Motivation . . . stochastic stability

```
x(n+1) = (a + b*randn)*x(n)
```

What happens as $n \to \infty$?

```
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Asymptotic: $x_n \to 0$ with prob. $1 \Leftrightarrow \mathbb{E}[\log |a + bN(0,1)|] < 0$

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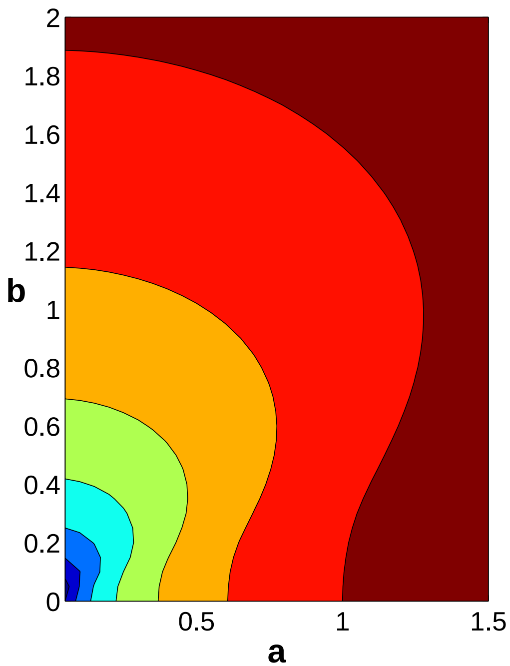

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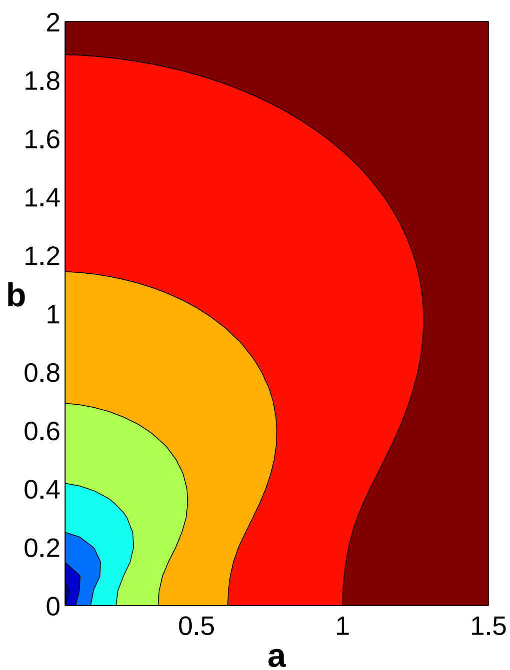

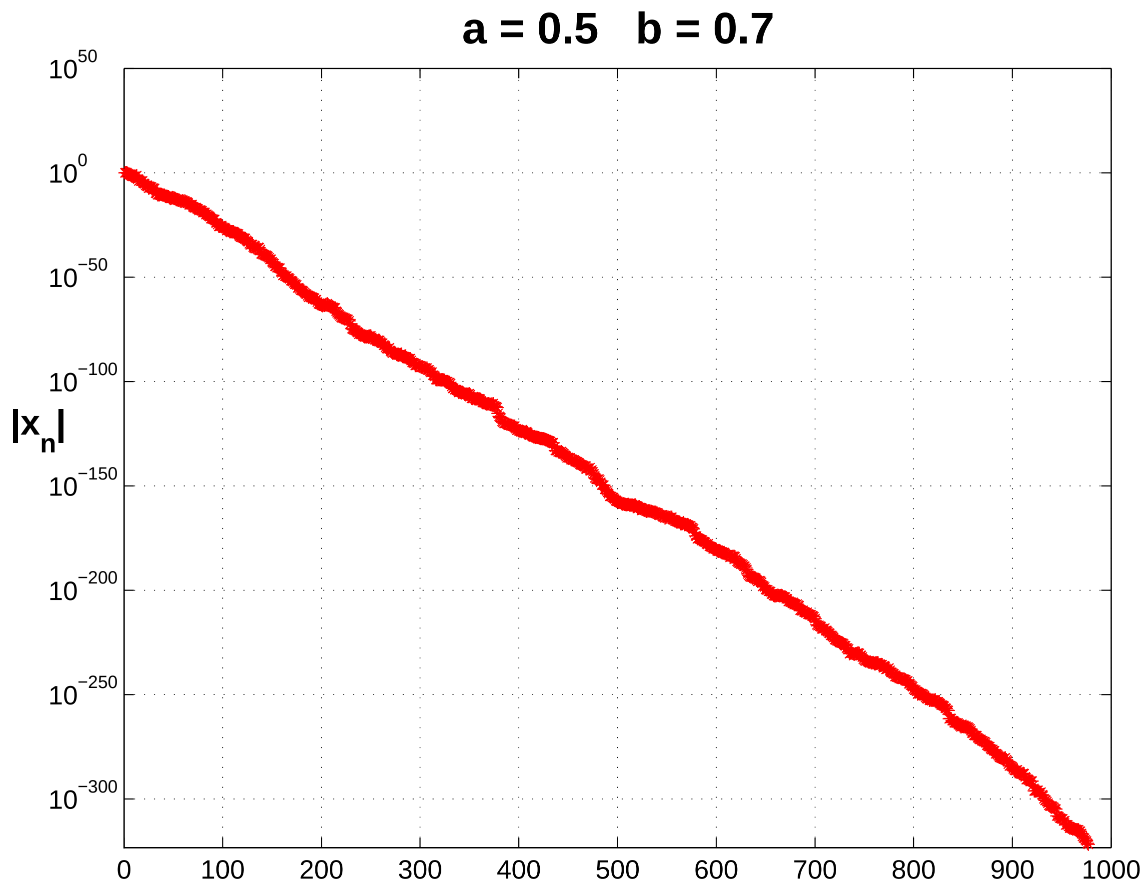

