A multi-symplectic numerical integrator for the two-component Camassa–Holm equation

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A new multi-symplectic formulation of the two-component Camassa-Holm equation (2CH) is presented, and the associated local conservation laws are shown to correspond to certain well-known Hamiltonian functionals. A multi-symplectic discretisation based on this new formulation is exemplified by means of the Euler box scheme. Furthermore, this scheme preserves exactly two discrete versions of the Casimir functions of 2CH. Numerical experiments show that the proposed numerical scheme has good conservation properties.

Keywords: Two-component Camassa–Holm equation; Hamiltonian PDE; Casimir function; Numerical discretisation; Multi-symplectic formulation; Multi-symplectic schemes; Euler box scheme.

1. Introduction

The goal of this note is to present a multi-symplectic formulation of the two-component Camassa–Holm equation (or 2CH system)

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx} - \kappa\rho\rho_x,$$
$$\rho_t + (u\rho)_x = 0 \tag{1.1}$$

which will allow us to numerically discretise this partial differential equation with a multi-symplectic integrator based on the Euler box scheme. In the above equations, we use the notations $u = u(x,t)$, $\rho = \rho(x,t)$ and κ denotes a real parameter. The case $\kappa > 0$ is the physical relevant

case. We consider initial conditions $(u(x,0),\rho(x,0)) = (u_0,\rho_0)$ and periodic boundary conditions $u(0,t) = u(L,t)$ and $\rho(0,t) = \rho(L,t)$ on a domain $[0,L]$, for an $L \in \mathbb{R}$.

The 2CH system is a generalisation of the famous Camassa–Holm equation (taking $\rho_0 = 0$ as initial value for (1.1)). This latter equation, which possesses a lot of interesting properties, has been extensively studied in the literature. One may consult for example the work [27], the recent review [16], and references therein.

Since its introduction in the seminal paper [26] (eq. (43)), the 2CH system (1.1) has also received considerable attention. The two-component Camassa–Holm equation approximates the governing equations for shallow water waves [7]. Furthermore, this system of partial differential equations has a rich mathematical structure: it is integrable; has a Lax pair; is bi-Hamiltonian with Hamiltonian functions

$$\mathscr{H}_1 = \frac{1}{2} \int (u^2 + u_x^2 + \kappa \rho^2) \, \mathrm{d}x \tag{1.2}$$

$$\mathscr{H}_2 = \frac{1}{2} \int (u^3 + u u_x^2 + \kappa u \rho^2) \, \mathrm{d}x \tag{1.3}$$

and with the respective compatible Poisson brackets

$$\{F,G\}_1 = -\int \left(\frac{\delta F}{\delta m}(m\partial + \partial m)\frac{\delta G}{\delta m} + \frac{\delta F}{\delta m}\rho \partial \frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta \rho}\partial \rho \frac{\delta G}{\delta m} \right) \mathrm{d}x,$$

$$\{F,G\}_2 = -\int \left(\frac{\delta F}{\delta m}(\partial - \partial^3)\frac{\delta G}{\delta m} + \frac{\delta F}{\delta \rho}\partial \frac{\delta G}{\delta \rho} \right) \mathrm{d}x,$$

where F and G are two functionals of the variables $m := u - u_{xx}$ and ρ; possesses the following two Casimir functions (for the second Poisson bracket $\{\cdot,\cdot\}_2$)

$$\mathscr{C}_1 = \int \rho \, \mathrm{d}x \tag{1.4}$$

$$\mathscr{C}_2 = \int (u - u_{xx}) \, \mathrm{d}x; \tag{1.5}$$

has global solutions for small initial data but wave breaking may also occur; solitary wave solutions; it can be seen as geodesic equations on some space; this system also appears in plasma theory models and in the theory of metamorphosis; etc. [3,7,8,13–15,17,18,21,24] (without being exhaustive).

The main objective of this note is to present an additional property of the two-component Camassa–Holm equation, namely the multi-symplectic structure of this partial differential equation, see Section 2. This new structure will then be used to derive a multi-symplectic numerical method (the Euler box scheme) in Section 3 in order to approximate solutions of the 2CH system. Furthermore, we will show that our scheme preserves exactly two discrete versions of the above Casimir functions. Numerical experiments are presented in Section 4 and finally, discussions and conclusions are drawn.

