Long-time analysis of nonlinearly perturbed wave equations via modulated Fourier expansions

D. COHEN[1,2], E. HAIRER[1] and CH. LUBICH[2]

April 26, 2006

[1] *Dept. de Mathématiques, Univ. de Genève, CH-1211 Genève 4, Switzerland.*
email: David.Cohen@math.unige.ch, Ernst.Hairer@math.unige.ch

[2] *Mathematisches Institut, Univ. Tübingen, D-72076 Tübingen, Germany.*
email: Cohen@na.uni-tuebingen.de, Lubich@na.uni-tuebingen.de

Abstract

A modulated Fourier expansion in time is used to show long-time near-conservation of the harmonic actions associated with spatial Fourier modes along the solutions of nonlinear wave equations with small initial data. The result implies the long-time near-preservation of the Sobolev-type norm that specifies the smallness condition on the initial data.

1 Introduction

We consider the one-dimensional nonlinear wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \rho u(x,t) + g(u(x,t)) = 0 \qquad (1)$$

for $t > 0$ and $-\pi \leq x \leq \pi$ subject to periodic boundary conditions. We assume $\rho > 0$ and a nonlinearity g that is a smooth real function with $g(0) = g'(0) = 0$. We take small initial data: in appropriate Sobolev norms, the initial data $u(\cdot, 0)$ and $\partial_t u(\cdot, 0)$ is bounded by a small parameter ε, but is not restricted otherwise. We are interested in studying the behaviour of the solutions over long times $t \leq \varepsilon^{-N}$, with fixed, but arbitrary positive integer N. We show that for each pair of spatial Fourier modes with the same frequency, the sum of its two harmonic actions remains nearly constant over such long times, as does the Sobolev-type norm specifying the smallness of the initial data. This result refines previous results by Bambusi [1] and Bourgain [2], using entirely different techniques.

The main novelty in the present paper lies in the technique of proof via a *modulated Fourier expansion* in time. This is a multiscale expansion that represents the solution as an asymptotic series of products of exponentials $e^{i\omega_j t}$ (with ω_j the frequencies of the linear equation) multiplied with coefficient functions that vary slowly in time. This approach was first used for showing long-time almost-conservation properties in [4], in that case of numerical methods for highly oscillatory Hamiltonian ordinary differential equations; also see [5, Ch. XIII] and further references therein. A modulated Fourier expansion appears similarly, and independently, in the work by Guzzo and Benettin [3] on the spectral formulation of the Nekhoroshev theorem for quasi-integrable Hamiltonian systems. In the context of wave equations, the expansion constructed here can be viewed as an extension to higher approximation order of a nonlinear geometric optics expansion given by Joly, Métivier, and Rauch [6].

In Section 2 we describe the technical framework and state the result on the long-time near-conservation of harmonic actions (Theorem 2.1). Section 3 gives the construction of the modulated Fourier expansion and proves the necessary bounds of its coefficients and of the remainder term, which are collected in Theorem 3.1. In Section 4 it is shown that the system determining the modulation functions has a Hamiltonian structure and a remarkable invariance property, which yields the existence of almost-invariants close to the harmonic actions (Theorems 4.1 and 4.2). Though the modulated Fourier expansion is constructed only as a short-time expansion (over time scale ε^{-1}), its almost-invariants can be patched together over very many short time intervals, which finally gives the long-time near-conservation of actions over times ε^{-N} with $N > 1$ as stated in Section 2.

We consider equation (1) only with periodic boundary conditions, but it appears that the problem with Dirichlet boundary conditions, as studied in [1, 2], can be treated in the same way. As in these previous works, an extension of the results to problems in more than one space dimension over time scales ε^{-N} with $N > 1$ does, however, not seem possible with the present techniques, mainly due to problems with small denominators.

Corresponding to the authors' research background, the present work was originally motivated by numerical analysis, with the aim of understanding the long-time behaviour of discretization schemes for the nonlinear wave equation (1). The approach to the long-time analysis of (1) via modulated Fourier expansions does not use nonlinear coordinate transforms and therefore turns out to be applicable also to numerical discretizations of (1), as will be shown in a companion paper to the present article.

2 Preparation and statement of result

In this section we describe basic concepts, introduce notation, formulate assumptions, and state the result on the long-time near-conservation of actions.

2.1 Modulated Fourier expansion

The spatially 2π-periodic solutions to the linear wave equation $\partial_t^2 u - \partial_x^2 u + \rho u = 0$ are superpositions of plane waves $\mathrm{e}^{\pm \mathrm{i}\omega_j t \pm \mathrm{i} j x}$, where j is an arbitrary integer and

$$\omega_j = \sqrt{\rho + j^2}$$

are the frequencies of the equation. If the nonlinearity g is evaluated at a superposition of plane waves, its Taylor expansion involves mixed products of such waves. This can be taken as a motivation to look for an approximation to the solution $u(x,t)$ of the nonlinear problem in the form of a *modulated Fourier expansion*, that is, a linear combination of products of plane waves with coefficient functions that change slowly in time, or more precisely, their derivatives with respect to the slow time $\tau = \varepsilon t$ are bounded independently of ε:

$$u(x,t) \approx \widetilde{u}(x,t) = \sum_{\|\mathbf{k}\| \le K} z^{\mathbf{k}}(x, \varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} = \sum_{\|\mathbf{k}\| \le K} \sum_{j=-\infty}^{\infty} z_j^{\mathbf{k}}(\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t + \mathrm{i} j x}. \quad (2)$$

Here, the sum is over all

$$\mathbf{k} = (k_\ell)_{\ell\ge 0} \quad \text{with integers } k_\ell \text{ and } \ \|\mathbf{k}\| := \sum_{\ell\ge 0} |k_\ell| \le K$$

(at most K of the k_ℓ are non-zero) and we write

$$\mathbf{k}\cdot\boldsymbol{\omega} = \sum_{\ell\ge 0} k_\ell\,\omega_\ell\,.$$

For $K = 2N$, we will obtain an expansion (2) with an approximation error of size $\mathcal{O}(\varepsilon^{N+1})$ in the same norm in which the initial data is assumed to be bounded by ε, uniformly over times $\mathcal{O}(\varepsilon^{-1})$.

In the construction, a special role is played by the modulation functions $z_j^{\mathbf{k}}$ for $\mathbf{k} = \pm\langle j\rangle$, with the notation (Kronecker delta)

$$\langle j\rangle := (\delta_{|j|,\ell})_{\ell\ge 0}\,.$$

The function $z_j^{\pm\langle j\rangle}$ is multiplied with $\mathrm{e}^{\pm\mathrm{i}\omega_j t+\mathrm{i}jx}$ in (2). The $z_j^{\pm\langle j\rangle}$ will be determined from first-order differential equations, whereas the other $z_j^{\mathbf{k}}$ are obtained from equations of the form $\big(\omega_j^2-(\mathbf{k}\cdot\boldsymbol{\omega})^2\big)z_j^{\mathbf{k}} = \dots$, where we need to divide by $\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}|$. If this denominator is too small in absolute value (less than $\varepsilon^{1/2}$, say), then this corresponds to a situation where we cannot safely distinguish the exponentials $\mathrm{e}^{\pm\mathrm{i}\omega_j t}$ and $\mathrm{e}^{\pm\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}$ and we just set $z_j^{\mathbf{k}} = 0$.