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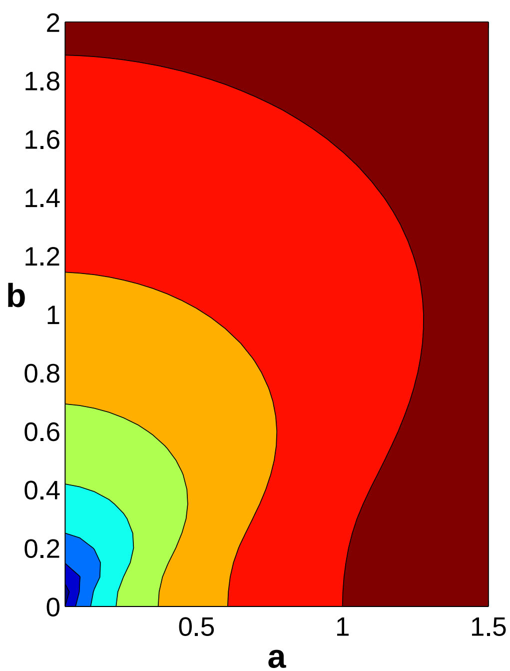

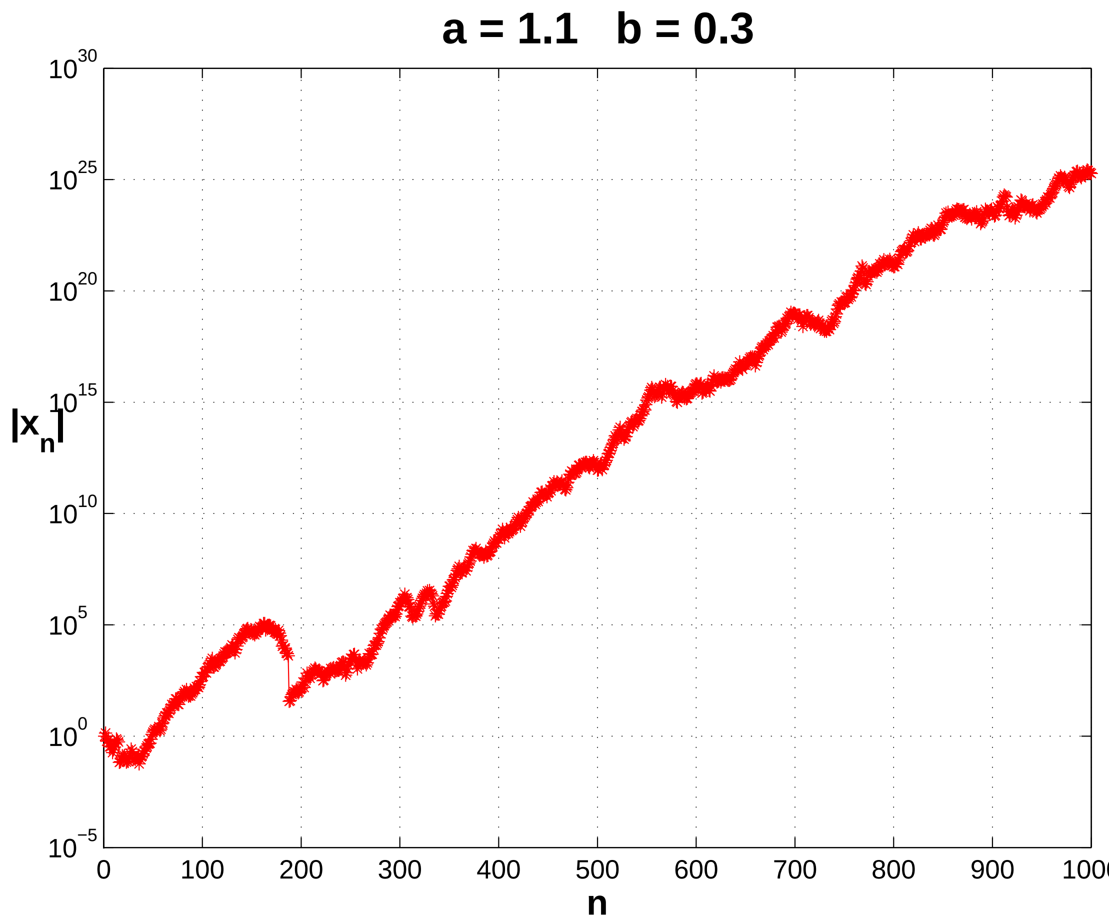

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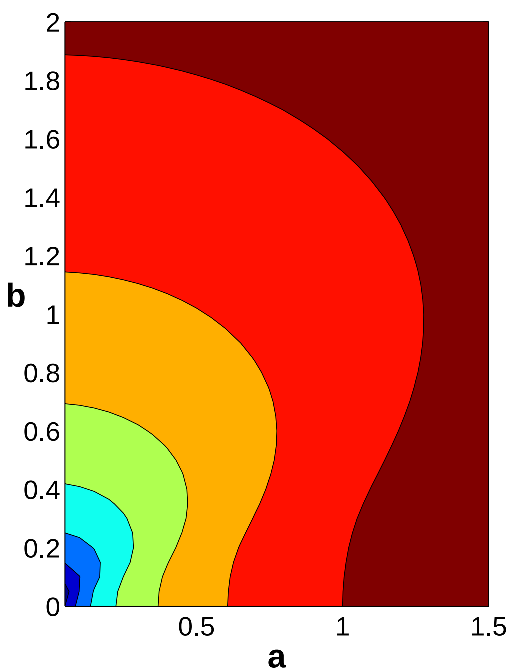

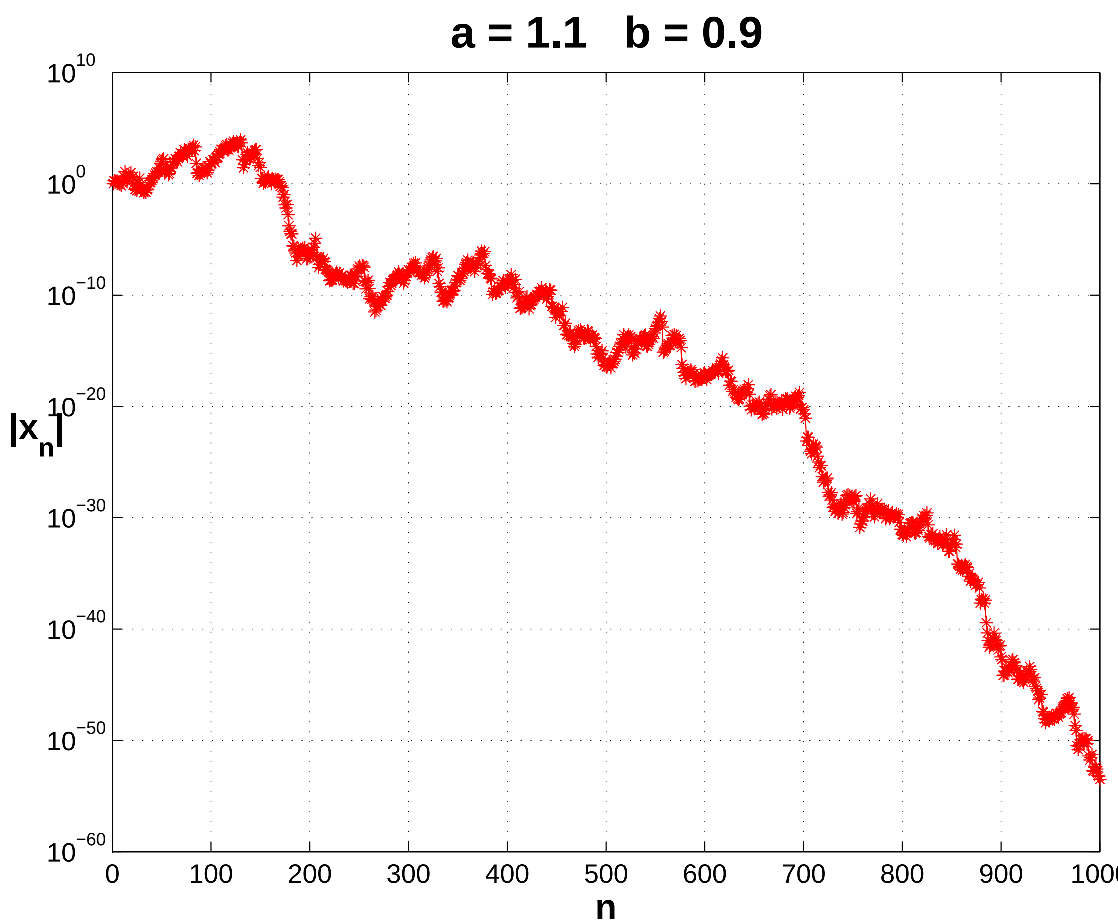

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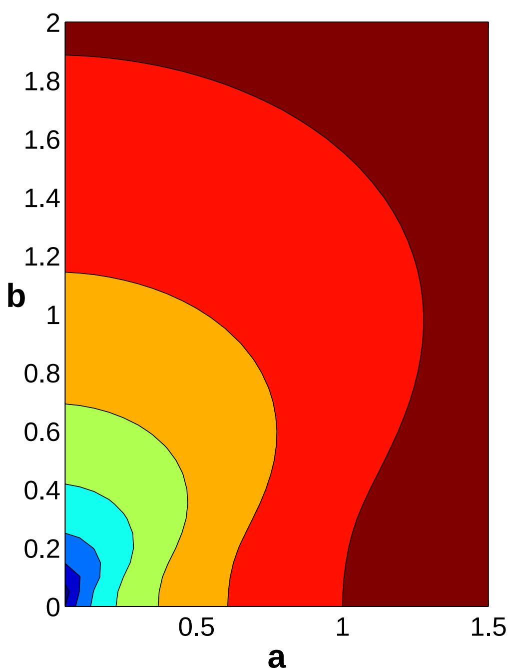

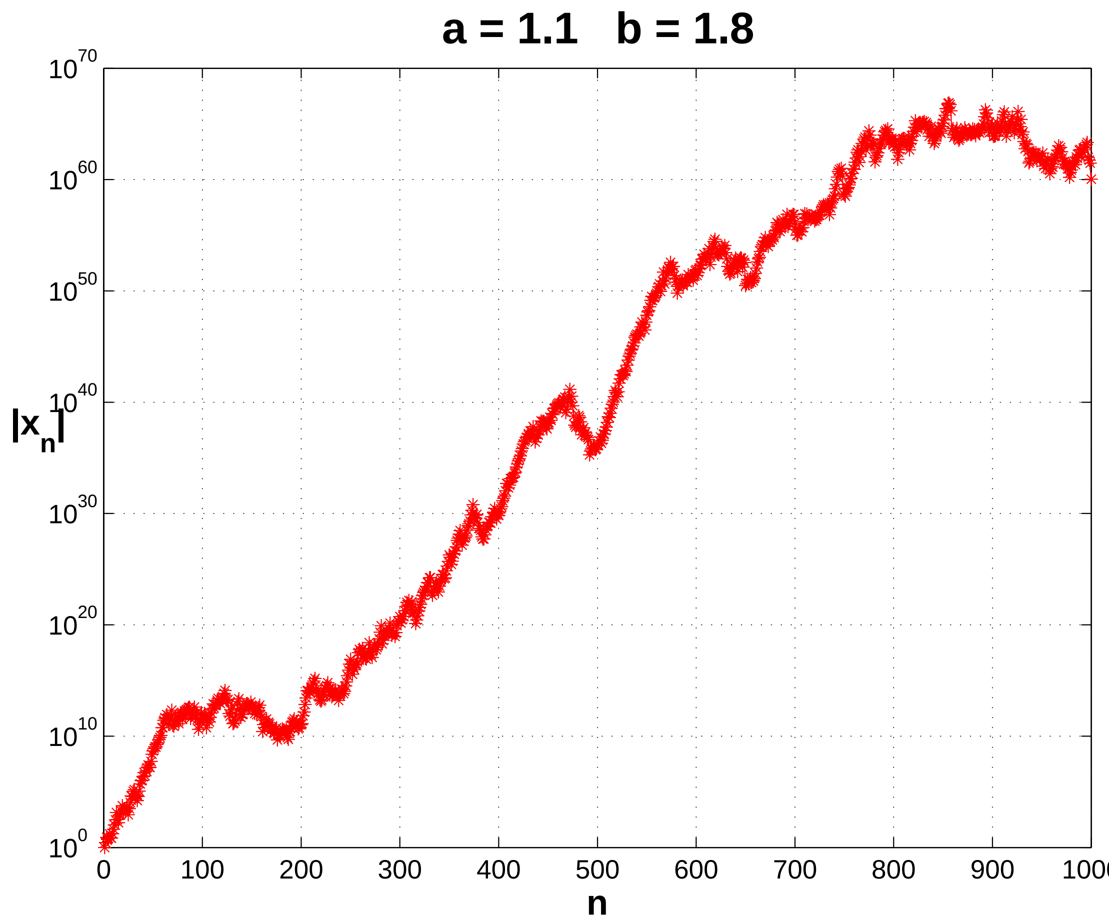

Overview

- Ito SDEs and the theta method
- linear mean-square stability
- generalizing B-stability

- jump-SDEs and implicit methods
