Euler–Maruyama

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Euler–Maruyama

- Definition of Euler–Maruyama Method
- Weak Convergence
- Strong Convergence

Recap: SDE

Given functions f and g, the stochastic process $\mathbf{X}(t)$ is a soluton of the SDE

$$d\mathbf{X}(t) = f(\mathbf{X}(t))dt + g(\mathbf{X}(t))d\mathbf{W}(t)$$

if $\mathbf{X}(t)$ solves the integral equation

$$\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t f(\mathbf{X}(s))\,ds + \int_0^t g(\mathbf{X}(s))\,d\mathbf{W}(s)$$

Discretize the interval $[0, T]$: let $\Delta t = T/N$ and $t_n = n\Delta t$
Compute $\mathbf{X}_n \approx \mathbf{X}(t_n)$
Initial value $\mathbf{X}_0$ is given

Euler–Maruyama

Exact solution:

$$\mathbf{X}(t_{n+1}) = \mathbf{X}(t_n) + \int_{t_n}^{t_{n+1}} f(\mathbf{X}(s))\,ds + \int_{t_n}^{t_{n+1}} g(\mathbf{X}(s))\,d\mathbf{W}(s)$$

Euler–Maruyama:

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t\, f(\mathbf{X}_n) + \Delta \mathbf{W}_n\, g(\mathbf{X}_n)$$

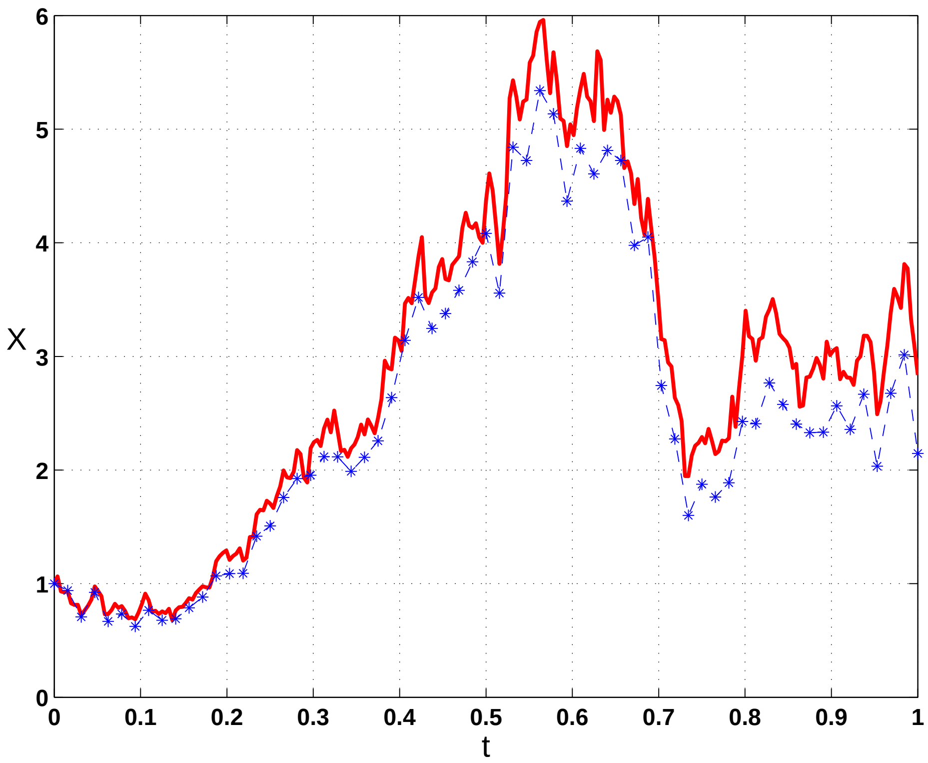
(Left endpoint Riemann sums)

In MATLAB, $\Delta \mathbf{W}_n$ becomes `sqrt(Dt)*randn`

$f(x) = \mu x$ and $g(x) = \sigma x$, $\mu = 2$, $\sigma = 0.1$, $X(0) = 1$

Solution: $\mathbf{X}(t) = \mathbf{X}(0) e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \mathbf{W}(t)}$

Disc. Brownian path with $\delta t = 2^{-8}$, E-M with $\Delta t = 4\delta t$:



$|\mathbf{X}_N - \mathbf{X}(T)| = 0.69$

Reducing to $\Delta t = 2\delta t$ gives $|\mathbf{X}_N - \mathbf{X}(T)| = 0.16$

Reducing to $\Delta t = \delta t$ gives $|\mathbf{X}_N - \mathbf{X}(T)| = 0.08$

Convergence?

$\mathbf{X}_n$ and $\mathbf{X}(t_n)$ are **random variables** at each t_n

In what sense does $|\mathbf{X}_n - \mathbf{X}(t_n)| \to 0$ as $\Delta t \to 0$?