Our derivations of a multi-symplectic formulation of (1.1) and of a multi-symplectic scheme for the 2CH system follow the lines of [6]. We obtain the same formulation, resp. numerical scheme, as the one presented in [6] when $\rho \equiv 0$ in (1.1). However, since we are not aware of any numerical schemes for the two-component Camassa–Holm equation, except the numerical simulations presented in [15], we think that the results presented in this short note are of interest and hope that they will initiate further numerical analysis of the 2CH system.

2. Multi-symplectic formulation of the 2CH system

There are two standard ways to obtain a multi-symplectic formulation of a partial differential equation. One approach consists of using the Lagrangian formulation of the problem, see the early references [10, 20] and references therein. The other approach is to write the partial differential equation as a system of equations containing only first-order derivatives in space and time, see equation (2.1) below, and then to extract the multi-symplectic structure, see the early references [1, 2] and references therein. In this section, we will follow the latter approach. Inspired by the multi-symplectic formulation of the Camassa–Holm equation found in [6], see also [5] for related problems, we can now derive a multi-symplectic formulation

$$M z_t + K z_x = \nabla_z S(z) \tag{2.1}$$

of the two-component Camassa–Holm system of equations

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} - \kappa \rho \rho_x,$$
$$\rho_t + (u\rho)_x = 0.$$

Here, $z \in \mathbb{R}^d$ is a vector of state variables, typically including the original variables u and ρ as one of its components, M and K are skew-symmetric $d \times d$-matrices and S is a smooth scalar function depending on z. Indeed, setting $z = [u, \phi, w, v, \eta, \rho, \gamma, \beta]$, using the following skew-symmetric matrices

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\kappa}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\kappa}{2} & 0 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa \\ 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & 0 \end{bmatrix}$$

and the gradient of the scalar function

$$S(z) = -wu - u^3/2 - u\eta^2/2 + \eta v - \kappa u\rho^2/2 + \kappa \gamma \rho$$

one obtains a multi-symplectic formulation (2.1) of the two-component Camassa–Holm equation (1.1). The above formulation can be written componentwise:

$$\frac{1}{2}\phi_t - \frac{1}{2}\eta_t - v_x = -w - \frac{1}{2}\eta^2 - \frac{\kappa}{2}\rho^2 - \frac{3}{2}u^2,$$
$$-\frac{1}{2}u_t + w_x = 0,$$
$$-\phi_x = -u,$$
$$u_x = \eta,$$
$$\frac{1}{2}u_t = -u\eta + v,$$
$$+\frac{\kappa}{2}\beta_t = \kappa\gamma - \kappa u\rho,$$
$$\kappa\beta_x = \kappa\rho,$$
$$-\frac{\kappa}{2}\rho_t - \kappa\gamma_x = 0.$$

At this point, we would like to comment on the fact that one could obtain the above system by considering the Lagrangian

$$\mathcal{L} := \frac{1}{2}u\phi_t + \frac{1}{2}\kappa\beta_t\rho + \frac{1}{2}\kappa u\rho^2 + \frac{1}{2}u^3 + \frac{1}{2}\eta u_t + \frac{1}{2}u\eta^2 + w(u - \phi_x) - \kappa\gamma(\rho - \beta_x) - v(\eta - u_x)$$

and taking variations with respect to the variables $(u, \phi, w, v, \eta, \rho, \gamma, \beta)$ [a]. A canonical way to find Lagrangians for shallow water waves can be found in [25]. In [22, Section 1.2], it is explained how to obtain a Lagrangian from a given multi-symplectic formulation.