2.2 Non-resonance condition

The effect of ignoring contributions from near-resonant indices $(j,\mathbf{k})$ for which $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$ (+ or −), should not spoil the $\mathcal{O}(\varepsilon^{N+1})$ remainder term in the modulated Fourier expansion. This requirement is fulfilled under a *non-resonance condition*. With the abbreviations

$$|\mathbf{k}| = (|k_\ell|)_{\ell\ge 0} \quad \text{and} \quad \boldsymbol{\omega}^{\sigma|\mathbf{k}|} = \prod_{\ell\ge 0} \omega_\ell^{\sigma|k_\ell|} \tag{3}$$

and the set of near-resonant indices

$$\mathcal{R} = \{(j,\mathbf{k}):\ j\in\mathbb{Z} \text{ and } \mathbf{k}\ne\pm\langle j\rangle,\ \|\mathbf{k}\|\le 2N \text{ with } |\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}\}\,, \tag{4}$$

the non-resonance condition can be formulated as follows: there are $\sigma > 0$ and a constant C_0 such that

$$\sup_{(j,\mathbf{k})\in\mathcal{R}} \frac{\omega_j^\sigma}{\boldsymbol{\omega}^{\sigma|\mathbf{k}|}}\,\varepsilon^{\|\mathbf{k}\|/2} \le C_0\,\varepsilon^N\,. \tag{5}$$

For $N = 1$, this condition is always satisfied for arbitrary $\sigma \ge 0$ and ρ in (1). For $N > 1$, it imposes a restriction on the choice of ρ, and the possible values of σ depend on N. The condition requires that a near-resonance can only appear with at least two large frequencies among the ω_ℓ with $k_\ell \ne 0$ (counted with their multiplicity $|k_\ell|$).

Condition (5) is implied, for sufficiently large σ, by the non-resonance condition of Bambusi [1], which reads as follows: for every positive integer r, there exist $\alpha = \alpha(r) > 0$ and $c > 0$ such that for all combinations of signs,

$$|\omega_j \pm \omega_k \pm \omega_{\ell_1} \pm \ldots \pm \omega_{\ell_r}| \geq c\, L^{-\alpha} \quad \text{for} \;\; j \geq k \geq L = \ell_1 \geq \ldots \geq \ell_r \geq 0, \tag{6}$$

provided that the sum does not vanish because the terms cancel pairwise. In [1] it is shown that for almost all (w.r.t. Lebesgue measure) ρ in a fixed interval of positive numbers there is a $c > 0$ such that condition (6) holds with $\alpha = 16\, r^5$. It is also noted in [1] that an analogous condition is typically not satisfied for wave equations in spatial dimension greater than 1. Under Bambusi's condition (6), a near-resonance $|\omega_j \pm \mathbf{k} \cdot \boldsymbol{\omega}| < \varepsilon^{1/2}$ can only appear with $L^{-\alpha} = \mathcal{O}(\varepsilon^{1/2})$, and choosing $\sigma = \max_{r+1<2N} (2N - r - 1)\, \alpha(r)$ then yields the bound (5). For $N = 2$, this choice already gives $\sigma = 512$. (The corresponding quantity in [1] is $s_* = 4M\, \alpha(2M)$ for $M = N + 3$, which for $N = 2$ results in $s_* = 32 \cdot 10^6$.) However, it should be noted that condition (5) may actually be satisfied with a much smaller exponent σ. This is suggested by testing (5) numerically for various values of ε, ρ, and N.

2.3 Functional-analytic setting: Sobolev algebras

For a 2π-periodic function $v \in L^2(\mathbb{T})$ (with the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$), we denote by $(v_j)_{j\in\mathbb{Z}}$ the sequence of Fourier coefficients of $v(x) = \sum_{-\infty}^{\infty} v_j\, \mathrm{e}^{\mathrm{i}jx}$. We will work with functions (or coefficient sequences) for which the weighted ℓ^2 norm

$$\|v\|_s = \Big(\sum_{j=-\infty}^{\infty} \omega_j^{2s}\, |v_j|^2 \Big)^{1/2}$$

is finite. We denote, for $s \geq 0$, the Sobolev space

$$H^s = \{ v \in L^2(\mathbb{T}) : \|v\|_s < \infty \} = \{ v : \; (-\partial_x^2 + \rho)^{s/2} v \in L^2 \}.$$

For $s > \frac{1}{2}$, we have $H^s \subset C(\mathbb{T})$ and H^s is a normed algebra:

$$\|vw\|_s \leq C_s\, \|v\|_s\, \|w\|_s\,. \tag{7}$$

It is convenient to rescale the norm such that $C_s = 1$.

2.4 Condition of small initial data

We assume that the initial position and velocity have small norms in H^{s+1} and H^s, resp., for an $s \geq \sigma + 1$ with σ of the non-resonance condition (5):

$$\Big(\|u(\cdot,0)\|_{s+1}^2 + \|\partial_t u(\cdot,0)\|_s^2 \Big)^{1/2} \leq \varepsilon. \tag{8}$$

This is equivalent to requiring

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s+1} \Big(\frac{1}{2\omega_j}\, |\partial_t u_j(0)|^2 + \frac{\omega_j}{2}\, |u_j(0)|^2 \Big) \leq \tfrac{1}{2}\, \varepsilon^2. \tag{9}$$

2.5 Long-time near-conservation of harmonic actions

Along every solution $u(x,t) = \sum_{j=-\infty}^{\infty} u_j(t)\,\mathrm{e}^{\mathrm{i}jx}$ to the linear wave equation $\partial_t^2 u - \partial_x^2 u + \rho u = 0$, the *actions* (energy divided by frequency)

$$I_j(t) = \frac{1}{2\omega_j}\,|\partial_t u_j(t)|^2 + \frac{\omega_j}{2}\,|u_j(t)|^2$$

remain constant in time. For the *nonlinear* equation (1) with a smooth nonlinearity satisfying $g(0) = g'(0) = 0$, and under conditions (5) and (8), we will show that the sums of actions corresponding to Fourier modes with the same frequency,

$$J_\ell(t) = I_\ell(t) + I_{-\ell}(t)\,, \qquad \ell \ge 1\,, \qquad J_0(t) = I_0(t),$$

and in fact also their weighted sums

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} J_\ell(t) = \|u(\cdot,t)\|_{s+1}^2 + \|\partial_t u(\cdot,t)\|_s^2\,,$$

remain constant up to small deviations over long times.

Theorem 2.1 *Under the non-resonance condition (5) and assumption (8) on the initial data with $s \ge \sigma + 1$, the estimate*

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1}\,\frac{|J_\ell(t) - J_\ell(0)|}{\varepsilon^2} \le C\varepsilon \quad \textit{for} \quad 0 \le t \le \varepsilon^{-N}$$

holds with a constant C which depends on s, N, and C_0, but not on ε.

This result is closely related to results by Bambusi [1] and Bourgain [2]. In particular, Bambusi shows that, under the non-resonance condition (6) and with the same assumption on the initial data, there is the estimate $|J_\ell(t) - J_\ell(0)|/\varepsilon^2 \le C\varepsilon\,\omega_\ell^{-2(s+1)}$, which is close to the above bound. Theorem 2.1 has the added charm of implying that the same norm that specifies the smallness condition on the initial data, remains nearly constant along the solution over long times: for $t \le \varepsilon^{-N}$,

$$\|u(\cdot,t)\|_{s+1}^2 + \|\partial_t u(\cdot,t)\|_s^2 = \|u(\cdot,0)\|_{s+1}^2 + \|\partial_t u(\cdot,0)\|_s^2 + \mathcal{O}(\varepsilon^3)\,.$$

We emphasize that the main novelty of the present work is not in the result, but in the technique of proof via invariance properties of the system of equations that determine the coefficient functions in the modulated Fourier expansion (2). This approach is completely different from the techniques in [1, 2] and turns out to be applicable also to numerical discretizations of (1), since it involves no transformations of coordinates.