There are many, non-equivalent, definitions of convergence for sequences of random variables

The two most common and useful concepts in numerical SDEs are

- **Weak convergence:** error of the mean
- **Strong convergence:** mean of the error

Weak Convergence

Weak convergence: capture the average behaviour

Given a function Φ, the **weak error** is

$$e_{\Delta t}^{\text{weak}} := \sup_{0 \le t_n \le T} \left| \mathbb{E}\left[\Phi(\mathbf{X}_n)\right] - \mathbb{E}\left[\Phi(\mathbf{X}(t_n))\right] \right|$$

Φ from e.g. set of polynomials of degree at most k

Converges weakly if $e_{\Delta t}^{\text{weak}} \to 0$, as $\Delta t \to 0$

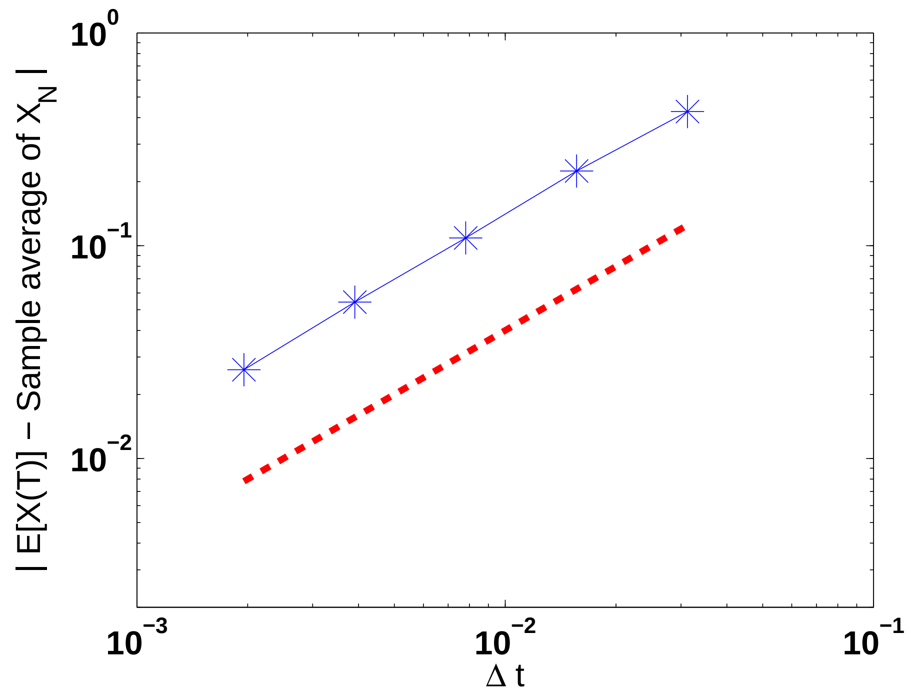
Weak order p if $e_{\Delta t}^{\text{weak}} \le K \Delta t^p$, for all $0 < \Delta t \le \Delta t^\star$

In practice we estimate $\mathbb{E}[\Phi(\mathbf{X}_n)]$ by Monte Carlo simulation over many paths $\Rightarrow$ "$1/\sqrt{M}$" sampling error

$f(x) = \mu x$ and $g(x) = \sigma x$, $\mu = 2$, $\sigma = 0.1$, $X(0) = 1$

Solution has $\mathbb{E}[X(t)] = e^{\mu t}$

Measure weak endpoint error $|a_M - e^{\mu T}|$ over $M = 10^5$ discretized Brownian paths. Try $\Delta t = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$



Least squares fit: power is 1.011

(Confidence intervals smaller than graphics symbols)

