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Long-time analysis of nonlinearly perturbed wave equations via modulated Fourier expansions

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SFB 382 *Verfahren und Algorithmen zur Simulation physikalischer Prozesse auf Höchstleistungsrechnern*

Joint work with E. Hairer & C. Lubich

The problem (I)

We consider

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \rho u(t,x) - g(u(t,x)) = 0\,,$$

where

- $t > 0$, $-\pi \leq x \leq \pi$, and periodic boundary conditions.
- $\rho > 0$ real.
- the smooth nonlinearity g with $g(0) = g'(0) = 0$.
- small initial data in an appropriate Sobolev space, say bounded by a small parameter ε (will be specified).

The problem (II)

Along every solution $u(x,t) = \sum_{j=-\infty}^{\infty} u_j(t)\mathrm{e}^{\mathrm{i}jx}$ of the linear wave equation $\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \rho u(x,t) = 0$, we have, for all j,

$$\ddot{u}_j(t) + (\rho + j^2)\, u_j(t) = 0.$$

The solution is $u_j(t) = \mathrm{e}^{\mathrm{i}\omega_j t}$, where we denote

$$\omega_j = \sqrt{\rho + j^2}$$

the frequencies of the linear equation.

The problem (III)

Still for the linear wave equation $\partial_t^2 u - \partial_x^2 u + \rho u = 0$, along every solution $u(x,t) = \sum_{j=-\infty}^{\infty} u_j(t) \mathrm{e}^{\mathrm{i}jx}$, the actions (energy divided by frequency)

$$I_j(t) = \frac{1}{2\omega_j} \left|\partial_t u_j(t)\right|^2 + \frac{\omega_j}{2} \left|u_j(t)\right|^2$$

remain constant in time.

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$$I_j(t) = \frac{1}{2\omega_j} |\partial_t u_j(t)|^2 + \frac{\omega_j}{2} |u_j(t)|^2$$

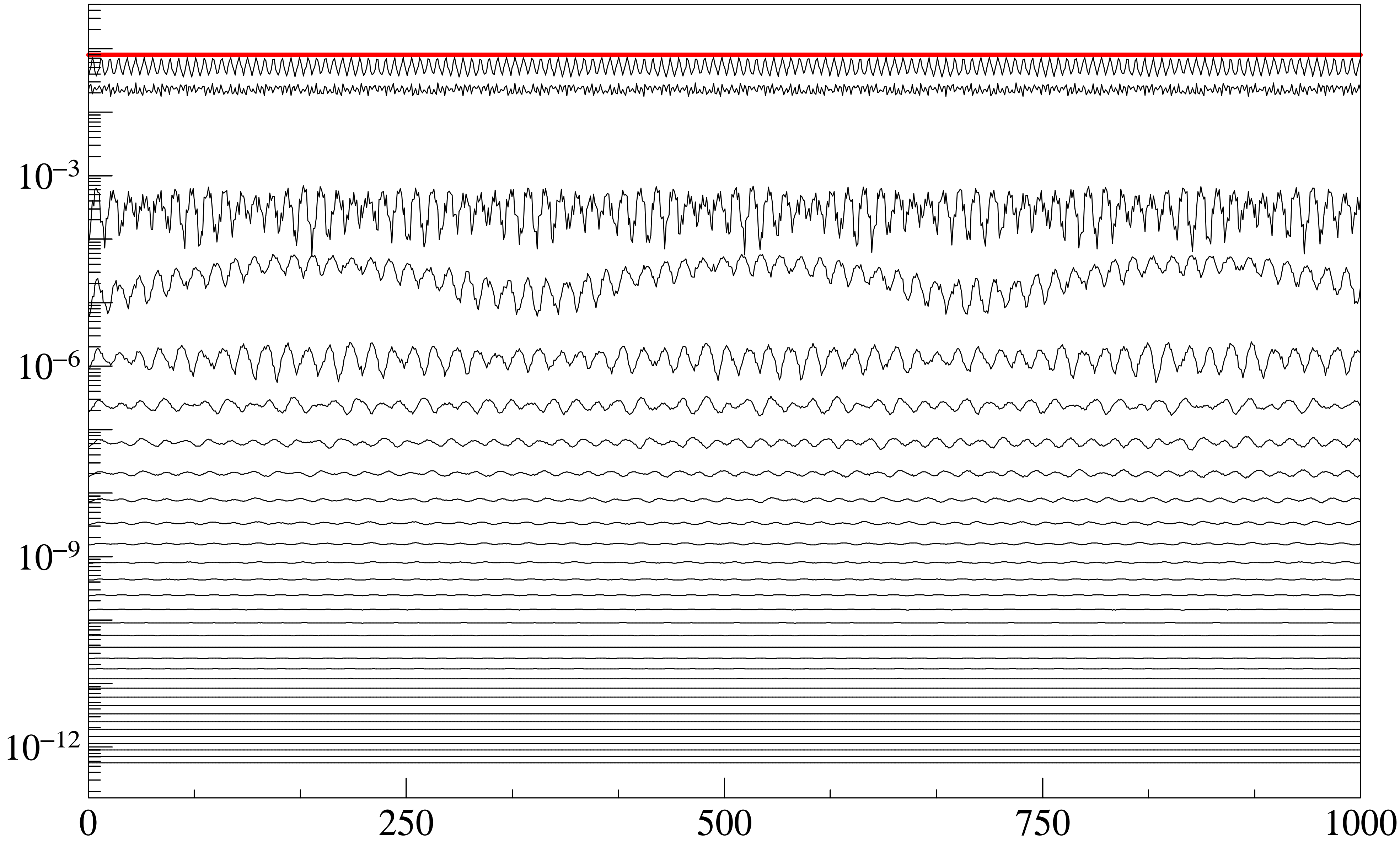
remain constant in time.

