\ & -1 & b & & & \\ & & -1 & b & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & & & & & \ddots & b \\ \epsilon & & & & & & -1 \end{bmatrix}$$

Eigenvalues equal to

$$-1 + e^{\frac{2\pi \mathbf{i} k}{N}} b^{\frac{N-1}{N}} \epsilon^{\frac{1}{N}}, \quad 0 \le k \le N-1$$

Eigenvalues of $\mathbf{A}_\epsilon$



Linear ODE system

$$\mathbf{A} = \begin{bmatrix} -1 & b & & \\ & \ddots & \ddots & \\ & & \ddots & b \\ & & & -1 \end{bmatrix}$$

$\frac{dy}{dt} = \mathbf{A}y$ has solution $e^{\mathbf{A}t}y(0)$

For $y(0) = [1, 1, \ldots, 1]^T$, we have

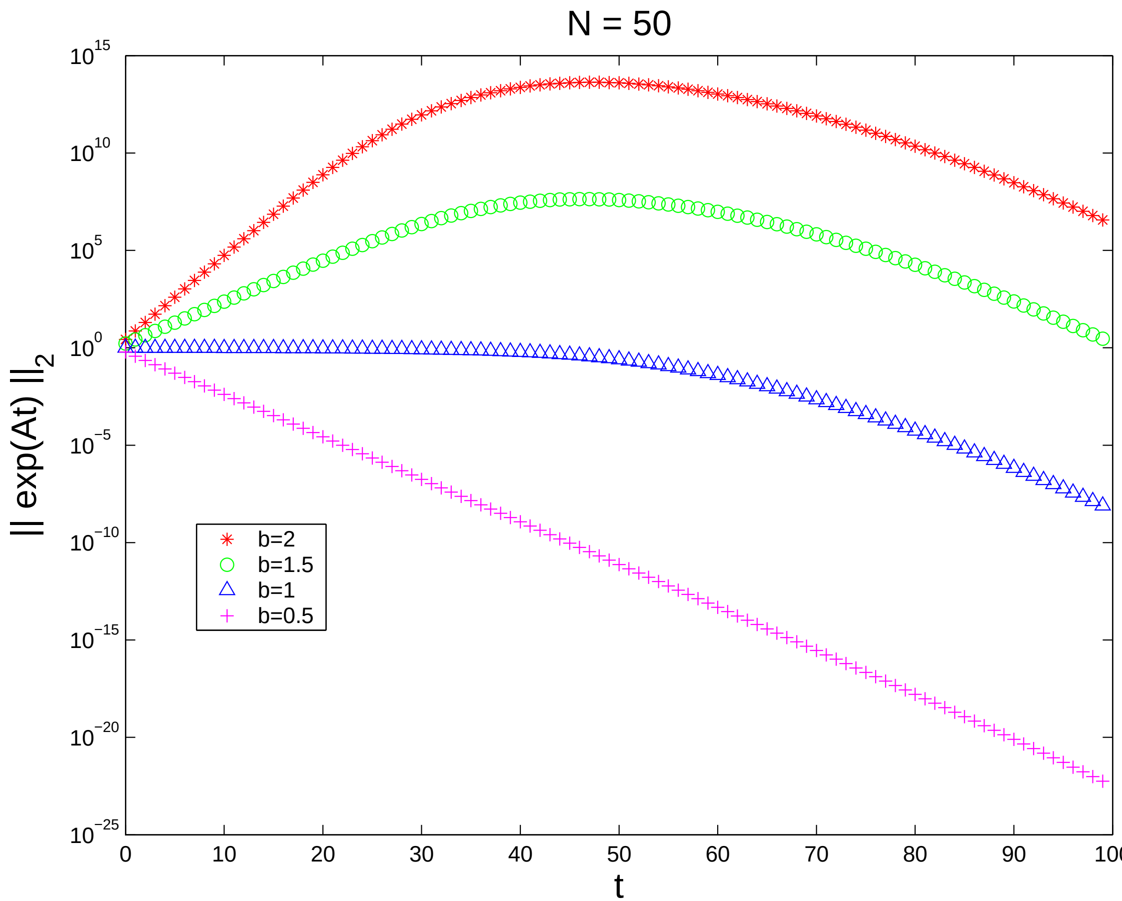
$$y_{N-k}(t) = e^{-t}\left(1 + bt + \frac{(bt)^2}{2!} + \frac{(bt)^3}{3!} + \cdots + \frac{(bt)^k}{k!}\right)$$