Suggests weak order $p = 1$

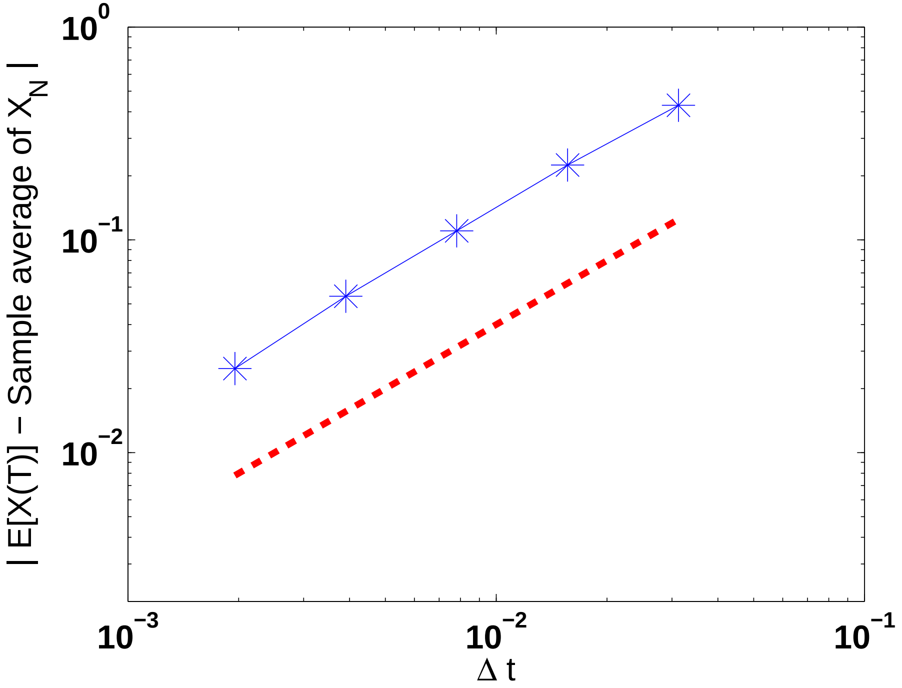
Weak Euler–Maruyama

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t f(\mathbf{X}_n) + \widehat{\Delta \mathbf{W}}_n \, g(\mathbf{X}_n)$$

where $\mathbb{P}\left(\widehat{\Delta \mathbf{W}}_n = \sqrt{\Delta t}\right) = \frac{1}{2} = \mathbb{P}\left(\widehat{\Delta \mathbf{W}}_n = -\sqrt{\Delta t}\right)$

E.g. use `sqrt(Dt)*sign(randn)`

or `sqrt(Dt)*sign(rand-0.5)`



Least squares fit: power is 1.03

Weak Euler–Maruyama

Generally, EM and weak EM have weak order $p = 1$ on appropriate SDEs for $\Phi(\cdot)$ with polynomial growth

Can prove via **Feynman-Kac formula** that relates SDEs to PDEs

Strong Convergence

Strong convergence: follow paths accurately

Strong error is

$$e_{\Delta t}^{\text{strong}} := \sup_{0 \le t_n \le T} \mathbb{E}\left[|\mathbf{X}_n - \mathbf{X}(t_n)| \right]$$

Converges strongly if $e_{\Delta t}^{\text{strong}} \to 0$, as $\Delta t \to 0$

Strong order p if $e_{\Delta t}^{\text{strong}} \le K \Delta t^p$, for all $0 < \Delta t \le \Delta t^\star$

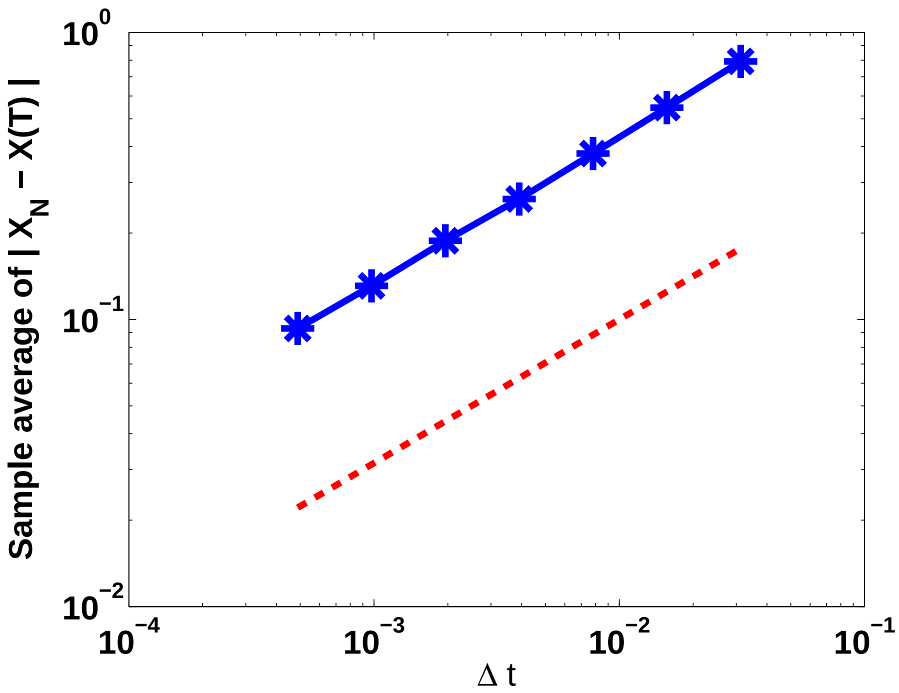
$f(x) = \mu x$ **and** $g(x) = \sigma x$**,** $\mu = 2$**,** $\sigma = 1$**,** $X(0) = 1$

Solution: $\mathbf{X}(t) = \mathbf{X}(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \mathbf{W}(t)}$