A key observation, [2], for the above multi-symplectic formulation of our problem is that the two skew-symmetric matrices M and K define symplectic structures on subspaces of $\mathbb{R}^8$

$$\omega = \mathrm{d}z \wedge M\mathrm{d}z, \qquad \zeta = \mathrm{d}z \wedge K\mathrm{d}z$$

thus resulting in the following multi-symplectic conservation law

$$\partial_t\omega + \partial_x\zeta = 0. \tag{2.2}$$

This is a local property of our problem and we thus hope that multi-symplectic numerical schemes, as derived in the next section, will render well local properties of equation (1.1). More explicitly, we have for any solutions of (2.1), the local conservation laws

$$\partial_t E(z) + \partial_x F(z) = 0 \qquad \text{and} \qquad \partial_t I(z) + \partial_x G(z) = 0, \tag{2.3}$$

with the density functions

$$E(z) = S(z) - \frac{1}{2}z_x^T K^T z, \qquad F(z) = \frac{1}{2}z_t^T K^T z,$$
$$G(z) = S(z) - \frac{1}{2}z_t^T M^T z, \qquad I(z) = \frac{1}{2}z_x^T M^T z.$$

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[a]We thank the anonymous referee for providing us with this Lagrangian.

Under the usual assumption on vanishing boundary terms for the functions $F(z)$ and $G(z)$ one obtains the following global conserved quantities

$$\mathscr{E}(z) = \int E(z)\,\mathrm{d}x \qquad \text{and} \qquad \mathscr{I}(z) = \int I(z)\,\mathrm{d}x. \tag{2.4}$$

These quantities correspond to the Hamiltonian functions (1.2) and (1.3) as we will see below. For our choice of the skew-symmetric matrices M and K, one thus obtains the density functions

$$E(z) = S(z) + \frac{1}{2}z_x^T K z = \frac{1}{4}\left(\phi_t u - u_{xt}u + u^3 + u_x u_t + u u_x^2 - u_t \phi - \kappa \rho^2 u - \kappa \rho_t \beta + 2\kappa \rho \gamma\right),$$

$$F(z) = -\frac{1}{2}z_t^T K z = \frac{1}{2}\left(u_t v - \phi_t w + \phi w_t - u v_t - \kappa \gamma_t \beta + \kappa \beta_t \gamma\right),$$

$$G(z) = S(z) + \frac{1}{2}z_t^T M z = \frac{1}{4}\phi_t u - \frac{3}{4}u_{xt}u - u^2 u_{xx} + u^3$$
$$+ \frac{1}{4}u_x u_t + \frac{1}{4}u_t \phi + \kappa \gamma \rho + \frac{\kappa}{4}\rho_t \beta - \frac{\kappa}{4}\beta_t \rho,$$

$$I(z) = -\frac{1}{2}z_x^T M z = \frac{1}{4}\left(-u_x \phi + u_x \eta + u\phi_x - u\eta_x - \kappa \rho_x \beta + \kappa \rho^2\right).$$

This will help us to derive the corresponding global invariants (2.4).

We first integrate the local conservation law $\partial_t I(z) + \partial_x G(z) = 0$ over the spatial domain and obtain, similarly to the computations done in [6], the invariant (1.2). Indeed, looking firstly only at terms involving κ in the above local conservation law, one has

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(-\kappa \rho_x \beta + \kappa \rho^2\right)\mathrm{d}x + \left[\kappa \gamma \rho + \frac{\kappa}{4}\rho_t \beta - \frac{\kappa}{4}\beta_t \rho\right],$$

where the square brackets denote the difference of the function evaluated at the upper and lower limit of the integral. Using one integration by parts and the periodic boundary conditions for u and ρ one thus gets (removing the factor κ for ease of presentation)

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\rho_x \beta - \rho^2\right)\mathrm{d}x + \left[-\gamma \rho - \frac{1}{4}\rho_t \beta + \frac{1}{4}\beta_t \rho\right]$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^2\,\mathrm{d}x + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho\beta\right] + \left[-\gamma \rho - \frac{1}{4}\rho_t \beta + \frac{1}{4}\beta_t \rho\right]$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^2\,\mathrm{d}x + \frac{1}{4}\left[\rho_t \beta\right] + \frac{1}{4}\left[\rho\beta_t\right] + \left[-\gamma \rho - \frac{1}{4}\rho_t \beta + \frac{1}{4}\beta_t \rho\right]$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^2\,\mathrm{d}x + \frac{1}{2}\left[\rho\beta_t\right] + \left[-\gamma \rho\right]$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^2\,\mathrm{d}x + \left[\rho\gamma - u\rho^2\right] + \left[-\gamma \rho\right]$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^2\,\mathrm{d}x.$$