2.6 Illustration of the near-conservation of actions

We consider the nonlinear wave equation (1) with $\rho = 1$ and nonlinearity $g(u) = u^2$, subject to periodic boundary conditions. As initial data we choose

$$u(x,0) = \varepsilon\Big(1 - \frac{x^2}{\pi^2}\Big)^2, \qquad \partial_t u(x,0) = 0 \qquad \text{for} \quad -\pi \le x \le \pi. \tag{10}$$

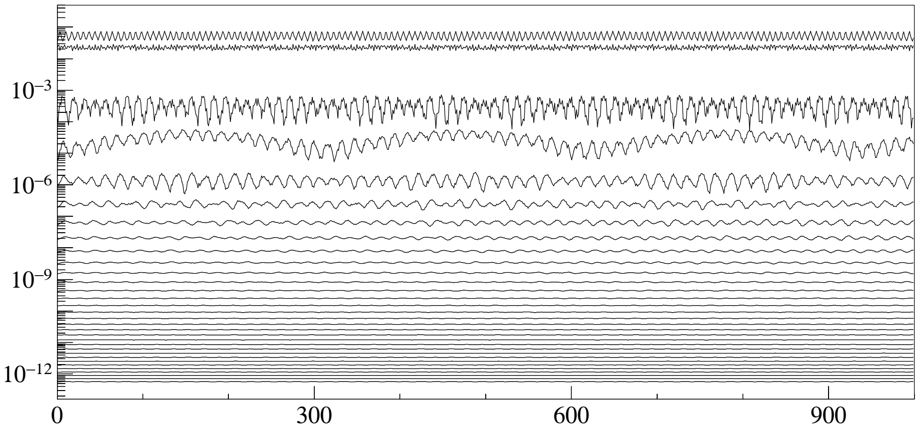

Figure 1: Near-conservation of actions; the first 32 actions $J_\ell(t)$ are plotted as functions of time.

The 2π-periodic extension of $u(x,0)$ has a jump in the third derivative, so that its Fourier coefficients $u_j(0)$ decay like j^{-4}. This function therefore lies in H^s with $s \le 3$.

Figure 1 shows the first 32 functions $J_\ell(t)$ on the time interval $[0, 1000]$ (they are computed numerically with high precision). We have chosen a large $\varepsilon = 0.5$, so that we are able to see oscillations at least in the low frequency modes. The higher the frequency, the better the relative error of the corresponding action is conserved. For ε smaller than 0.1 only horizontal straight lines could be observed. Further experiments with this example have shown that the qualitative behaviour of Fig. 1 is insensitive with respect to the value of ρ, as long as it is not too small, and the good conservation holds on much longer time intervals.

3 The modulated Fourier expansion

Our principal tool for the long-time analysis of the nonlinearly perturbed wave equation is a short-time expansion constructed in this section.

3.1 Statement of result

We will prove the following result, where we use the abbreviation (3) and, for $\mathbf{k} = (k_\ell)_{\ell \ge 0}$ with integers k_ℓ and $\|\mathbf{k}\| = \sum_\ell |k_\ell|$, we set

$$[\![\mathbf{k}]\!] = \begin{cases} \frac{1}{2}(\|\mathbf{k}\| + 1), & \mathbf{k} \ne 0 \\ \frac{3}{2}\,, & \mathbf{k} = 0. \end{cases} \tag{11}$$

Theorem 3.1 *Consider the nonlinear wave equation (1) with frequencies ω_j satisfying the non-resonance condition (5), and with small initial data bounded by (8) with $s \ge \sigma + 1$. Then, the solution u admits an expansion (2),*

$$u(x,t) = \sum_{\|\mathbf{k}\| \le 2N} z^{\mathbf{k}}(x, \varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} + r(x,t), \tag{12}$$

where the remainder is bounded by

$$\|r(\cdot,t)\|_{s+1} + \|\partial_t r(\cdot,t)\|_s \le C_1\,\varepsilon^{N+1} \qquad \textit{for} \quad 0 \le t \le \varepsilon^{-1}. \tag{13}$$

On this time interval, the modulation functions $z^{\mathbf{k}}$ *are bounded by*

$$\sum_{\|\mathbf{k}\|\le 2N} \left(\frac{\boldsymbol{\omega}^{s|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}} \|z^{\mathbf{k}}(\cdot,\varepsilon t)\|_1 \right)^2 \le C_2\,. \tag{14}$$

Bounds of the same type hold for any fixed number of derivatives of $z^{\mathbf{k}}$ *with respect to the slow time* $\tau = \varepsilon t$. *Moreover, the modulation functions satisfy* $z_{-j}^{-\mathbf{k}} = \overline{z_j^{\mathbf{k}}}$. *The constants* C_1 *and* C_2 *are independent of* ε, *but depend on* N, s, *on* C_0 *of (5), and on bounds of derivatives of the nonlinearity* g.

3.2 Formal modulation equations

Formally inserting the ansatz (2) into (1), equating terms with the same exponential $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t+\mathrm{i}jx}$ and Taylor expansion of g lead to the condition

$$\begin{aligned} \big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big) z_j^{\mathbf{k}} \;&+\; 2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot z_j^{\mathbf{k}} \;+\; \varepsilon^2 \ddot z_j^{\mathbf{k}} \\ &+\; \mathcal{F}_j \sum_m \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \frac{1}{m!}\, g^{(m)}(0)\, z^{\mathbf{k}^1} \ldots z^{\mathbf{k}^m} = 0\,. \end{aligned} \tag{15}$$

Here, $\mathcal{F}_j v = v_j$ denotes the jth Fourier coefficient of a function $v \in L^2(\mathbb{T})$, and the dots $(\dot\cdot)$ on $z_j^{\mathbf{k}}(\tau)$ symbolize derivatives with respect to $\tau = \varepsilon t$. From this formal consideration, it becomes obvious that there will be three groups of modulation functions $z_j^{\mathbf{k}}$: for $\mathbf{k} = \pm\langle j\rangle$, the first term vanishes and the second term with the time derivative $\dot z_j^{\mathbf{k}}$ can be viewed as the dominant term. For $\mathbf{k} \ne \pm\langle j\rangle$, the first term is dominant if $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \ge \varepsilon^{1/2}$. Else, the non-resonance condition (5) will ensure that the defect in simply setting $z_j^{\mathbf{k}} \equiv 0$ is of size $\mathcal{O}(\varepsilon^{N+1})$ in an appropriate Sobolev-type norm.

In addition, the initial conditions $\widetilde u(\cdot,0) = u(\cdot,0)$ and $\partial_t \widetilde u(\cdot,0) = \partial_t u(\cdot,0)$ need to be taken care of. They will yield the initial conditions for the functions $z_j^{\pm\langle j\rangle}$:

$$\sum_{\mathbf{k}} z_j^{\mathbf{k}}(0) = u_j(0)\,, \qquad \sum_{\mathbf{k}} \Big(\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega}) z_j^{\mathbf{k}}(0) + \varepsilon \dot z_j^{\mathbf{k}}(0) \Big) = \partial_t u_j(0). \tag{16}$$

3.3 Reverse Picard iteration

We now turn to an iterative construction of the functions $z_j^{\mathbf{k}}$ such that after $2N$ iteration steps, the defect in conditions (15) and (16) is of size $\mathcal{O}(\varepsilon^{N+1})$ in the H^s norm. The iteration procedure we employ can be viewed as a reverse Picard iteration on (15) and (16): for $\mathbf{k} = \pm\langle j\rangle$ we set

$$\pm 2\mathrm{i}\varepsilon\omega_j \Big[\dot z_j^{\pm\langle j\rangle} \Big]^{n+1} = -\Big[\varepsilon^2 \ddot z_j^{\pm\langle j\rangle} + \mathcal{F}_j \sum_{m=2}^{N} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\pm\langle j\rangle} \frac{1}{m!}\, g^{(m)}(0)\, z^{\mathbf{k}^1} \ldots z^{\mathbf{k}^m} \Big]^n$$

and for $\mathbf{k} \neq \pm\langle j\rangle$ and j with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \geq \varepsilon^{1/2}$ we set

$$\begin{aligned}\left(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\right)\left[z_j^{\mathbf{k}}\right]^{n+1} &= -\Big[2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot z_j^{\mathbf{k}} + \varepsilon^2 \ddot z_j^{\mathbf{k}} \\ &\quad + \mathcal{F}_j \sum_{m=2}^{N} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \frac{1}{m!}\, g^{(m)}(0)\, z^{\mathbf{k}^1}\dots z^{\mathbf{k}^m}\Big]^n,\end{aligned}$$

whereas we let $z_j^{\mathbf{k}} = 0$ for $\mathbf{k} \neq \pm\langle j\rangle$ with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$.