- linear mean-square stability
- compensated Poisson process
- generalizing B-stability

Stochastic Theta Method

$$\mathbf{X}_{n+1} = \mathbf{X}_n + (1-\theta)\Delta t f(\mathbf{X}_n) + \theta \Delta t f(\mathbf{X}_{n+1}) + g(\mathbf{X}_n)\Delta \mathbf{W}_n$$

where $\Delta \mathbf{W}_n = W(t_{n+1}) - W(t_n)$,
so $\Delta \mathbf{W}_n = \sqrt{\Delta t}\mathbf{V}_n$, with $\mathbf{V}_n \sim$ Normal$(0,1)$ i.i.d.

$\mathbf{X}_n \approx \mathbf{X}(t_n)$ in the SDE (Itô)

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t), \quad \mathbf{X}(0) = \mathbf{X}_0$$

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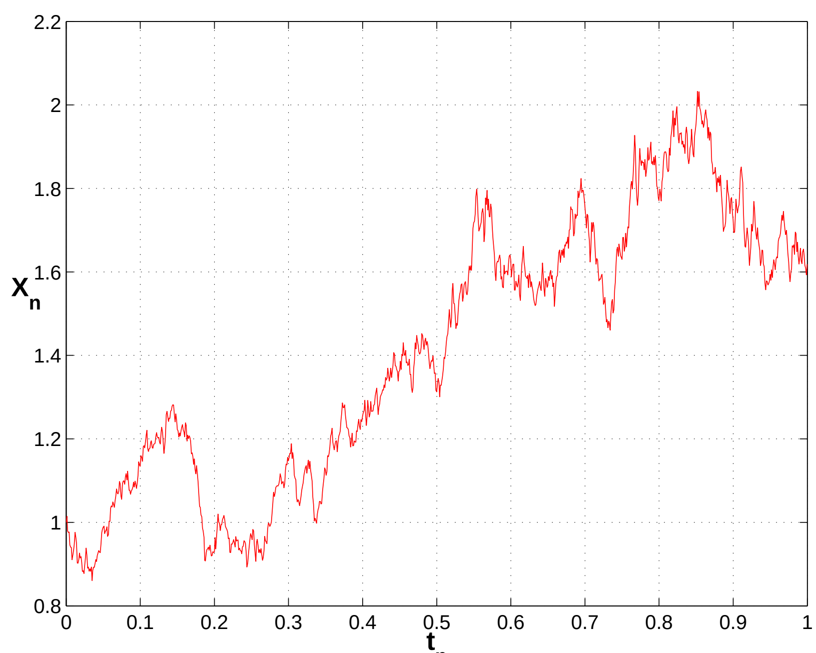


Stochastic Test Equation

$$d\mathbf{X}(t) = \mu \mathbf{X}(t)dt + \sigma \mathbf{X}(t)d\mathbf{W}(t)$$

(Asset model in math-finance)

Mean-square stability

$$\lim_{t \to \infty} \mathbb{E}(\mathbf{X}(t)^2) = 0 \Leftrightarrow 2\mu + \sigma^2 < 0$$

STM gives $\mathbf{X}_{n+1} = (a + b\mathbf{V}_n)\mathbf{X}_n$, with