so

$$y_1(t) = e^{-t}\left(1 + bt + \frac{(bt)^2}{2!} + \frac{(bt)^3}{3!} + \cdots + \frac{(bt)^{N-1}}{(N-1)!}\right)$$

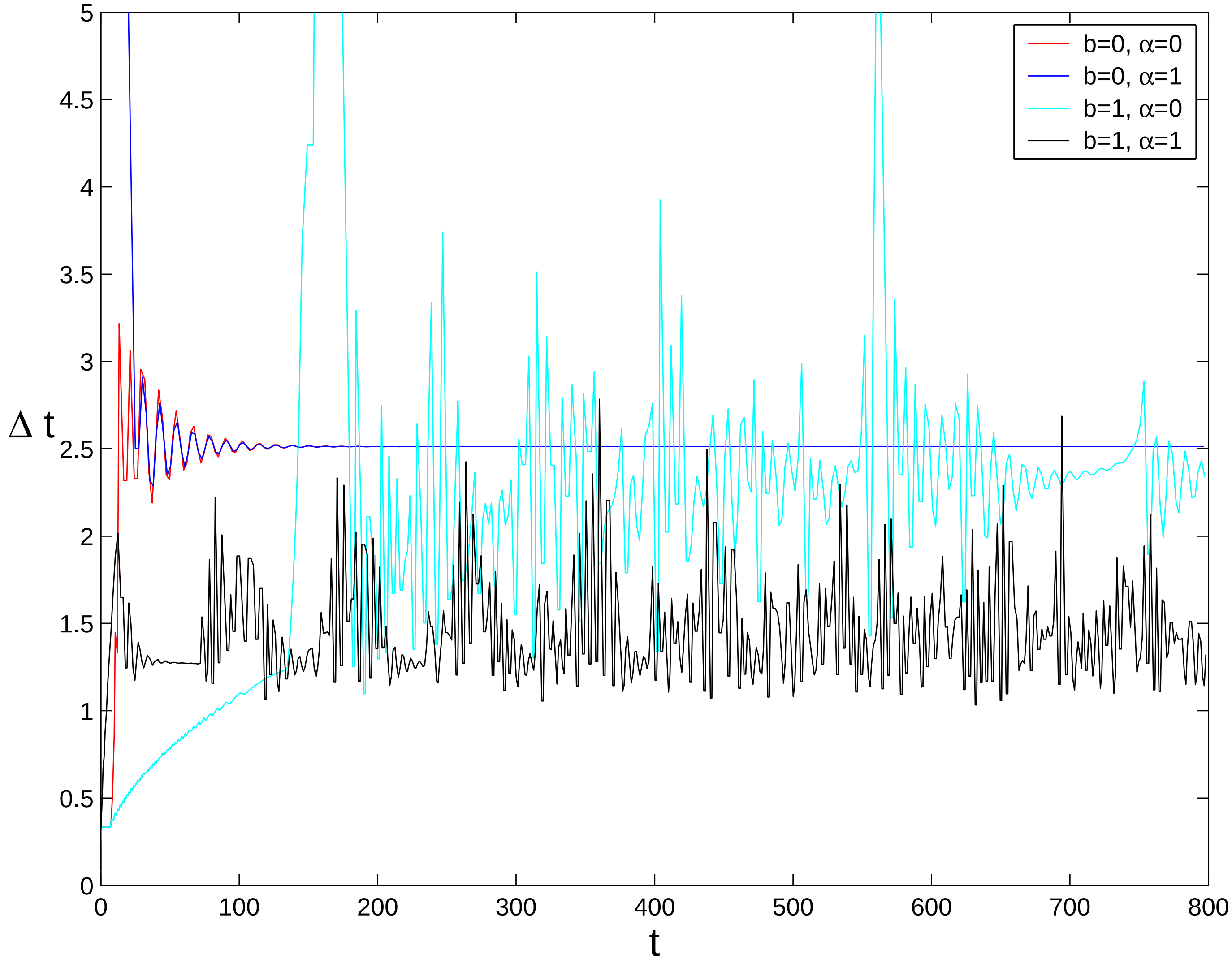
Behaviour of $\|e^{\mathbf{A}t}\|_2$

$$\mathbf{A} = \begin{bmatrix} -1 & b & & \\ & \ddots & \ddots & \\ & & \ddots & b \\ & & & -1 \end{bmatrix}$$



Stepsizes from ode45

$$\frac{dy}{dt} = \mathbf{A}y + \alpha \cos(10^{-4}t)$$



Numerical Timestepping

Key message:

When the **eigenvalues** of $\mathbf{A}$ are **very sensitive** to perturbations in $\mathbf{A}$, the **pseudospectra** (eigenvalues of $\mathbf{A} + \mathbf{E}$ for small $\mathbf{E}$) are more relevant than the spectrum of $\mathbf{A}$

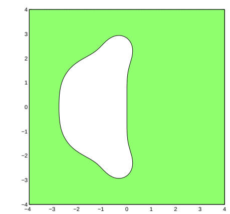
Can be made rigorous:

Transplant the **Kreiss matrix theorem** from the unit disk the **stability region** of the method . . .

Numerical Timestepping ... cont'd

Explicit Runge–Kutta method on $\frac{dy}{dt} = \mathbf{A}y$ produces

$$y_n = p(\mathbf{A}\Delta t)^n y_0$$