$M = 5,000$ disc. Brownian paths over $[0,1]$ with $\delta t = 2^{-11}$

For each path apply EM with $\Delta t = \delta t, 2\delta t, 4\delta t, 16\delta t, 32\delta t, 64\delta t$

Record $\mathbb{E}\left[|\mathbf{X}_N - \mathbf{X}(1)|\right]$ for each δt



Least squares fit: power is 0.51

Strong Convergence

Generally EM has strong order $p = \frac{1}{2}$ on appropriate SDEs

Can prove using Ito's Lemma, Ito isometry and Gronwall

Note: **strong convergence** $\Rightarrow$ **weak convergence**,
but this doesn't recover the optimal weak order

Strong Convergence

Euler–Maruyama has

$$\mathbb{E}\left[|\mathbf{X}_n - \mathbf{X}(t_n)|\right] \leq K \Delta t^{\frac{1}{2}}$$

Markov inequality says

$$\mathbb{P}\left(|\mathbf{X}| > a\right) \leq \frac{\mathbb{E}[|\mathbf{X}|]}{a}, \qquad \text{for any } a > 0$$

Taking $a = \Delta t^{\frac{1}{4}}$ gives $\mathbb{P}\left(|\mathbf{X}_n - \mathbf{X}(t_n)| \geq \Delta t^{\frac{1}{4}}\right) \leq K \Delta t^{\frac{1}{4}}$, i.e.

$$\mathbb{P}\left(|\mathbf{X}_n - \mathbf{X}(t_n)| < \Delta t^{\frac{1}{4}}\right) \geq 1 - K \Delta t^{\frac{1}{4}}$$

Along any path error is small with high prob.

Higher Strong Order

If $g(x)$ is constant, then EM has strong order $p = 1$

More generally, strong order $p = 1$ is achieved by the **Milstein** method

$$\begin{aligned} \mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t f(\mathbf{X}_n) + \Delta \mathbf{W}_n \, g(\mathbf{X}_n) \\ + \tfrac{1}{2} g(\mathbf{X}_n) g'(\mathbf{X}_n) \left(\Delta \mathbf{W}_n^2 - \Delta t \right) \end{aligned}$$

(More complicated for SDE systems.)

Even Higher Strong Order: Warning!

Numerical methods for stochastic differential equations
Joshua Wilkie
Physical Review E, 2004

Claims to derive arbitrarily high (strong?) order methods, with a Runge–Kutta approach.
But using only Brownian increments, $\Delta \mathbf{W}_n$, rather than more general integrals like

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} d\mathbf{W}_1(s) d\mathbf{W}_2(t)$$

there is an order barrier of $p = 1$ (Rümelin, 1982).

Nonlinear SDEs

There is a limited amount of theory regarding convergence on nonlinear SDEs for which global Lipschitz conditions do not hold

em.m: part 1

```
%EM  Euler-Maruyama method on linear SDE
%
% SDE is  dX = lambda*X dt + mu*X dW,   X(0) = Xzero,
%      where lambda = 2, mu = 1 and Xzero = 1.
%
% Discretized Brownian path over [0,1] has dt = 2^(-8).
% Euler-Maruyama uses timestep R*dt.

clf
randn('state',100)                   % set state of randn
lambda = 2; mu = 1; Xzero = 1;    % problem parameters
T = 1; N = 2^8; dt = T/N;
dW = sqrt(dt)*randn(1,N);            % Brownian increments
W = cumsum(dW);                      % disc. Brownian path

Xtrue = Xzero*exp((lambda-0.5*mu^2)*([dt:dt:T])+mu*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on
```

em.m: part 2

```
R = 4; Dt = R*dt; L = N/R; % L EM steps of size Dt = R*dt
Xem = zeros(1,L);           % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
   Winc = sum(dW(R*(j-1)+1:R*j));
   Xtemp = Xtemp + Dt*lambda*Xtemp + mu*Xtemp*Winc;
   Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'r--*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment',

emerr = abs(Xem(end)-Xtrue(end))
```

Least Squares Fit

$$\text{Xerr}_i = C \Delta t_i^q \Rightarrow \log(\text{Xerr}_i) = \log(C) + q \log(\Delta t_i)$$

This is

$$\begin{bmatrix} 1 & \log(\Delta t_1) \\ 1 & \log(\Delta t_2) \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \log(C) \\ q \end{bmatrix} = \begin{bmatrix} \log(\text{Xerr}_1) \\ \log(\text{Xerr}_2) \\ \vdots \end{bmatrix}$$

```
%%%% Least squares fit of error = C * Dt^q %%%
A = [ones(p,1),log(Dtvals)']; rhs = log(Xerr);
sol = A\rhs; q = sol(2)
resid = norm(A*sol - rhs)
```