Numerical Simulation
of
Stochastic Differential Equations
Lecture Notes on a Strong Convergence Proof

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1 Introduction

These notes accompany my lectures on the numerical simulation of stochastic differential equations. I have produced this handout because some of the details are messy to write down, and hence not all of this may get written on the blackboard.

We will prove that Euler–Maruyama gives strong convergence with order at least $\frac{1}{2}$. The proof is based on that of Theorem 9.6.2 of KP. [Our proof is specialised to E–M, and we spell out the key steps more slowly.]

2 Strong Convergence Proof

We need the following Lemmas:

Lemma 2.1 ***(Trivial)*** *For any $a, b, c, d \in \mathbb{R}$*

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2).$$

Proof This follows from $(x+y)^2 = x^2+2xy+y^2 \leq 2(x^2+y^2)$. ∎

Lemma 2.2 ***(Cauchy-Schwartz)***

$$\mathbb{E}\left[\left(\int_0^t S(r)dr\right)^2\right] \leq t\int_0^t \mathbb{E}\left[S(r)^2\right] dr,$$

for suitable stochastic processes $S(r)$.

Proof Cauchy-Schwartz on a deterministic integral gives

$$\left(\int_0^t h(r)dr\right)^2 \leq \int_0^t 1^2 dr \int_0^t h(r)^2 dr = t\int_0^t h(r)^2 dr.$$

For a stochastic $h(r)$, take expected values and sneak $\mathbb{E}$ inside the integral. ∎

Lemma 2.3 ***(Itô Isometry)***

$$\mathbb{E}\left[\left(\int_0^t S(r)dW(r)\right)^2\right] = \mathbb{E}\left[\int_0^t S(r)^2 dr\right] = \int_0^t \mathbb{E}\left[S(r)^2\right] dr,$$

for suitable stochastic processes $S(r)$.

Proof This is the *Itô Isometry*. See, for example, BZ page 182—185, or KP page 85. ∎

Lemma 2.4 ***(Lyapunov)*** *For suitable random variables X*

$$\mathbb{E}[|X|] \leq \sqrt{\mathbb{E}[X^2]}.$$

Proof This is a consequence of Jensen's inequality (KP, page 16), and also a special case of Lyapunov's inequality (KP, page 17). ∎

Lemma 2.5 ***(Gronwall)*** *If $\alpha(t)$ is integrable and*

$$0 \leq \alpha(t) \leq A + B\int_0^t \alpha(r)dr, \qquad 0 \leq t \leq T,$$

for some constants $A, B > 0$, then

$$\alpha(t) \leq Ae^{Bt}, \qquad 0 \leq t \leq T.$$

Proof This is Gronwall's Lemma. (See, for example, *Numerical Solution of Ordinary Differential Equations*, by L. F. Shampine, Chapman & Hall, 1994.) ∎

Notation for convergence theorem

Recall that we have the (scalar, autonomous) SDE

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \tag{1}$$

for $0 \leq t \leq T$, with $X(0) = X_0$ given. Applying Euler–Maruyama gives

$$X_{n+1} = X_n + f(X_n)\Delta t + g(X_n)\Delta W_n,$$

where $\Delta W_n := W(t_{n+1}) - W(t_n)$.

We will let $\overline{X}(t)$ denote a piecewise constant extension of $\{X_n\}$, so that

$$\overline{X}(t) = X_n, \quad \text{for } t_n \leq t < t_{n+1}.$$

Also, given a point $s \in [0,T]$, we let n_s denote the integer such that $s \in [t_{n_s}, t_{n_s+1})$. (In other words, $\overline{X}(s) = X_{n_s}$.)

We will assume that the functions f and g in (1) satisfy a global Lipschitz condition; that is, there is a constant K such that

$$|f(x) - f(y)| \leq K|x-y| \quad \text{and} \quad |g(x) - g(y)| \leq K|x-y|, \tag{2}$$

for all $x, y \in \mathbb{R}$.

We also assume that the SDE satisfies other conditions (such as those in Theorem 4.5.3 of KP) to guarantee that a unique solution exists with bounded second moment.

Theorem 2.1 *Under appropriate assumptions about the SDE, including the global Lipschitz condition (2), Euler–Maruyama gives strong convergence of order at least* $\frac{1}{2}$.