Stability region: $S := \{z \in \mathbb{C} : |p(z)| < 1\}$
Then

$$C_1\mathcal{K} \leq \sup_{n\geq 0} \|\left(p(\mathbf{A}\Delta t)\right)^n\|_2 \leq C_2\mathcal{K},$$

where

$$\mathcal{K} := \sup_{\epsilon>0} \epsilon^{-1}\,\overline{\mathrm{dist}}(\Lambda_\epsilon(\mathbf{A}\Delta t), S),$$

$\overline{\mathrm{dist}}(A, B)$ means $\sup_{z\in A} \mathrm{dist}(z, B)$,
C_1 depends on RK method,
C_2 depends on RK method and (linearly) on N

Stochastic ODEs and Nonnormality

Idea:

If $\mathbf{A}$ is nonnormal, behaviour of $\frac{dy}{dt} = \mathbf{A}y$
might be **very sensitive**
to **small noise** perturbations

Aim:

Show that a **family of stable problems** $\frac{dy}{dt} = \mathbf{A}y$
can be made **unstable** by a noise perturbation that
shrinks to zero as **nonnormality increases**

Linear SDE

$$dx = \mathbf{A}x\,dt + \mathbf{G}x\,dw$$

$\mathbf{A}, \mathbf{G} \in \mathbb{R}^{N \times N}$

Mao, 1997

> "In general, the fundamental matrix ... cannot be given explicitly"