We next observe that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \kappa \rho^2\,\mathrm{d}x$$

is the time derivative of the third integrand present in the Hamiltonian (1.2). The two other integrands in (1.2) are obtained considering the remaining terms in the integral form of the local conservation law $\partial_t I(z) + \partial_x G(z) = 0$ as this was done in [6]. This thus gives us the first Hamiltonian (1.2) of the two-component Camassa–Holm equation.

Similarly, integrating the local conservation law $\partial_t E(z) + \partial_x F(z) = 0$, one obtains

$$0 = \frac{1}{4}\frac{d}{dt}\int \left((\eta_t + 2v_x - 2w - \eta^2 - \kappa\rho^2 - 3u^2)u - u_{xt}u + u^3 + u_x u_t + uu_x^2 - 2w_x\phi\right.$$
$$\left. - \kappa\rho^2 u - \kappa\rho_t\beta + 2\kappa\rho\gamma\right)dx - \frac{\kappa}{2}\left[\gamma_t\beta - \beta_t\gamma\right] + \frac{1}{2}\left[u_t v - \phi_t w + \phi w_t - uv_t\right].$$

Using two integrations by parts, one gets

$$0 = \frac{1}{4}\frac{d}{dt}\int \left(2u^2 u_{xx} + 2uu_x^2 - 2u^3 - 2\kappa\rho^2 u + 2\kappa(\gamma\beta)_x\right)dx$$
$$+ \frac{1}{4}\frac{d}{dt}\left[uu_t - 2w\phi\right] - \frac{\kappa}{2}\left[\ldots\right] + \frac{1}{2}\left[\ldots\right]$$
$$= -\frac{1}{2}\frac{d}{dt}\int \left(u^3 + uu_x^2 + \kappa\rho^2 u\right)dx + \frac{\kappa}{2}\frac{d}{dt}\int (\gamma\beta)_x dx + \frac{1}{4}\frac{d}{dt}\left[\ldots\right] - \frac{\kappa}{2}\left[\ldots\right] + \frac{1}{2}\left[\ldots\right].$$

Finally, using the periodicity of the functions u and ρ (together with the periodicity of $\gamma, w, v, \phi_t, \beta_t$), the second integral and the expressions in the square brackets cancel. Finally, one obtains the second Hamiltonian (1.3) of the two-component Camassa–Holm equation.

3. An Euler box scheme for the 2CH system

In this section, we will derive a numerical scheme based on the multi-symplectic formulation (2.1) of the two-component Camassa–Holm equation (1.1).

Following [23], see also [19, Chap. 12], one may obtain an integrator satisfying a discrete multi-symplectic conservation law by applying the classical symplectic Euler method to each independent variables in (2.1). One then obtains the so-called Euler box scheme. To do this, we first introduce finite differences. We set $\Delta x = x_{n+1} - x_n, n \in \mathbb{Z}$, and $\Delta t = t_{i+1} - t_i, i \geq 0$. Moreover, we define the forward and backward differences in time

$$\delta_t^+ Z^{n,i} = \frac{Z^{n,i+1} - Z^{n,i}}{\Delta t} \qquad \text{and} \qquad \delta_t^- Z^{n,i} = \frac{Z^{n,i} - Z^{n,i-1}}{\Delta t},$$

and similarly for differences in space. The Euler box scheme uses a splitting of the two skew-symmetric matrices M and K in (2.1): $M = M_+ + M_-$ and $K = K_+ + K_-$ where $M_+^T = -M_-$ and $K_+^T = -K_-$. This numerical scheme then reads

$$M_+\delta_t^+ z^{n,i} + M_-\delta_t^- z^{n,i} + K_+\delta_x^+ z^{n,i} + K_-\delta_x^- z^{n,i} = \nabla_z S(z^{n,i}), \tag{3.1}$$

where $z^{n,i} \approx z(x_n, t_i)$ on a uniform rectangular grid.