For the nonlinear wave equation, the sum of actions

$$J_\ell(t) = I_\ell(t) + I_{-\ell}(t)\,, \qquad \ell \geq 1\,, \qquad\qquad J_0(t) = I_0(t)$$

remain almost constant for long times.

The problem (IV)



$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + u(t,x) - u(t,x)^2 = 0$$

Results

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Bambusi 2003, Bourgain 1996.

Working spaces

For $v \in L^2$, 2π-periodic, $v(x) = \sum_{j=-\infty}^{\infty} v_j \mathrm{e}^{\mathrm{i}jx}$, we consider, for $s \geq 0$, the norm

$$\|v\|_s = \left(\sum_{j=-\infty}^{\infty} \omega_j^{2s} \, |v_j|^2 \right)^{1/2}$$

and the Sobolev-type space

$$H^s = \{ v \in L^2 : \|v\|_s < \infty \}.$$

Some precisions . . . (I)

For s large enough (depends on the non-resonance condition), we assume that the initial data satisfy

$$\Big(\|u(\cdot,0)\|_{s+1}^2 + \|\partial_t u(\cdot,0)\|_s^2 \Big)^{1/2} \leq \varepsilon.$$

Or in other words,

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s+1} \Big(\frac{1}{2\omega_j} \, |\partial_t u_j(0)|^2 + \frac{\omega_j}{2} \, |u_j(0)|^2 \Big) \leq \tfrac{1}{2} \, \varepsilon^2.$$

Some precisions . . . (II)

Theorem. *Under a non-resonance condition for* ω_j*, for small enough initial data, we have for an arbitrary (large) integer* N

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \frac{|J_\ell(t) - J_\ell(0)|}{\varepsilon^2} \le C\varepsilon \quad \textit{for} \quad 0 \le t \le \varepsilon^{-N}$$

where the constant C *depends on* N*, but not on* ε.

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where the constant C *depends on* N*, but not on* ε*.*

Corollary. *In the same norm that specifies the smallness condition on the initial data, the solution remains nearly constant for* $t \leq \varepsilon^{-N}$*:*

$$\|u(\cdot,t)\|_{s+1}^2 + \|\partial_t u(\cdot,t)\|_s^2 = \|u(\cdot,0)\|_{s+1}^2 + \|\partial_t u(\cdot,0)\|_s^2 + \mathcal{O}(\varepsilon^3)\,.$$

The modulated Fourier expansion (I)

The spatially 2π-periodic solutions of

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \rho u(t,x) = 0$$

are superpositions of plane waves $\mathrm{e}^{\pm \mathrm{i}\omega_j \pm \mathrm{i} j x}$ where j is an arbitrary integer and

$$\omega_j = \sqrt{\rho + j^2}$$

are the frequencies of the problem.