Consider **mean-square stability**, i.e. behaviour of $\mathbb{E}\|x(t)\|_2^2$

Linear SDE

$$dx = \mathbf{A}x\,dt + \mathbf{G}x\,dw$$

Let

$$v(x) := x^T\mathbf{Q}x$$

with $\mathbf{Q}$ pos. def.
Stochastic calculus gives

$$dv = x^T\mathbf{M}x\,dt + x^T\mathbf{N}x\,dw$$

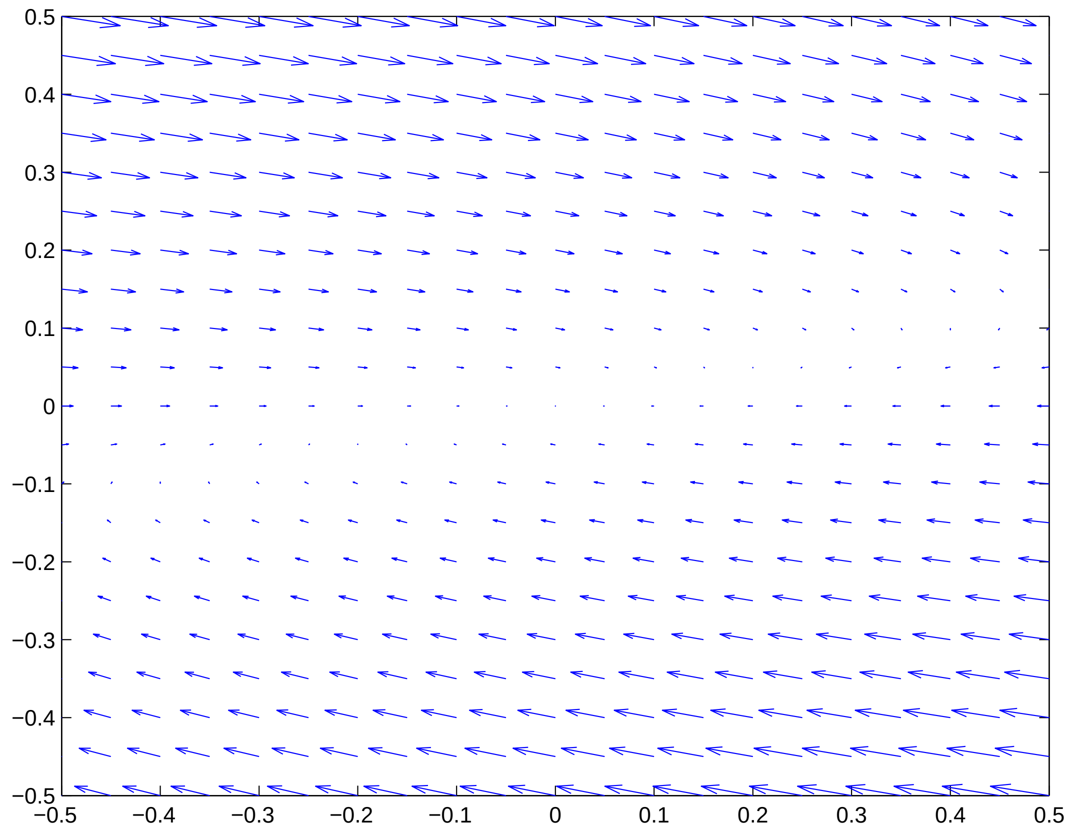
where

$$\mathbf{M} := \mathbf{Q}\mathbf{A} + \mathbf{A}^T\mathbf{Q} + \mathbf{G}^T\mathbf{Q}\mathbf{G}$$
$$\mathbf{N} := 2\mathbf{Q}\mathbf{G}$$

Lyapunov functions like $v(x)$ are used to establish **(in)stability** results, see, e.g., Mao, 1997

Vector Field from $A \in \mathbb{R}^{2\times 2}$

Deterministic ODE will look like this:



Linear SDE example

$$dx = \mathbf{A}x\,dt + \mathbf{G}x\,dw$$

$$\mathbf{A} = \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}, \quad \sigma = b^{-\frac{1}{4}}$$

Set $\mathbf{Q} = \begin{bmatrix} \frac{1}{2}b^{-2} & \frac{1}{4}b^{-1} \\ \frac{1}{4}b^{-1} & \frac{1}{4} + \frac{1}{2}b^{-4} \end{bmatrix}$

$\mathbf{Q}$ is pos. def. with eig's $\approx \frac{1}{4}b^{-2}$ and $\frac{1}{4}$

Linear SDE example

$$dx = \mathbf{A}x\,dt + \mathbf{G}x\,dw$$

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$\mathbf{Q}$ is pos. def. with eig's $\approx \frac{1}{4}b^{-2}$ and $\frac{1}{4}$

$$\mathbf{QA} + \mathbf{A}^T\mathbf{Q} = \begin{bmatrix} -\frac{1}{2}b^{-2} & 0 \\ 0 & -\frac{1}{4}b^{-4} \end{bmatrix}$$