In this note, we will only consider the following matrices $M_+ = \frac{1}{2}M$ and $K_+ = \frac{1}{2}K$ for the above splitting, keeping in mind that the above splitting of the matrices is not unique. With this particular choice, the numerical method (3.1) now reads

$$M\delta_t z^{n,i} + K\delta_x z^{n,i} = \nabla_z S(z^{n,i})$$

with the centered differences $\delta_t = \frac{1}{2}(\delta_t^+ + \delta_t^-)$, and $\delta_x = \frac{1}{2}(\delta_x^+ + \delta_x^-)$.

For ease of implementation, we can now express the Euler box scheme (3.1) only in terms of the variables u and ρ and the centered divided differences δ_t and δ_x reads

$$\delta_t u^{n,i} - \delta_t \delta_x^2 u^{n,i} + \frac{1}{2}\delta_x\big((\delta_x u^{n,i})^2\big) - \delta_x^2(u^{n,i}\delta_x u^{n,i}) + \frac{\kappa}{2}\delta_x\big((\rho^{n,i})^2\big) + \frac{3}{2}\delta_x\big((u^{n,i})^2\big) = 0$$
$$\delta_t \rho^{n,i} + \delta_x(u^{n,i}\rho^{n,i}) = 0. \tag{3.2}$$

The multi-symplecticity of the Euler box scheme is interpreted in the sense that, recall (2.2),

$$\delta_t^+ \omega^{n,i} + \delta_x^+ \zeta^{n,i} = 0,$$

where $\omega^{n,i} = dz^{n,i-1} \wedge M_+ dz^{n,i}$ and $\zeta^{n,i} = dz^{n-1,i} \wedge K_+ dz^{n,i}$.

Finally, we can observe that discrete versions of the above Casimir functions are preserved along the numerical solutions given by the Euler box scheme. Indeed, from equation (3.2) and using the fact that $\sum_n \delta_x v_n = 0$ for any *periodic* sequence v_n, we have

$$\delta_t\Big(\sum_n (u^{n,i} - \delta_x^2 u^{n,i})\Big) = 0,$$

and the approximation of the Casimir function $\mathscr{C}_2$ given by $\sum_n (u^n - \delta_x^2 u^n)$ is preserved in time. Similarly, we have $\delta_t\big(\sum_n \rho^{n,i}\big) = 0$ and the approximation $\sum_n \rho^n$ of the Casimir function $\mathscr{C}_1$ is preserved in time along the numerical solutions.

4. Numerical experiments

4.1. *Preservation of Hamiltonians*

In the first numerical experiment that we consider, we are interested in the preservation of the invariants (1.2)-(1.3) by the numerical scheme presented in Section 3. For the initial value u_0, we take a peakon $u_0(x) = \exp(-|x|)$ and set $\rho_0 = 0.5$. The computational domain is set to be $[0, 20]$ and the problem is integrated on the time interval $[0, 5]$. Figure 1 displays snapshots of the numerical solution and the discretised Hamiltonians (corresponding to the Hamiltonians (1.2)-(1.3))

$$\mathscr{H}_{1,\Delta x} = \frac{\Delta x}{2}\sum_n \big((u^n)^2 + (\delta_x u^n)^2 + \kappa(\rho^n)^2\big),$$

$$\mathscr{H}_{2,\Delta x} = \frac{\Delta x}{2}\sum_n \big((u^n)^3 + u^n(\delta_x u^n)^2 + \kappa u^n(\rho^n)^2\big)$$

along the numerical solution (u^n, ρ^n) of (1.1) with $\kappa = 1$ given by the Euler box scheme (3.2). Excellent conservation properties by the numerical integrator is observed. Furthermore, one can observe that the relative errors in the discrete Hamiltonians is diminishing. One interpretation could be that the numerical scheme introduces artificial numerical dissipation for this non-smooth solution. This is not the case when considering smooth solutions as the one presented at the end of this section, see the dam-break initial conditions.