On the initial conditions we iterate by

$$\begin{aligned}\left[z_j^{\langle j\rangle}(0) + z_j^{-\langle j\rangle}(0)\right]^{n+1} &= \Big[u_j(0) - \sum_{\mathbf{k}\neq\pm\langle j\rangle} z_j^{\mathbf{k}}(0)\Big]^n \\ \mathrm{i}\omega_j\left[z_j^{\langle j\rangle}(0) - z_j^{-\langle j\rangle}(0)\right]^{n+1} &= \Big[\partial_t u_j(0) - \sum_{\mathbf{k}\neq\pm\langle j\rangle} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega}) z_j^{\mathbf{k}}(0) - \varepsilon \sum_{\|\mathbf{k}\|\leq K} \dot z_j^{\mathbf{k}}(0)\Big]^n.\end{aligned}$$

In all the above formulas, it is tacitly assumed that $\|\mathbf{k}\| \leq K = 2N$ and $\|\mathbf{k}^1\| + \cdots + \|\mathbf{k}^m\| \leq K$. In each iteration step, we thus have an initial value problem of first-order differential equations for $z_j^{\pm\langle j\rangle}$ ($j \in \mathbb{Z}$) and algebraic equations for $z_j^{\mathbf{k}}$ with $\mathbf{k} \neq \pm\langle j\rangle$.

The starting iterates ($n = 0$) are chosen as $z_j^{\mathbf{k}} = 0$ for $\mathbf{k} \neq \pm\langle j\rangle$, and $z_j^{\pm\langle j\rangle}(\tau) = z_j^{\pm\langle j\rangle}(0)$ with $z_j^{\pm\langle j\rangle}(0)$ determined from the above formula with right-hand sides $u(0)$ and $\partial_t u(0)$.

3.4 Inequalities for the frequencies

We collect a few inequalities involving the frequencies ω_ℓ, which are needed later on. A first observation is that for $s > \frac{1}{2}$,

$$\sum_{\|\mathbf{k}\|\leq K} \boldsymbol{\omega}^{-2s|\mathbf{k}|} \leq C_{K,s} < \infty, \tag{17}$$

where we have used the short-hand notation (3). This just follows from rewriting

$$\sum_{\|\mathbf{k}\|\leq K} \boldsymbol{\omega}^{-2s|\mathbf{k}|} = \sum_{n=0}^{K} \sum_{\ell_1\geq 0} \dots \sum_{\ell_n\geq 0} \omega_{\ell_1}^{-2s} \dots \omega_{\ell_n}^{-2s}$$

and the facts that $\omega_\ell \sim \ell$ and $\sum_{\ell\geq 1} \ell^{-2s} < \infty$. For $s > \frac{1}{2}$ and $m \geq 2$, we have

$$\sup_{\|\mathbf{k}\|\leq K} \sum_{\mathbf{k}^1+\ldots+\mathbf{k}^m=\mathbf{k}} \frac{\boldsymbol{\omega}^{-2s(|\mathbf{k}^1|+\cdots+|\mathbf{k}^m|)}}{\boldsymbol{\omega}^{-2s|\mathbf{k}|}} \leq C_{m,K,s} < \infty, \tag{18}$$

where it is again assumed that $\|\mathbf{k}^1\| + \cdots + \|\mathbf{k}^m\| \leq K$. We begin by illustrating the idea of the proof of this bound for the particular sum with $\mathbf{k} = \langle j\rangle$ and $m = 3$ and $K = 6$. In this case the sequences $\mathbf{k}^i$ are formed by partitioning the $K - 1 = 5$ basis sequences $\langle j\rangle$, $\langle j_1\rangle$, $-\langle j_1\rangle$, $\langle j_2\rangle$, $-\langle j_2\rangle$, (with arbitrary integers j_1, j_2) into $m = 3$ subsets and summing up the sequences in each subset. For each of these partitions the infinite sum over $j_1, j_2 \in \mathbb{Z}$ is bounded by a constant times ω_j^{-2s}, which yields the desired result for $\mathbf{k} = \langle j \rangle$. A general $\mathbf{k}$ with $\|\mathbf{k}\| \le K$ is a sum of at most K basis sequences $\pm\langle j \rangle$, and the result follows by the same combinatorial argument, noting that each factor in the denominator appears also in the numerator and the extra factors are summable. We omit the details of the full formal proof, which would need cumbersome notation.

As a further bound we note

$$\sup_{\|\mathbf{k}\| \le K} \frac{\sum_{\ell \ge 0} |k_\ell|\, \omega_\ell^{2(s+1)}}{\boldsymbol{\omega}^{2s|\mathbf{k}|}\, (1 + |\mathbf{k} \cdot \boldsymbol{\omega}|)^2} \le C_{K,s} < \infty \,. \tag{19}$$

The proof is based on the observation that if the numerator is large and $|\mathbf{k} \cdot \boldsymbol{\omega}|$ is small, then at least two ω_ℓ with $k_\ell \ne 0$ must be large, so that the product $\boldsymbol{\omega}^{2s|\mathbf{k}|}$ becomes the dominant term.

3.5 Rescaling and estimation of the nonlinear terms

Since we aim for (14), for the following analysis it is convenient to work with rescaled functions

$$c_j^{\mathbf{k}} = \frac{\boldsymbol{\omega}^{s|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}}\, z_j^{\mathbf{k}} \,, \qquad c^{\mathbf{k}}(x) = \sum_{j=-\infty}^{\infty} c_j^{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}jx} = \frac{\boldsymbol{\omega}^{s|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}}\, z^{\mathbf{k}}(x) \,, \tag{20}$$

where we use the notation (11) and (3). The superscripts $\mathbf{k}$ are in

$$\mathcal{K} = \{\mathbf{k} = (k_\ell)_{\ell \ge 0} \text{ with integers } k_\ell \,:\; \|\mathbf{k}\| \le K = 2N\},$$

and we will work in the Hilbert space

$$\mathbf{H}^1 := (H^1)^{\mathcal{K}} = \{\mathbf{c} = (c^{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}} : c^{\mathbf{k}} \in H^1\}$$
$$\text{with norm} \quad |\!|\!|\mathbf{c}|\!|\!|_1^2 = \sum_{\mathbf{k} \in \mathcal{K}} \|c^{\mathbf{k}}\|_1^2 = \sum_{\mathbf{k} \in \mathcal{K}} \sum_{j=-\infty}^{\infty} \omega_j^2 |c_j^{\mathbf{k}}|^2.$$