Proof We have

$$\begin{aligned}
\overline{X}(s) - X(s) &= X_{n_s} - X(s) \\
&= X_{n_s} - \left(X_0 + \int_0^s f(X(r))dr + \int_0^s g(X(r))dW(r) \right) \\
&= \sum_{i=0}^{n_s-1} (X_{i+1} - X_i) - \int_0^s f(X(r))dr - \int_0^s g(X(r))dW(r) \\
&= \sum_{i=0}^{n_s-1} f(X_i)\Delta t + \sum_{i=0}^{n_s-1} g(X_i)\Delta W_i - \int_0^s f(X(r))dr - \int_0^s g(X(r))dW(r) \\
&= \int_0^{t_{n_s}} f(\overline{X}(r))dr + \int_0^{t_{n_s}} g(\overline{X}(r))dW(r) - \int_0^s f(X(r))dr - \int_0^s g(X(r))dW(r) \\
&= \int_0^{t_{n_s}} f(\overline{X}(r)) - f(X(r))dr + \int_0^{t_{n_s}} g(\overline{X}(r)) - g(X(r))dW(r) \\
&\quad - \int_{t_{n_s}}^s f(X(r))dr - \int_{t_{n_s}}^s g(X(r))dW(r). \qquad (3)
\end{aligned}$$

Squaring, taking expected values and using Lemma 2.1, it follows that

$$\begin{aligned}
\mathbb{E}\left[\left(\overline{X}(s) - X(s) \right)^2 \right] \leq 4\{ \quad & \mathbb{E}\left[\left(\int_0^{t_{n_s}} f(\overline{X}(r)) - f(X(r))dr \right)^2 \right] \\
+ \quad & \mathbb{E}\left[\left(\int_0^{t_{n_s}} g(\overline{X}(r)) - g(X(r))dW(r) \right)^2 \right] \\
+ \quad & \mathbb{E}\left[\left(\int_{t_{n_s}}^s f(X(r))dr \right)^2 \right]
\end{aligned}$$

$$+ \quad \mathbb{E}\left[\left(\int_{t_{n_s}}^{s} g(X(r))dW(r)\right)^2\right]\}. \qquad (4)$$

We will now bound the four terms on the right-hand side of (4). We introduce the notation

$$Z(t) := \sup_{0\le s\le t} \mathbb{E}\left[\left(\overline{X}(s) - X(s)\right)^2\right].$$

For the first term, using Lemma 2.2 (Cauchy-Schwartz) and the Lipschitz condition (2), we have

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^{t_{n_s}} f(\overline{X}(r)) - f(X(r))dr\right)^2\right] &\le K^2 T \int_0^s \mathbb{E}\left[\left(\overline{X}(r)) - X(r)\right)^2\right] dr \\ &\le K^2 T \int_0^s Z(r)dr. \qquad (5)\end{aligned}$$

For the second term in (4), using Lemma 2.3 (Itô Isometry) and the Lipschitz condition (2), we have

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^{t_{n_s}} g(\overline{X}(r)) - g(X(r))dW(r)\right)^2\right] &\le K^2 \int_0^s \mathbb{E}\left[\left(\overline{X}(r) - X(r)\right)^2\right] dr \\ &\le K^2 \int_0^s Z(r)dr. \qquad (6)\end{aligned}$$

Lemmas 2.2 and 2.3 and the Lipschitz condition (2) also allow us to bound the third and fourth terms:

$$\begin{aligned}\mathbb{E}\left[\left(\int_{t_{n_s}}^s f(X(r))dr\right)^2\right] + \mathbb{E}\left[\left(\int_{t_{n_s}}^s g(X(r))dW(r)\right)^2\right] &\le (\Delta t + 1)K^2 \int_{t_{n_s}}^s \mathbb{E}\left[X(r)^2\right] dr \\ &\le (\Delta t + 1)K^2 K_M \Delta t, \qquad (7)\end{aligned}$$

where the constant K_M is an upper bound for the second moment of $X(r)$.

Inserting the bounds (5), (6) and (7) into (4) we find that

$$Z(t) \le 4K^2(T+1)\int_0^t Z(r)dr + 4K^2 K_M(1+\Delta t)\Delta t.$$

So, from Lemma 2.5 (Gronwall),

$$Z(t) \le 4K^2 K_M e^{4K^2(T+1)T}(1+\Delta t)\Delta t.$$

Hence, there is a constant C for which

$$Z(t) \le C\Delta t,$$

for sufficiently small Δt.

We have thus shown that

$$\mathbb{E}\left[\left(\overline{X}(t) - X(t)\right)^2\right] \leq C\Delta t, \quad \text{for all } 0 \leq t \leq T.$$

In particular, we have

$$\mathbb{E}\left[(X_n - X(n\Delta t))^2\right] \leq C\Delta t, \quad \text{for all } 0 \leq n\Delta t \leq T.$$

Lemma 2.4 (Lyapunov) then gives

$$\mathbb{E}\left[|X_n - X(n\Delta t)|\right] \leq \sqrt{\mathbb{E}\left[(X_n - X(n\Delta t))^2\right]} \leq \sqrt{C\Delta t},$$

which establishes the strong order result. ∎

Remarks:

1. Further analysis shows that E–M has strong order exactly equal to $\frac{1}{2}$ in general (as indicated by our numerical experiments).
2. Strong order $\frac{1}{2}$ implies weak order of at least $\frac{1}{2}$. In fact, E–M has weak order equal to 1 in general, as we observed in our numerical experiments. (This can be proved via PDE theory.)
3. In the deterministic case, $g \equiv 0$, the right-hand side of (7) becomes $O(\Delta t^2)$ and following the proof above leads to the correct classical deterministic order of 1 for Euler's method.