The discrete versions of the Casimir functions (1.4)-(1.5) studied previously are not displayed in our plots since these quantities are exactly preserved along numerical solutions given by our multi-symplectic scheme.
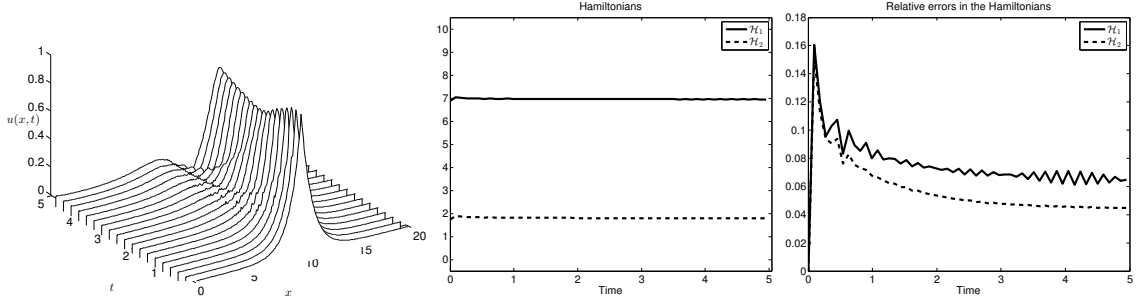
Figure 1. Peakon of 2CH: Snapshots of the numerical profile of u (left plot). Discretised Hamiltonians (middle plot) and relative errors in the discrete Hamiltonians (right plot) along the numerical solution given by the Euler box scheme with $\Delta x = 0.1, \Delta t = 0.09$.

4.2. *Traveling wave*

Here, we present a derivation of a periodic traveling wave for (1.1) with $\kappa = 1$. We look for a solution of the form $\rho(t,x) = \xi(x - ct)$ and $u(t,x) = \phi(x - ct)$. We denote $\mu = \phi - \phi''$. From (1.1), we obtain that ξ and ϕ satisfy

$$\mu'(\phi - c) + 2\phi'\mu + \kappa\xi\xi' = 0,$$
$$-c\xi' + (\phi\xi)' = 0.$$

We introduce $\tilde{\phi} = \phi - c$ and, correspondingly, $\tilde{\mu} = \mu - c$. The system above, after multiplying the first equation with $\tilde{\phi}$, can the be rewritten as

$$(\tilde{\phi}^2(\tilde{\mu} + c))' + \kappa\tilde{\phi}\xi\xi' = 0,$$
$$(\tilde{\phi}\xi)' = 0.$$

We integrate the second equation and obtain

$$\tilde{\phi}\xi = A \tag{4.1}$$

for some real constant of integration A. We plug this result into the first equation and get

$$(\tilde{\phi}^2(\tilde{\mu} + c))' + (\kappa A\xi)' = 0$$

which, after integration, yields

$$\tilde{\phi}^2\tilde{\mu} + c\tilde{\phi}^2 + \kappa A\xi = B$$

for a constant of integration B. Using (4.1) and the definition of μ, we finally obtain the following second order differential equation for ϕ

$$\phi'' = \kappa\frac{A^2}{(\phi - c)^3} - \frac{B}{(\phi - c)^2} + \phi. \tag{4.2}$$

Let $f(\phi) = \kappa\frac{A^2}{(\phi-c)^3} - \frac{B}{(\phi-c)^2} + \phi$. Only periodic solutions to the above differential equation will give us traveling waves for 2CH. For $\kappa = 1$ and $c = -A = -B = 2$, we numerically check that the solution of (4.2) for $\phi(0) = 0.5$ and $\phi'(0) = 0$ is periodic with period 5.1475. Using (4.1), we obtain
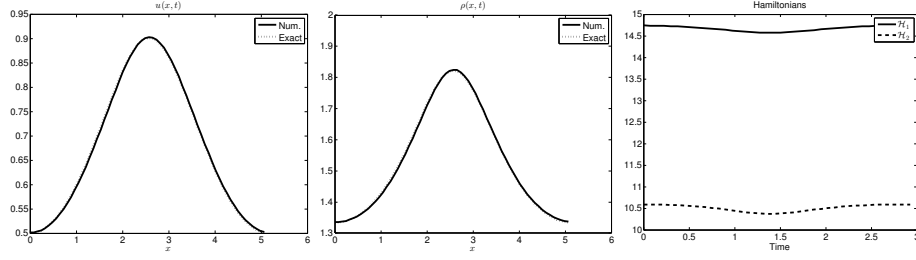
Figure 2. Traveling wave of 2CH: Exact and numerical profiles of $u(x,t)$ and $\rho(x,t)$ at time $T = 3$ and discrete Hamiltonians (right plot).

