## Computer practicals (in MatLab) 1 - Mean-square approximation of SDEs

## Tasks

1. Using the in-built function $\operatorname{randn}()$ for generating random numbers with standard normal distribution in MatLab, simulate trajectories of the Wiener process. Plot the trajectories and zoom-in to visualise the fractal nature of the Wiener process.
2. Consider the linear SDE

$$
\begin{equation*}
d S=r S d t+\sigma S d w, \quad S(0)=S_{0} \tag{1}
\end{equation*}
$$

where $r$ and $\sigma$ are some positive constants. This is an SDE for GBM. Apply the Euler scheme and Milstein scheme to (1) and realise the corresponding algorithms in MatLab. Solve (1) exactly. Run MatLab simulations of (1) using the exact formula for $S\left(t_{k}\right)$ and the two methods along the same Wiener paths and compute the corresponding errors $\left|S\left(t_{k}\right)-X_{k}\right|$. Look experimentally at the a.s. convergence of the Euler and Milstein methods.
3. Continue the previous exercise and experimentally study the mean square errors of the Euler and Milstein methods. Use the Monte Carlo technique to evaluate the errors, i.e.

$$
\sqrt{E\left|S\left(t_{k}\right)-S_{k}\right|^{2}} \approx \sqrt{\frac{1}{M} \sum_{m=1}^{M}\left|S^{(m)}\left(t_{N}\right)-S_{N}^{(m)}\right|^{2}}
$$

where $S^{(m)}\left(t_{N}\right)$ and $S_{N}^{(m)}$ are realisations of the exact solution to (1) and of one of the two numerical methods, respectively, along $m$ th independent Wiener path. To observe the expected mean-square order, $M$ should be sufficiently large to make the Monte Carlo error negligible.
4. The system of SDEs in the sense of Stratonovich (Kubo oscillator)

$$
\begin{gather*}
d X^{1}=-a X^{2} d t-\sigma X^{2} \circ d w(t), \quad X^{1}(0)=x^{1}  \tag{2}\\
d X^{2}=a X^{1} d t+\sigma X^{1} \circ d w(t), \quad X^{2}(0)=x^{2}
\end{gather*}
$$

is often used for testing numerical methods. Here $a$ and $\sigma$ are constants and $w(t)$ is a one-dimensional standard Wiener process. The phase flow of this system preserves symplectic structure. Moreover, the quantity $\mathcal{H}\left(x^{1}, x^{2}\right)=\left(x^{1}\right)^{2}+\left(x^{2}\right)^{2}$ is conservative for this system, i.e.,

$$
\mathcal{H}\left(X^{1}(t), X^{2}(t)\right)=\mathcal{H}\left(x^{1}, x^{2}\right) \text { for } t \geq 0
$$

This means that a phase trajectory of (2) belongs to the circle with center at the origin and of radius $\sqrt{\mathcal{H}\left(x^{1}, x^{2}\right)}$.
Consider two methods of order $1 / 2$ for (2): the explicit Euler scheme

$$
\begin{align*}
& X_{k+1}^{1}=X_{k}^{1}-a X_{k}^{2} h-\frac{\sigma^{2}}{2} X_{k}^{1} h-\sigma X_{k}^{2} \Delta_{k} w  \tag{3}\\
& X_{k+1}^{2}=X_{k}^{2}+a X_{k}^{1} h-\frac{\sigma^{2}}{2} X_{k}^{2} h+\sigma X_{k}^{1} \Delta_{k} w
\end{align*}
$$

and the symplectic partitioned Runge-Kutta method ${ }^{2}$ :

$$
\begin{gather*}
X_{k+1}^{1}=X_{k}^{1}-a X_{k}^{2} h-\frac{\sigma^{2}}{2} X_{k+1}^{1} h-\sigma X_{k}^{2} \Delta_{k} w  \tag{4}\\
X_{k+1}^{2}=X_{k}^{2}+a X_{k+1}^{1} h+\frac{\sigma^{2}}{2} X_{k}^{2} h+\sigma X_{k+1}^{1} \Delta_{k} w
\end{gather*}
$$

[^0]Plot trajectories produces by these methods in the phase space over relatively long simulation time.
Observe that despite both methods being of the same order of accuracy, (4) is much more accurate in long time simulations than (3). The symplectic partitioned Runge-Kutta method (4) is an example of stochastic geometric integrators ${ }^{2}$.


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    ${ }^{2}$ See [G.N. Milstein, Yu.M. Repin, M.V. Tretyakov. Numerical methods for stochastic systems preserving symplectic structure. SIAM J. Num. Anal. 40 (2002), 1583-1604] and [G.N. Milstein, M.V. Tretyakov. Stochastic Numerics for Mathematical Physics. Springer, 2004].