We now express the map $(z^{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}} \mapsto (v^{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}}$ with

$$v^{\mathbf{k}} = \sum_{m=2}^{N} \frac{1}{m!}\, g^{(m)}(0) \sum_{\mathbf{k}^1 + \cdots + \mathbf{k}^m = \mathbf{k}} z^{\mathbf{k}^1} \ldots z^{\mathbf{k}^m}$$

in rescaled variables as the map

$$\Phi : \mathbf{H}^1 \to \mathbf{H}^1 :\; \mathbf{c} = (c^{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}} \mapsto \mathbf{f} = (f^{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}}$$

given by

$$f^{\mathbf{k}} = \frac{\boldsymbol{\omega}^{s|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}} \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1 + \cdots + \mathbf{k}^m = \mathbf{k}} \frac{\varepsilon^{[\![\mathbf{k}^1]\!] + \cdots + [\![\mathbf{k}^m]\!]}}{\boldsymbol{\omega}^{s(|\mathbf{k}^1| + \cdots + |\mathbf{k}^m|)}}\, c^{\mathbf{k}^1} \ldots c^{\mathbf{k}^m}.$$

Using the triangle inequality, the inequality $(\sum_{m=1}^N a_m)^2 \le N \sum_{m=1}^N a_m^2$, and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|||\mathbf{f}|||_1^2 &= \sum_{\|\mathbf{k}\|\le K} \|f^{\mathbf{k}}\|_1^2 \\
&\le \sum_{\|\mathbf{k}\|\le K} N \sum_{m=2}^N \Big(\frac{g^{(m)}(0)}{m!}\Big)^2 \sum_{\mathbf{k}^1+\dots+\mathbf{k}^m=\mathbf{k}} \Big(\frac{\varepsilon^{[[\mathbf{k}^1]]+\dots+[[\mathbf{k}^m]]}}{\varepsilon^{[[\mathbf{k}]]}} \frac{\boldsymbol{\omega}^{-s(|\mathbf{k}^1|+\dots+|\mathbf{k}^m|)}}{\boldsymbol{\omega}^{-s|\mathbf{k}|}} \Big)^2 \\
&\qquad\qquad \times \sum_{\mathbf{k}^1+\dots+\mathbf{k}^m=\mathbf{k}} \|c^{\mathbf{k}^1}\dots c^{\mathbf{k}^m}\|_1^2 .
\end{aligned}
$$

Since H^1 is a normed algebra, and since we have the bound (18) and the obvious lower estimate $[[\mathbf{k}^1]]+\dots+[[\mathbf{k}^m]] \ge \frac{m-1}{2} + [[\mathbf{k}]]$, this is further estimated as

$$
\begin{aligned}
|||\mathbf{f}|||_1^2 &\le N \sum_{m=2}^N \Big(\frac{g^{(m)}(0)}{m!}\Big)^2 \varepsilon^{m-1} C_{m,K,s} \sum_{\|\mathbf{k}\|\le K} \sum_{\mathbf{k}^1+\dots+\mathbf{k}^m=\mathbf{k}} \|c^{\mathbf{k}^1}\|_1^2 \dots \|c^{\mathbf{k}^m}\|_1^2 \\
&\le N \sum_{m=2}^N \Big(\frac{g^{(m)}(0)}{m!}\Big)^2 \varepsilon^{m-1} C_{m,K,s} \Big(\sum_{\|\mathbf{k}\|\le K} \|c^{\mathbf{k}}\|_1^2 \Big)^m = \varepsilon\, P(|||\mathbf{c}|||_1^2) \qquad (21)
\end{aligned}
$$

where the polynomial $P(\mu) = N \sum_{m=2}^N \Big(\frac{g^{(m)}(0)}{m!}\Big)^2 C_{m,K,s}\, \varepsilon^{m-2} \mu^m$ has coefficients bounded independently of ε. For $\mathbf{k} = \pm\langle j\rangle$ we note that if $m \ge 2$ and $\mathbf{k}^1+\dots+\mathbf{k}^m = \pm\langle j\rangle$, then necessarily $[[\mathbf{k}^1]]+\dots+[[\mathbf{k}^m]] \ge \frac{5}{2}$. Hence, for the restriction to this case the bound improves to a factor ε^3 instead of ε:

$$
\sum_{j=-\infty}^{\infty} \|f^{\pm\langle j\rangle}\|_1^2 \le \varepsilon^3\, \widetilde{P}(|||\mathbf{c}|||_1^2),
$$

where $\widetilde{P}$ is another polynomial with coefficients bounded independently of ε.

Since H^1 is a normed algebra and the map Φ is an absolutely convergent sum of polynomials in the functions $c^{\mathbf{k}}$, we also obtain that Φ is arbitrarily differentiable with correspondingly bounded derivatives on bounded subsets of $\mathbf{H}^1$.

Instead of (20), we could also have worked with a different rescaling:

$$
\widehat{c}_j^{\mathbf{k}} = \frac{\boldsymbol{\omega}^{|\mathbf{k}|}}{\varepsilon^{[[\mathbf{k}]]}} z_j^{\mathbf{k}}, \qquad \widehat{c}^{\mathbf{k}}(x) = \sum_{j=-\infty}^{\infty} \widehat{c}_j^{\mathbf{k}} \mathrm{e}^{\mathrm{i}jx} = \frac{\boldsymbol{\omega}^{|\mathbf{k}|}}{\varepsilon^{[[\mathbf{k}]]}} z^{\mathbf{k}}(x) \qquad (22)
$$

considered in the space $\mathbf{H}^s = (H^s)^{\mathcal{K}}$ with norm $|||\widehat{\mathbf{c}}|||_s^2 = \sum_{\|\mathbf{k}\|\le K} \|\widehat{c}^{\mathbf{k}}\|_s^2$. For $\widehat{f}^{\mathbf{k}}$ defined in the same way as $f^{\mathbf{k}}$ above, but with the exponent 1 in place of s, we then have the bound

$$
|||\widehat{\mathbf{f}}|||_s^2 \le \varepsilon P(|||\widehat{\mathbf{c}}|||_s^2) . \qquad (23)
$$

3.6 Abstract reformulation of the iteration

For $\mathbf{c} = (c_j^{\mathbf{k}}) \in \mathbf{H}^1$ with $c_j^{\mathbf{k}} = 0$ for all $\mathbf{k} \ne \pm\langle j\rangle$ with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$, we split the components of $\mathbf{c}$ corresponding to $\mathbf{k} = \pm\langle j\rangle$ and $\mathbf{k} \ne \pm\langle j\rangle$ and collect them in $\mathbf{a} = (a_j^{\mathbf{k}}) \in \mathbf{H}^1$ and $\mathbf{b} = (b_j^{\mathbf{k}}) \in \mathbf{H}^1$, respectively:

$$
\begin{aligned}
a_j^{\mathbf{k}} &= c_j^{\mathbf{k}} \quad \text{if } \mathbf{k} = \pm\langle j\rangle, \quad \text{and 0 else} \\
b_j^{\mathbf{k}} &= c_j^{\mathbf{k}} \quad \text{if } |\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \ge \varepsilon^{1/2}, \quad \text{and 0 else.}
\end{aligned} \qquad (24)
$$

We then have $\mathbf{a}+\mathbf{b}=\mathbf{c}$ and $|||\mathbf{a}|||_1^2+|||\mathbf{b}|||_1^2=|||\mathbf{c}|||_1^2$. We define the multiplication operator on $\mathbf{H}^1$,

$$(\Omega^{-1}\mathbf{c})_j^{\mathbf{k}} = \frac{1}{\omega_j + |\mathbf{k}\cdot\boldsymbol{\omega}|}\, c_j^{\mathbf{k}} \quad \text{for} \quad \mathbf{c}\in\mathbf{H}^1.$$