$\xi = \frac{A}{\phi - c}$. We use this solution as reference solution in our code. A similar derivation of traveling waves for the two-component Camassa–Holm equation was obtained in [7].

Figure 2 displays the exact and numerical profiles of $u(x,t)$ and $\rho(x,t)$ at time $T = 3$ and also the computed discrete versions of the Hamiltonians (1.2)-(1.3) using the Euler box scheme (3.2) with step sizes $\Delta t = 0.06$ and $\Delta x = 0.09$. One may notice that the numerical solution agrees very well with the exact ones and also good conservation properties of the numerical scheme.

4.3. *Peakon anti-peakon solution*

Since the Camassa–Holm equation is obtained from (1.1) setting $\rho \equiv 0$, it is interesting to consider how peakon anti-peakon solution initial value behaves for the system (1.1) with $\kappa = 1$. In Figure 3, we thus consider the following initial value (see also [12])

$$u_0 = \begin{cases} 1/\sinh(1/4)\sinh(x) & \text{if} \quad x \geq 0 \quad \text{and} \quad x \leq 1/4 \\ \sinh(x-1/2)/\sinh(-1/2) & \text{if} \quad x > 1/4 \quad \text{and} \quad x \leq 3/4 \\ 1/\sinh(1/4)\sinh(x-1) & \text{if} \quad x > 3/4 \quad \text{and} \quad x < 1 \end{cases}$$

$$\rho_0 = 1.5$$

and displays the numerical solution obtained by our multi-symplectic scheme with meshes $\Delta t = 0.003$ and $\Delta x = 0.004$ on the periodic domain $[0,1]$. Since $\rho_0 > 0$, it can be shown that $\rho(t)$ remains strictly positive and that the solution retains the regularity of the initial data, see [11]. In particular $\rho(t)$ remains bounded. As expected, we observe that the numerical approximation of ρ concentrates around the collision point of the peakon anti-peakon case of the CH equation (case $\rho = 0$). It corresponds to the fact that, at that time, the total energy distribution $u^2 + u_x^2 + \rho^2$ becomes mainly supported by the variable ρ. Despite this concentration phenomena, we still observe good preservation properties for the numerical scheme.

4.4. *Dam-break initial conditions*

Finally, we consider problem (1.1) with $\kappa = 1$ on the periodic domain $[-6,6]$ augmented with the dam-break initial conditions from [15]