$$a := \frac{1 + (1-\theta)\mu \Delta t}{1 - \theta \mu \Delta t}, \qquad b := \frac{\sigma \sqrt{\Delta t}}{1 - \theta \mu \Delta t}$$

Mean-square stability

Saito & Mitsui, SIAM J Num Anal 1996

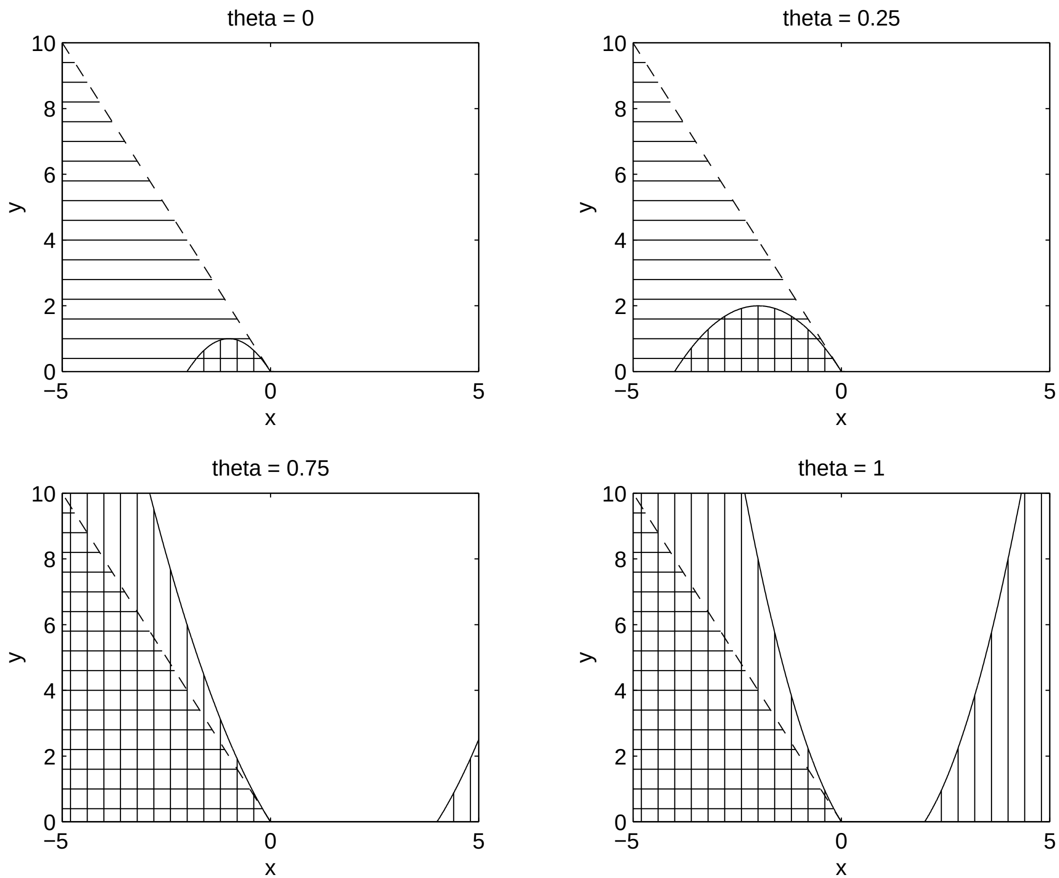
$0 \le \theta < \frac{1}{2}$: SDE **stable** $\Rightarrow$ method **stable** iff

$$\Delta t < \frac{|2\mu + \sigma^2|}{\mu^2(1-2\theta)}$$

$\theta = \frac{1}{2}$: SDE **stable** $\Leftrightarrow$ method **stable** $\forall \Delta t > 0$
$\frac{1}{2} < \theta \le 1$: SDE **stable** $\Rightarrow$ method **stable** $\forall \Delta t > 0$

Stability Regions

Let $x := \Delta t\mu$ and $y := \Delta t\sigma^2$
SDE **stable** $\Leftrightarrow y < -2x$
Method **stable** $\Leftrightarrow y < (2\theta - 1)x^2 - 2x$



Linear Stability Issues?

Deterministic analysis popular:

- describes behavior around a fixed point
- models the propagation of errors
- scalar problem is relevant to systems
- leads the way to nonlinear theories (G-stability, B-stability,...)

How much of this remains true for SDEs?