In terms of $\mathbf{a}$ and $\mathbf{b}$, the iteration of Subsection 3.3 then becomes of the form

$$\begin{aligned}\dot{\mathbf{a}}^{(n+1)} &= \varepsilon\Omega^{-1}A\ddot{\mathbf{a}}^{(n)} + \varepsilon\Omega^{-1}F(\mathbf{a}^{(n)}+\mathbf{b}^{(n)})\\ \mathbf{b}^{(n+1)} &= \varepsilon^{1/2}B_1\dot{\mathbf{b}}^{(n)} + \varepsilon^{3/2}\Omega^{-1}B_2\ddot{\mathbf{b}}^{(n)} + \Omega^{-1}G(\mathbf{a}^{(n)}+\mathbf{b}^{(n)}),\end{aligned} \tag{25}$$

where, in view of the preceding subsection, F and G are arbitrarily differentiable maps with bounded derivatives on bounded subsets of $\mathbf{H}^1$, and A, B_1, B_2 are bounded[1] linear operators. The loss of a factor $\varepsilon^{1/2}$ in the equation for $\mathbf{b}$ results from the condition $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \ge \varepsilon^{1/2}$ in (24). The initial value for $\mathbf{a}^{(n+1)}$ is determined by an equation of the form

$$\mathbf{a}^{(n+1)}(0) = \mathbf{v} + Q_0\mathbf{b}^{(n)}(0) + \varepsilon Q_1\dot{\mathbf{b}}^{(n)}(0) + \varepsilon P_1\dot{\mathbf{a}}^{(n)}(0) \tag{26}$$

with bounded operators Q_0, Q_1, P_1 and with bounded $\mathbf{v}$. The starting iterate is $\mathbf{a}^{(0)} = \mathbf{v} = (\varepsilon^{-[\![\mathbf{k}]\!]}\boldsymbol{\omega}^{s|\mathbf{k}|}w_j^{\mathbf{k}})$ with $w_j^{\mathbf{k}} = \frac{1}{2}u_j(0) \pm \frac{1}{2}(\mathrm{i}\omega_j)^{-1}\partial_t u_j(0)$ if $\mathbf{k} = \pm\langle j\rangle$ and 0 else, and $\mathbf{b}^{(0)} = 0$.

3.7 Bounds of the modulation functions

The iterates $\mathbf{a}^{(n)}$ and $\mathbf{b}^{(n)}$ and their derivatives with respect to the slow time $\tau = \varepsilon t$ are thus bounded in $\mathbf{H}^1$ for $0 \le \tau \le 1$ and $n \le 2N+2$: more precisely, the $(2N+2)$-th iterates satisfy

$$\begin{aligned}&\mathbf{a}^{(2N+2)} = \mathbf{a}^{(0)} + \mathbf{e}^{(2N+2)} \quad \text{with} \quad |||\Omega\mathbf{e}^{(2N+2)}|||_1 \le C\,,\\ &|||\Omega\dot{\mathbf{a}}^{(2N+2)}|||_1 \le C\varepsilon\,, \qquad |||\Omega\mathbf{b}^{(2N+2)}|||_1 \le C\,.\end{aligned} \tag{27}$$

We also obtain the bound $|||\Omega\dot{\mathbf{b}}^{(2N+2)}|||_1 \le C$ and similarly for higher derivatives with respect to $\tau = \varepsilon t$. For $z_j^{\mathbf{k}} = \varepsilon^{[\![\mathbf{k}]\!]}\boldsymbol{\omega}^{-s|\mathbf{k}|}\, c_j^{\mathbf{k}}$ with $(c_j^{\mathbf{k}}) = \mathbf{c}^{(2N+2)} = \mathbf{a}^{(2N+2)} + \mathbf{b}^{(2N+2)}$, the bounds for $\mathbf{a}$ and $\mathbf{b}$ together yield the bound (14).

In addition to these bounds, we also obtain that the map

$$B_\varepsilon \subset H^{s+1}\times H^s \to \mathbf{H}^1 :\ (u(\cdot,0), \partial_t u(\cdot,0)) \mapsto \mathbf{c}(0)$$

(with B_ε the ball of radius ε centered at 0) is Lipschitz continuous with a Lipschitz constant independent of ε.

With the alternative scaling (22) we obtain in the same way, for $\tau = \varepsilon t \le 1$,

$$\begin{aligned}&\widehat{\mathbf{a}}^{(2N+2)} = \widehat{\mathbf{a}}^{(0)} + \widehat{\mathbf{e}}^{(2N+2)} \quad \text{with} \quad |||\Omega\widehat{\mathbf{e}}^{(2N+2)}|||_s \le C\,,\\ &|||\Omega\dot{\widehat{\mathbf{a}}}^{(2N+2)}|||_s \le C\varepsilon\,, \qquad |||\Omega\widehat{\mathbf{b}}^{(2N+2)}|||_s \le C\,.\end{aligned} \tag{28}$$

These bounds imply $|||\widehat{\mathbf{c}}^{(2N+2)}|||_{s+1} \le C$ and hence give a bound of the expansion

[1] *Bounded* here always means *bounded in* $\mathbf{H}^1$ *with a bound that is independent of* ε.

(2) in the H^{s+1} norm:

$$\begin{aligned} \|\widetilde{u}\|_{s+1}^2 &= \sum_{j=-\infty}^{\infty} \omega_j^{2(s+1)} \Big| \sum_{\|\mathbf{k}\|\le K} z_j^{\mathbf{k}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t+\mathrm{i}jx} \Big|^2 \\ &\le \sum_{j=-\infty}^{\infty} \omega_j^{2(s+1)} \Big(\sum_{\|\mathbf{k}\|\le K} \varepsilon^{[\![\mathbf{k}]\!]} \boldsymbol{\omega}^{-|\mathbf{k}|} \, |\widehat{c}_j^{\mathbf{k}}| \Big)^2 \\ &\le C_{K,1}\, \varepsilon^2 \sum_{j=-\infty}^{\infty} \omega_j^{2(s+1)} \sum_{\|\mathbf{k}\|\le K} |\widehat{c}_j^{\mathbf{k}}|^2 = C_{K,1}\varepsilon^2 |\!|\!|\widehat{\mathbf{c}}|\!|\!|_{s+1}^2 , \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and (17) in the last inequality. So we have

$$\|\widetilde{u}(\cdot,t)\|_{s+1} \le C\varepsilon \quad \text{for} \quad t \le \varepsilon^{-1}. \tag{29}$$

3.8 Defects

We consider the defect $\mathbf{d} = (d_j^{\mathbf{k}})$ in (15),

$$\begin{aligned} d_j^{\mathbf{k}} = \big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big) z_j^{\mathbf{k}} \quad &+ \quad 2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot{z}_j^{\mathbf{k}} \; + \; \varepsilon^2 \ddot{z}_j^{\mathbf{k}} \\ &+ \quad \mathcal{F}_j \sum_m \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \frac{1}{m!}\, g^{(m)}(0)\, z^{\mathbf{k}^1} \ldots z^{\mathbf{k}^m} . \end{aligned} \tag{30}$$

The approximation $\widetilde{u}$ given by (2) inserted into the wave equation (1) yields the defect

$$\delta = \partial_t^2 \widetilde{u} - \partial_x^2 \widetilde{u} + \rho \widetilde{u} + g(\widetilde{u}) \tag{31}$$

with

$$\delta(x,t) = \sum_{\|\mathbf{k}\|\le K} d^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} + R_{N+1}(\widetilde{u}(x,t)),$$

where R_{N+1} is the remainder term of the Taylor expansion of g after N terms. By (29), we have $\|R_{N+1}(\widetilde{u})\|_{s+1} \le C\varepsilon^{N+1}$. We need to bound

$$\begin{aligned} &\Big\| \sum_{\|\mathbf{k}\|\le K} d^{\mathbf{k}}(\cdot,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \Big\|_s^2 = \sum_{j=-\infty}^{\infty} \omega_j^{2s} \Big| \sum_{\|\mathbf{k}\|\le K} d_j^{\mathbf{k}}(\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \Big|^2 \\ &\le C_{K,1} \sum_{j=-\infty}^{\infty} \sum_{\|\mathbf{k}\|\le K} \omega_j^{2s} \Big| \boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}}(\varepsilon t) \Big|^2 = C_{K,1} \sum_{\|\mathbf{k}\|\le K} \Big\| \boldsymbol{\omega}^{|\mathbf{k}|}\, d^{\mathbf{k}}(\cdot,\varepsilon t) \Big\|_s^2 . \end{aligned} \tag{32}$$

For the inequality we have used (17) with 1 in place of s and the Cauchy-Schwarz inequality.