If the nonlinearity g is evaluated at superposition of plane waves, its Taylor expansion involves mixed products of such waves.

The modulated Fourier expansion (II)

We search for an approximation of the solution $u(x,t)$

$$u(x,t) \approx \sum_{\|\mathbf{k}\| \le K} z^{\mathbf{k}}(x, \varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} = \sum_{\|\mathbf{k}\| \le K} \sum_{j=-\infty}^{\infty} z_j^{\mathbf{k}}(\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t + \mathrm{i}jx}.$$

The sum is over all

$\mathbf{k} = (k_\ell)_{\ell \ge 0}$ with integers k_ℓ and $\|\mathbf{k}\| := \sum_{\ell \ge 0} |k_\ell| \le K := 2N$

and we write

$$\mathbf{k}\cdot\boldsymbol{\omega} = \sum_{\ell \ge 0} k_\ell\, \omega_\ell\, .$$

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$\sum_{\mathbf{k}} z^{\mathbf{k}}(x, \varepsilon t) \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} = \sum_{\mathbf{k}} \sum_j z_j^{\mathbf{k}}(\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t + \mathrm{i}jx}$ into the PDE $\partial_t^2 u - \partial_x^2 u + \rho u - g(u) = 0$, Taylor expansion, and compare the coefficients of $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t + \mathrm{i}jx}$:

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$$\big(\omega_j^2 \;-\; (\mathbf{k}\cdot\boldsymbol{\omega})^2\big)\, z_j^{\mathbf{k}}(\tau) + 2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\,\dot z_j^{\mathbf{k}}(\tau) \;+\; \varepsilon^2\,\ddot z_j^{\mathbf{k}}(\tau)$$

$$+\;\; \mathcal{F}_j \sum_m \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \frac{1}{m!}\, g^{(m)}(0)\, z^{\mathbf{k}^1}\ldots z^{\mathbf{k}^m} = 0\,.$$

$\mathcal{F}_j(v) = v_j$ is the j^{th} Fourier coeff. of v and $(\cdot)$ is the derivatives with respect to $\tau = \varepsilon t$.

The modulated Fourier expansion (IV)

- $\mathbf{k} = \pm\langle j \rangle = (\ldots, 0, \pm 1, 0, \ldots)$ –> ODEs for $\dot{z}_j^{\pm\langle j \rangle}$:

$$\pm 2\mathrm{i}\varepsilon\omega_j \dot{z}_j^{\pm\langle j \rangle} = -\varepsilon^2 \ddot{z}_j^{\pm\langle j \rangle} - \mathcal{F}_j \sum_m \ldots .$$

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- Algebraic equations for $z_j^{\mathbf{k}}$:

$$\left(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\right) z_j^{\mathbf{k}} = -2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot{z}_j^{\mathbf{k}} - \varepsilon^2 \ddot{z}_j^{\mathbf{k}} - \mathcal{F}_j \sum_m \ldots.$$

Need to divide by $\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}| \Longrightarrow$ use a non-resonance condition.

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Need to divide by $\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}| \Longrightarrow$ use a non-resonance condition.
If this denominator is too small (say $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$), set $z_j^{\mathbf{k}} = 0$.