Non globally Lipschitz example

Scalar SDE $d\mathbf{X}(t) = (-\mathbf{X}(t) - \mathbf{X}(t)^3)dt + \mathbf{X}(t)d\mathbf{W}(t)$ has

$$\mathbb{E}\mathbf{X}(t)^2 \leq e^{-t}\,\mathbb{E}\mathbf{X}(0)^2$$

Euler–Maruyama ($\theta = 0$) for any $0 < \Delta t \leq 2$: if

$$(\mathbb{E}\mathbf{X}(0)^2)^2 \geq \frac{6}{\Delta t^2}$$

then $\mathbb{E}\mathbf{X}_n^2 \geq 2^n\,\mathbb{E}\mathbf{X}_0^2$ and hence $\lim_{n\to\infty}\mathbb{E}\mathbf{X}_n^2 = \infty$

Euler–Maruyama does not preserve stability

One-sided Lipschitz Condition

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

Conditions:

$$\langle u - v, f(u) - f(v) \rangle \leq \nu |u - v|^2, \quad \text{and}$$

$$|g(u) - g(v)|^2 \leq L_g |u - v|^2, \quad \forall u, v \in \mathbb{R}^n$$

$$\Rightarrow \mathbb{E}|\mathbf{X}(t) - \mathbf{Y}(t)|^2 \leq e^{(2\nu + L_g)t}\, \mathbb{E}|\mathbf{X}(0) - \mathbf{Y}(0)|^2$$

Result: For any $\Delta t > 0$, **Backward Euler** method

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t f(\mathbf{X}_{n+1}) + g(\mathbf{X}_n)\Delta \mathbf{W}_n$$

gives $\mathbb{E}|\mathbf{X}_n - \mathbf{Y}_n|^2 \leq e^{\widehat{\gamma}(\Delta t) n \Delta t}\, \mathbb{E}|\mathbf{X}_0 - \mathbf{Y}_0|^2$, where

$$\widehat{\gamma}(\Delta t) := \frac{1}{\Delta t} \log \left[\frac{1 + L_g \Delta t}{1 - 2\nu \Delta t} \right]$$

Note $\widehat{\gamma}(\Delta t) < 0$ when $2\nu + L_g < 0$

Jump-Stochastic Theta Method

$$\mathbf{X}_{n+1} = \mathbf{X}_n + (1-\theta)\Delta t f(\mathbf{X}_n) + \theta \Delta t f(\mathbf{X}_{n+1}) + g(\mathbf{X}_n)\Delta \mathbf{W}_n + h(\mathbf{X}_n)\Delta \mathbf{N}_n$$

where $\Delta \mathbf{N}_n = \mathbf{N}(t_{n+1}) - \mathbf{N}(t_n)$
$\mathbf{N}(t)$ is a **Poisson process** with intensity λ:

$$\mathbb{P}\left(\Delta \mathbf{N}_n = k\right) = e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Jump-Stochastic Theta Method

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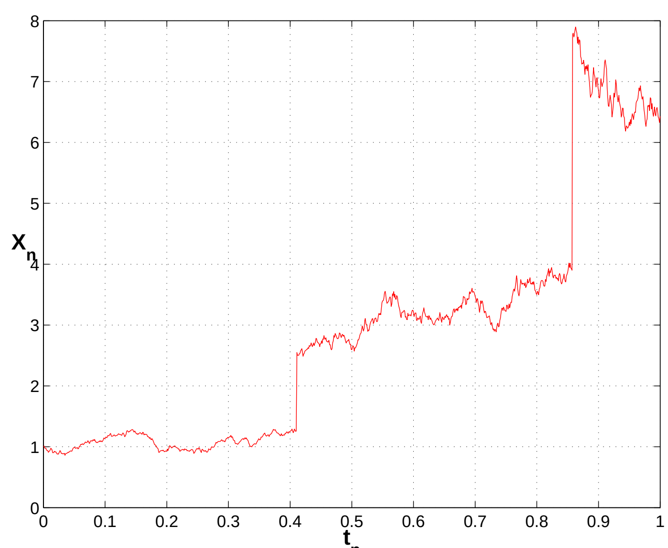

$$\mathbb{E}(\#Jumps) = \lambda T$$

Jump-Stochastic Theta Method

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$\mathbf{X}_n \approx \mathbf{X}(t_n)$ in the jump-SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t) + h(\mathbf{X}(t))d\mathbf{N}(t)$$

Hanson & Westman, SIAM book, forthcoming

Appl'ns: economics, biology, physics, engineering, ...