3.9 Defect in the near-resonant modes

For $(j,\mathbf{k})$ in the set $\mathcal{R}$ of near-resonances defined by (4) we have set $z_j^{\mathbf{k}} = 0$. The defect corresponding to the near-resonant modes is thus

$$d_j^{\mathbf{k}} = \mathcal{F}_j \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} z^{\mathbf{k}^1} \ldots z^{\mathbf{k}^m} = \varepsilon^{[\![\mathbf{k}]\!]}\, \boldsymbol{\omega}^{-s|\mathbf{k}|}\, f_j^{\mathbf{k}}$$

with $|||\mathbf{f}|||_1^2 \le \widehat{C}\varepsilon$ by (14) and (21). We then have

$$\begin{aligned}\sum_{(j,\mathbf{k})\in\mathcal{R}} \omega_j^{2s}\big|\boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}}\big|^2 &= \sum_{(j,\mathbf{k})\in\mathcal{R}} \frac{\omega_j^{2(s-1)}}{\boldsymbol{\omega}^{2(s-1)|\mathbf{k}|}}\, \varepsilon^{2[\![\mathbf{k}]\!]}\, \omega_j^2 |f_j^{\mathbf{k}}|^2 \\ &\le \widehat{C} \sup_{(j,\mathbf{k})\in\mathcal{R}} \frac{\omega_j^{2(s-1)}\, \varepsilon^{2[\![\mathbf{k}]\!]+1}}{\boldsymbol{\omega}^{2(s-1)|\mathbf{k}|}}\,.\end{aligned}$$

Condition (5) is formulated such that the supremum is bounded by $C_0^2\,\varepsilon^{2(N+1)}$, and hence

$$\sum_{(j,\mathbf{k})\in\mathcal{R}} \omega_j^{2s}\big|\boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}}\big|^2 \le C\varepsilon^{2(N+1)}. \tag{33}$$

3.10 Defect in the non-resonant modes

We take a different rescaling, again denoted $\mathbf{c} = (c_j^{\mathbf{k}})$, now with

$$c_j^{\mathbf{k}} = \boldsymbol{\omega}^{s|\mathbf{k}|} z_j^{\mathbf{k}}\,, \qquad c^{\mathbf{k}}(x) = \sum_{j=-\infty}^{\infty} c_j^{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}jx} = \boldsymbol{\omega}^{s|\mathbf{k}|} z^{\mathbf{k}}(x)\,,$$

without the factor $\varepsilon^{-[\![\mathbf{k}]\!]}$, considered in the space $\mathbf{H}^1$. Splitting the rescaled $\mathbf{c}$ as $\mathbf{c} = \mathbf{a} + \mathbf{b} \in \mathbf{H}^1$ in the same way as in (24), the reverse Picard iteration of Section 3.3 is again of the form (25)–(26), where A, B_1, B_2 are bounded linear operators on $\mathbf{H}^1$, and F and G have derivatives bounded on $\mathbf{H}^1$ by $\mathcal{O}(\varepsilon^{1/2})$ in an $\mathbf{H}^1$-neighbourhood of 0 where the bound (14) holds. An induction argument shows that the defect in the iteration,

$$\begin{aligned}\varepsilon\Omega^{-1}A\ddot{\mathbf{a}}^{(n)} + \varepsilon\Omega^{-1}F(\mathbf{a}^{(n)} + \mathbf{b}^{(n)}) - \dot{\mathbf{a}}^{(n)} &= \dot{\mathbf{a}}^{(n+1)} - \dot{\mathbf{a}}^{(n)} \\ \varepsilon^{1/2}B_1\dot{\mathbf{b}}^{(n)} + \varepsilon^{3/2}\Omega^{-1}B_2\ddot{\mathbf{b}}^{(n)} + \Omega^{-1}G(\mathbf{a}^{(n)} + \mathbf{b}^{(n)}) - \mathbf{b}^{(n)} &= \mathbf{b}^{(n+1)} - \mathbf{b}^{(n)},\end{aligned}$$

and in the initial values,

$$\mathbf{v} + Q_0\mathbf{b}^{(n)}(0) + \varepsilon Q_1\dot{\mathbf{b}}^{(n)}(0) + \varepsilon P_1\dot{\mathbf{a}}^{(n)}(0) - \mathbf{a}^{(n)}(0) = \mathbf{a}^{(n+1)}(0) - \mathbf{a}^{(n)}(0),$$

is bounded by $\mathcal{O}(\varepsilon^{n/2})$ in the $\mathbf{H}^1$ norm for $0 \le \tau \le 1$ and $n \le 2N+2$.

We translate this bound back to the iteration of Section 3.3. Since the iteration concerns only the non-resonant modes $(j,\mathbf{k}) \notin \mathcal{R}$, it is convenient to set, only in this subsection, $d_j^{\mathbf{k}} = 0$ for $(j,\mathbf{k}) \in \mathcal{R}$. For $(z_j^{\mathbf{k}}) = \mathbf{z}^{(2N+2)}$ defined by the $(2N+2)$-th iterate, the above bound yields the following bounds of the defect in (15):

$$\Big(\sum_{\|\mathbf{k}\|\le K} \|\boldsymbol{\omega}^{s|\mathbf{k}|} d^{\mathbf{k}}(\cdot,\tau)\|_1^2\Big)^{1/2} \le C\varepsilon^{N+1} \quad \text{for} \quad \tau \le 1\,. \tag{34}$$

With the alternative scaling $\widehat{c}_j^{\mathbf{k}} = \boldsymbol{\omega}^{|\mathbf{k}|} z_j^{\mathbf{k}}$ we obtain in the same way

$$\Big(\sum_{\|\mathbf{k}\|\le K} \|\boldsymbol{\omega}^{|\mathbf{k}|} d^{\mathbf{k}}(\cdot,\tau)\|_s^2\Big)^{1/2} \le C\varepsilon^{N+1} \quad \text{for} \quad \tau \le 1. \tag{35}$$

For the defect in the initial conditions (16) we obtain

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s} \Big| \omega_j \sum_{\|\mathbf{k}\|\le K} z_j^{\mathbf{k}}(0) - \omega_j\, u_j(0) \Big|^2 \le C\,\varepsilon^{2(N+1)} \tag{36}$$

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s} \Big| \sum_{\|\mathbf{k}\|\le K} \Big(\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega}) z_j^{\mathbf{k}}(0) + \varepsilon \dot z_j^{\mathbf{k}}(0) \Big) - \partial_t u_j(0) \Big|^2 \le C\,\varepsilon^{2(N+1)}\,. \tag{37}$$