The modulated Fourier expansion (V)

Iterative construction of the functions $z_j^{\mathbf{k}}$ such that after $2N$ iterations the defect is of size $\mathcal{O}(\varepsilon^{N+1})$:

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- $\mathbf{k} = \pm\langle j\rangle$ we set

$$\pm 2\mathrm{i}\varepsilon\omega_j \left[\dot{z}_j^{\pm\langle j\rangle}\right]^{n+1} = -\left[\varepsilon^2 \ddot{z}_j^{\pm\langle j\rangle} + \mathcal{F}_j \sum_{m=2}^{N} \dots\right]^n$$

- $\mathbf{k} \neq \pm\langle j\rangle$ and j with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \geq \varepsilon^{1/2}$ we set

$$\left(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\right)\left[z_j^{\mathbf{k}}\right]^{n+1} = -\left[2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot{z}_j^{\mathbf{k}} + \varepsilon^2\ddot{z}_j^{\mathbf{k}} + \mathcal{F}_j \sum_{m=2}^{N} \dots\right]^n,$$

- $z_j^{\mathbf{k}} = 0$ for $\mathbf{k} \neq \pm\langle j\rangle$ with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$.

The modulated Fourier expansion (VI)

Theorem. *For the solution* $u(x,t)$ *of the nonlinear wave equation, we have*

$$u(x,t) = \sum_{\|\mathbf{k}\| \le 2N} z^{\mathbf{k}}(x, \varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} + r(x,t),$$

where the remainder is bounded by

$$\|r(\cdot,t)\|_{s+1} + \|\partial_t r(\cdot,t)\|_s \le C\,\varepsilon^{N+1} \qquad \text{for} \quad 0 \le t \le \varepsilon^{-1}.$$

Moreover, the modulation functions $z^{\mathbf{k}}$ *are bounded.*

Almost-invariants (I)

We introduce the notation

$$\mathbf{y} = (y^{\mathbf{k}})_{\|\mathbf{k}\| \le K} \text{ with } y^{\mathbf{k}}(x,t) = z^{\mathbf{k}}(x, \varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}.$$

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By construction, we have

$$\partial_t^2 y^{\mathbf{k}} - \partial_x^2 y^{\mathbf{k}} + \rho y^{\mathbf{k}} + \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} y^{\mathbf{k}^1}\ldots y^{\mathbf{k}^m} = e^{\mathbf{k}},$$

where the defects $e^{\mathbf{k}}(x,t) = d^{\mathbf{k}}(x,\varepsilon t)\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} = \mathcal{O}(\varepsilon^{N+1})$.

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where the defects $e^{\mathbf{k}}(x,t) = d^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} = \mathcal{O}(\varepsilon^{N+1})$.
We can rewrite it as

$$\partial_t^2 y^{\mathbf{k}} - \partial_x^2 y^{\mathbf{k}} + \rho y^{\mathbf{k}} + \nabla^{-\mathbf{k}} \mathcal{U}(\mathbf{y}) = e^{\mathbf{k}}.$$

Almost-invariants (II)

We find almost-invariants for this system:

$$\sum_{\ell\geq 0} \omega_\ell^{2s+1} \left| \frac{d}{dt} \mathcal{J}_\ell(\mathbf{y}(t), \partial_t \mathbf{y}(t)) \right| \leq C\,\varepsilon^{N+2} \quad \text{for} \quad t \leq \varepsilon^{-1}.$$

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These almost-invariants are close to the actions $J_\ell\big(u(t), \partial_t u(t)\big)$:

$$\mathcal{J}_\ell\big(\mathbf{y}(t), \partial_t \mathbf{y}(t)\big) = J_\ell\big(u(t), \partial_t u(t)\big) + \gamma_\ell(t)\, \varepsilon^3$$

for $t \leq \varepsilon^{-1}$ and for all $\ell \geq 0$, with $\sum_{\ell\geq 0} \omega_\ell^{2s+1} \gamma_\ell(t) \leq C$.

Almost-invariants (III)

We use these results repeatedly on intervals of length ε^{-1} for modulated Fourier expansions corresponding to different starting values

$$(u(t_n), \partial_t u(t_n)) \ \text{at} \ t_n = n\varepsilon^{-1}.$$

We can thus patch many short time intervals together and obtain

Theorem. *Under a non-resonance condition for ω_j, for small initial data, we have for an arbitrary (large) integer N*

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \frac{|J_\ell(t) - J_\ell(0)|}{\varepsilon^2} \leq C\varepsilon \ \textit{for} \ 0 \leq t \leq \varepsilon^{-N}.$$