Existence and **strong finite-time convergence** goes through

Stability ......

Linear jump-SDE Test Equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t) + \gamma X(t)dN(t)$$

(Asset model in math-finance)

Mean-square stability

$$\lim_{t\to\infty} \mathbb{E}(X(t)^2) = 0 \Leftrightarrow 2\mu + \sigma^2 + \lambda\gamma(2+\gamma) < 0$$

- **symmetric** about $\gamma = -1$
- for $-2 < \gamma < 0$, jump term is **stabilizing**

Jump-Stochastic Theta Method

Euler $\theta = 0$:

1. problem **stable** $\Rightarrow$ method **stable** for

$$\Delta t < \frac{|2\mu + \sigma^2 + \lambda\gamma(2+\gamma)|}{(\mu + \gamma\lambda)^2}$$

2. problem **unstable** $\Rightarrow$ method **unstable** $\forall \Delta t > 0$

General $0 < \theta \leq 1$:

1. there exist $\{\mu, \sigma, \lambda, \gamma\}$ for which problem **stable** and method **stable** $\forall \Delta t > 0$
2. given any $\epsilon > 0$, there exist $\{\mu, \sigma, \lambda, \gamma\}$ for which problem **unstable**, yet method **stable** $\forall \Delta t > \epsilon$

A-stability

Use the term **A-stability** to mean

problem stable $\Rightarrow$ method stable $\forall \Delta t > 0$

For $\gamma \geq 0$:

method **A-stable** whenever $\theta \geq \frac{1}{2}$

For general $\gamma \in \mathbb{R}$:

method **not A-stable** for any θ

Compensated Poisson Process

$\widetilde{\mathbf{N}}(t) := \mathbf{N}(t) - \lambda t$ **(martingale)**

Write jump-SDE as

$$d\mathbf{X}(t) = f_\lambda(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t) + h(\mathbf{X}(t))d\widetilde{\mathbf{N}}(t)$$

where $f_\lambda(\cdot) := f(\cdot) + \lambda h(\cdot)$

CSSBE method:

$$\mathbf{X}_n^\star = \mathbf{X}_n + f_\lambda(\mathbf{X}_n^\star)\Delta t$$
$$\mathbf{X}_{n+1} = \mathbf{X}_n^\star + g(\mathbf{X}_n^\star)\Delta\mathbf{W}_n + h(\mathbf{X}_n^\star)\Delta\widetilde{\mathbf{N}}_n$$

where $\Delta\widetilde{\mathbf{N}}_n = \widetilde{\mathbf{N}}(t_{n+1}) - \widetilde{\mathbf{N}}(t_n)$

Existence and **strong finite-time convergence** follow under OSL on f

Contractivity . . . . . .

Mean-Square Contractivity: Jump-SDE

$$\langle u - v, f(u) - f(v) \rangle \le \nu |u - v|^2$$
$$|g(u) - g(v)|^2 \le L_g |u - v|^2$$
$$|h(u) - h(v)|^2 \le L_h |u - v|^2$$

gives

$$\mathbb{E}|\mathbf{X}(t) - \mathbf{Y}(t)|^2 \le e^{\alpha t}\, \mathbb{E}|\mathbf{X}_0 - \mathbf{Y}_0|^2$$

where

$$\alpha := 2\nu + L_g + \lambda \sqrt{L_h} \left(\sqrt{L_h} + 2 \right)$$

Mean-Square Contractivity: CSSBE

If $\alpha < 0$ then for all $\Delta t > 0$ CSSBE gives

$$\mathbb{E}|\mathbf{X}_n - \mathbf{Y}_n|^2 \leq e^{\widehat{\alpha}(\Delta t) n \Delta t}\, \mathbb{E}|\mathbf{X}_0 - \mathbf{Y}_0|^2$$

where

$$\widehat{\alpha}(\Delta t) := \frac{1}{\Delta t} \log \left(\frac{1 + \Delta t (L_g + \lambda L_h)}{1 - 2\Delta t(\nu + \lambda \sqrt{L_h})} \right) < 0$$

Further,

$$\widehat{\alpha}(\Delta t) = \alpha + O(\Delta t)$$

Remarks

- "**Uncompensated**" version requires $\Delta t < -\alpha/(L_h\lambda^2)$
- CSSBE also gives **A-stability** for all $\gamma \in \mathbb{R}$