$$u(x,0) = 0, \quad \rho(x,0) = 1 + \tanh(x+0.1) - \tanh(x-0.1).$$

Figure 4 displays the evolution of $u(x,t)$ and $\rho(x,t)$ on the time interval $[0,20]$ together with the computed discrete versions of the Hamiltonians (1.2)-(1.3) for the numerical solution given by the
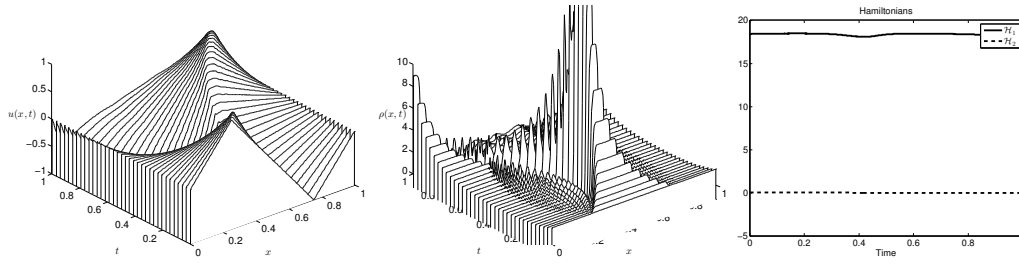
Figure 3. Peakon anti-peakon of 2CH: Evolution of the numerical profiles of $u(x,t)$ and $\rho(x,t)$ on the time interval $[0,1]$ and discrete Hamiltonians (right plot).
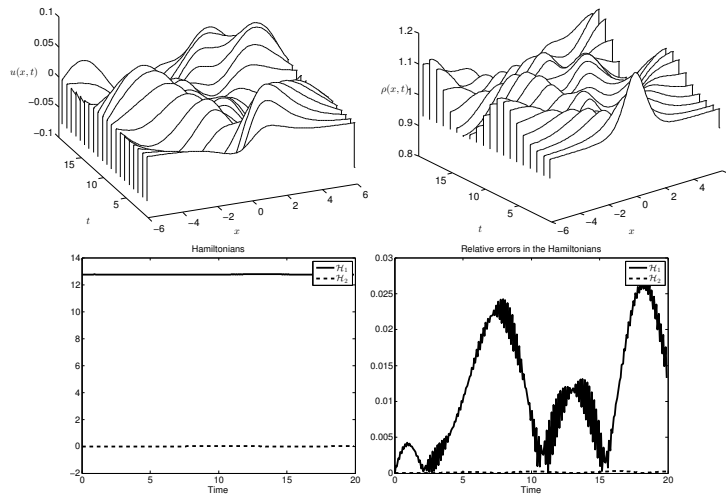

Figure 4. Dam-break of 2CH: Evolution of the numerical profiles of $u(x,t)$ and $\rho(x,t)$ on the time interval $[0,20]$ (top). Discrete Hamiltonians and relative errors in the discrete Hamiltonians (bottom).

Euler box scheme using step sizes $\Delta t = 0.08$ and $\Delta x = 0.09$. Once again, the numerical solution conserves very well the discrete Hamiltonians of the 2CH system. Furthermore, in contrast with the first numerical experiment on the peakon solution, the relative errors in this case do not diminish since we consider smooth solution.

5. Conclusion and open problems

With this note, we have presented the first multi-symplectic formulation of the two-component Camassa–Holm equation. This motivates the use of multi-symplectic integrators for the numerical discretisation of this system of partial differential equations. Furthermore, good conservation properties in terms of energies were illustrated by the (multi-symplectic) Euler box scheme. Finally, it was shown that the above numerical method exactly preserves two discrete versions of the Casimir functions of the problem.

Using the framework of discrete variational derivative methods [9], one could in principle construct $\mathscr{H}_{1,\Delta x}$- or $\mathscr{H}_{2,\Delta x}$-preserving numerical schemes for 2CH. Such numerical schemes will however not be multi-symplectic in general. It would be of interest to get more insight into the (long-time) behaviour of such energy preserving schemes in comparison to the multi-symplectic Euler box scheme presented here. Numerical comparisons of both types of methods is proposed in [5] for Hunter–Saxton like equations.

Furthermore, one could extend the multi-symplectic Euler box scheme derived in the present article to numerically integrate the generalised two-component Camassa–Holm equation [4]

$$u_t - u_{txx} + 3uu_x - Au_x = 2u_xu_{xx} + uu_{xxx} - \kappa\rho\rho_x,$$
$$\rho_t + (u\rho)_x = 0,$$

where $A \geq 0$ without additional difficulties. It is however not clear to the authors if one can get a second multi-symplectic formulation of our problem based on the reformulation of the 2CH system [11]

$$u_t + uu_x + P_x = 0,$$
$$\rho_t + (u\rho)_x = 0,$$
$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2 + \kappa\frac{1}{2}\rho^2,$$
$$\mu_t + (u\mu)_x = (u^3 - 2Pu)_x,$$

with the measure $\mu := u^2 + u_x^2 + \kappa\rho^2$. Such formulation would allow to compute the conservative solutions of the 2CH equation, as it was done in [6] for the CH equation. Conservative solutions are global, weak, energy preserving solutions, defined beyond the blows up which may naturally arise for this equation, see [12, 13].

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