Linear SDE example

$$dx = \mathbf{A}x\,dt + \mathbf{G}x\,dw$$

$$\mathbf{A} = \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}, \quad \sigma = b^{-\frac{1}{4}}$$

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$\mathbf{Q}$ is pos. def. with eig's $\approx \frac{1}{4}b^{-2}$ and $\frac{1}{4}$

$$\mathbf{QA} + \mathbf{A}^T\mathbf{Q} = \begin{bmatrix} -\frac{1}{2}b^{-2} & 0 \\ 0 & -\frac{1}{4}b^{-4} \end{bmatrix}$$

$\mathbf{M} := \mathbf{QA} + \mathbf{A}^T\mathbf{Q} + \mathbf{G}^T\mathbf{QG}$ has eig's $\approx \frac{1}{4}b^{-\frac{5}{2}}$ and $\frac{1}{4}b^{-\frac{1}{2}}$

Stability Analysis

$$\frac{\mathbb{E}\, v(t+h) - \mathbb{E}\, v(t)}{h} = \mathbb{E}\, \frac{1}{h} \int_t^{t+h} x(s)^T \mathbf{M} x(s)\, ds$$

gives

$$\begin{aligned} \frac{d}{dt} \mathbb{E}\, v(t) &= \mathbb{E}\, x(t)^T \mathbf{M} x(t) \\ &\geq \lambda_{\mathbf{M}}^{\mathbf{min}} \mathbb{E}\, \|x(t)\|_2^2 \\ &\geq \lambda_{\mathbf{M}}^{\mathbf{min}} \frac{\mathbb{E}\, x(t)^T Q x(t)}{\lambda_{\mathbf{Q}}^{\mathbf{max}}} \\ &= \frac{\lambda_{\mathbf{M}}^{\mathbf{min}}}{\lambda_{\mathbf{Q}}^{\mathbf{max}}} \mathbb{E}\, v(t) \end{aligned}$$

Stability Analysis . . . cont'd

So

$$\mathbb{E}\, v(t) \geq e^{(\lambda_{\mathbf{M}}^{\mathbf{min}}/\lambda_{\mathbf{Q}}^{\mathbf{max}})t}\mathbb{E}\, v(0)$$

Hence

$$\begin{aligned}\mathbb{E}\, \|x(t)\|_2^2 &\geq \frac{1}{\lambda_{\mathbf{Q}}^{\mathbf{max}}}\mathbb{E}\, v(t) \\ &\geq \frac{\mathbb{E}\, v(0)}{\lambda_{\mathbf{Q}}^{\mathbf{max}}} e^{(\lambda_{\mathbf{M}}^{\mathbf{min}}/\lambda_{\mathbf{Q}}^{\mathbf{max}})t}\end{aligned}$$

$\Rightarrow$ Result

$$dx = \mathbf{A}x\,dt + \mathbf{G}x\,dw$$

$$\mathbf{A} = \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}$$

- $\sigma = 0$ gives $\|x(t)\|_2^2 \leq Ke^{-t}$
- $\sigma = b^{-\frac{1}{4}}$ gives $\mathbb{E}\,\|x(t)\|_2^2 \geq C_b e^{\delta_b t}$, $\quad C_b, \delta_b > 0$

As nonnormality increases, a **vanishingly small noise term** can change the **second moment Lyapunov exponent** from -1 to something **positive**

Wish List . . .

- **asymptotic stability** result: $\|x(t)\|_2 \to 0$ with prob. 1
- result for **fixed** $b > 1$ and **large** N (e.g. $\sigma = \frac{1}{N}$)
- result for **more general, nonnormal** $\mathbf{A}$
- result for **numerical methods**, e.g.

$$x_{n+1} = (I + \Delta t\mathbf{A} + \Delta W_n \mathbf{G})\, x_n$$

or, more simply,

$$x_{n+1} = \left(I + \Delta t\mathbf{A} \pm \sqrt{\Delta t}\mathbf{G}\right) x_n$$

[**random matrix products**]