3.11 Defect in the wave equation

We estimate the defect δ of (31). By (32), (33), and (35), we now have

$$\Big\| \sum_{\|\mathbf{k}\|\le K} d^{\mathbf{k}}(\cdot,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \Big\|_s \le C\varepsilon^{N+1} \quad \text{for} \quad t \le \varepsilon^{-1}\,,$$

so that indeed

$$\|\delta(\cdot,t)\|_s \le C\varepsilon^{N+1} \quad \text{for} \quad t \le \varepsilon^{-1}. \tag{38}$$

We also note that, by (36)–(37), the deviations in the initial values are bounded by

$$\|\widetilde u(\cdot,0) - u(\cdot,0)\|_{s+1} + \|\partial_t \widetilde u(\cdot,0) - \partial_t u(\cdot,0)\|_s \le C\,\varepsilon^{N+1}. \tag{39}$$

3.12 Remainder term of the modulated Fourier expansion

Using the well-posedness of the nonlinear wave equation in $H^{s+1}\times H^s$, we now conclude from a small defect to a small error by a standard argument: we rewrite (1) and (31) in terms of the Fourier coefficients as

$$\begin{aligned} \partial_t^2 u_j + \omega_j^2\, u_j + \mathcal{F}_j g(u) &= 0 \\ \partial_t^2 \widetilde u_j + \omega_j^2\, \widetilde u_j + \mathcal{F}_j g(\widetilde u) &= \delta_j \end{aligned}$$

and subtract the equations. With the variation-of-constants formula, the error $r_j = u_j - \widetilde u_j$ satisfies

$$\begin{aligned} \begin{pmatrix} r_j(t) \\ \omega_j^{-1}\dot r_j(t) \end{pmatrix} &= \begin{pmatrix} \cos(\omega_j t) & \sin(\omega_j t) \\ -\sin(\omega_j t) & \cos(\omega_j t) \end{pmatrix} \begin{pmatrix} r_j(0) \\ \omega_j^{-1}\dot r_j(0) \end{pmatrix} \\ &\quad - \int_0^t \omega_j^{-1} \begin{pmatrix} \sin(\omega_j(t-\theta)) \\ \cos(\omega_j(t-\theta)) \end{pmatrix} \Big(\mathcal{F}_j g(u(\cdot,\theta)) - \mathcal{F}_j g(\widetilde u(\cdot,\theta)) - \delta_j(\cdot,\theta) \Big) d\theta\,. \end{aligned}$$

The Taylor expansion of the nonlinearity g at 0 and the fact that H^s is a normed algebra, yield the Lipschitz bound

$$\|g(v) - g(w)\|_s \le C\varepsilon\, \|v-w\|_s \quad \text{for } v,w\in H^s \text{ with } \|v\|_s \le M\varepsilon,\ \|w\|_s \le M\varepsilon.$$

Comparing the solution u with 0, this Lipschitz bound and the Gronwall inequality give $\|u(\cdot,t)\|_{s+1} \le M\varepsilon$ for $t\le\varepsilon^{-1}$. Comparing u and $\widetilde u$ gives, together with (38) and (39),

$$\|\widetilde u(\cdot,t) - u(\cdot,t)\|_{s+1} + \|\partial_t \widetilde u(\cdot,t) - \partial_t u(\cdot,t)\|_s \le C\varepsilon^{N+1} \quad \text{for } t\le\varepsilon^{-1}. \tag{40}$$

This completes the proof of Theorem 3.1.

3.13 Remark

The analysis of the modulated Fourier expansion could be done more neatly in weighted Wiener algebras $W^s = \{v \in C(\mathbb{T}) : \sum_{-\infty}^{\infty} \omega_j^s \, |v_j| < \infty\}$. Unfortunately, this ℓ^1 framework is not suited for the analysis of the almost-invariants studied in the next section, which are quadratic quantities and therefore require an ℓ^2-based framework.

4 Almost-invariants

We now show that the system of equations determining the modulation functions has almost-invariants close to the actions. The arguments are modelled after those of [5, Ch. XIII] for finite-dimensional oscillatory Hamiltonian systems.

4.1 The extended potential

Corresponding to the modulation functions $z^{\mathbf{k}}(x, \varepsilon t)$ we introduce

$$\mathbf{y} = (y^{\mathbf{k}})_{\|\mathbf{k}\| \le K} \quad \text{with} \quad y^{\mathbf{k}}(x,t) = z^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \tag{41}$$

and denote the Fourier coefficients of $y^{\mathbf{k}}(x,t)$ by $y_j^{\mathbf{k}}(t)$. By construction, the functions $y^{\mathbf{k}}$ satisfy

$$\partial_t^2 y^{\mathbf{k}} - \partial_x^2 y^{\mathbf{k}} + \rho y^{\mathbf{k}} + \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} y^{\mathbf{k}^1} \ldots y^{\mathbf{k}^m} = e^{\mathbf{k}}\,, \tag{42}$$

where the defects $e^{\mathbf{k}}(x,t) = d^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}$ are bounded by $C\varepsilon^{N+1}$ in $\mathbf{H}^s$, see (33) and (35). In (1), the nonlinearity $g(u)$ is the gradient of the potential $U(u) = \int_0^u g(v)\,dv$. The sum in (42) is recognized as the functional gradient $\nabla^{-\mathbf{k}} \mathcal{U}(\mathbf{y})$ with respect to $y^{-\mathbf{k}}$ of the *extended potential* $\mathcal{U} : \mathbf{H}^1 \to \mathbb{R}$ defined, for $\mathbf{y} = (y^{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathbf{H}^1$, by

$$\mathcal{U}(\mathbf{y}) = \sum_{m=2}^{N} \frac{U^{(m+1)}(0)}{(m+1)!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^{m+1}=\mathbf{0}} \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{\mathbf{k}^1} \ldots y^{\mathbf{k}^{m+1}}\, \mathrm{d}x\,, \tag{43}$$

where we note that by Parseval's formula,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} y^{\mathbf{k}^1} \ldots y^{\mathbf{k}^{m+1}}\, \mathrm{d}x = \sum_{j_1+\cdots+j_{m+1}=0} y_{j_1}^{\mathbf{k}^1} \ldots y_{j_{m+1}}^{\mathbf{k}^{m+1}}\,.$$

Hence, the modulation system (42) can be rewritten as

$$\partial_t^2 y^{\mathbf{k}} - \partial_x^2 y^{\mathbf{k}} + \rho y^{\mathbf{k}} + \nabla^{-\mathbf{k}} \mathcal{U}(\mathbf{y}) = e^{\mathbf{k}}, \tag{44}$$

or equivalently in terms of the Fourier coefficients,

$$\partial_t^2 y_j^{\mathbf{k}} + \omega_j^2\, y_j^{\mathbf{k}} + \nabla_{-j}^{-\mathbf{k}} \mathcal{U}(\mathbf{y}) = e_j^{\mathbf{k}}\,,$$

where $\nabla_{-j}^{-\mathbf{k}} \mathcal{U}$ is the partial derivative of $\mathcal{U}$ with respect to $y_{-j}^{-\mathbf{k}}$.

4.2 Invariance property of the extended potential

The key to the existence of almost-invariants for the system (44) is, in the spirit of Noether's theorem, the *invariance of the extended potential under continuous group actions*: for an arbitrary real sequence $\boldsymbol{\mu} = (\mu_\ell)_{\ell \geq 0}$ and for $\theta \in \mathbb{R}$, let

$$S_{\boldsymbol{\mu}}(\theta)\mathbf{y} = \left(\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu})\theta} y^{\mathbf{k}}\right)_{\|\mathbf{k}\| \leq K} . \tag{45}$$

Since the sum in the definition of $\mathcal{U}$ is over $\mathbf{k}^1 + \cdots + \mathbf{k}^{m+1} = \mathbf{0}$, we have

$$\mathcal{U}(S_{\boldsymbol{\mu}}(\theta)\mathbf{y}) = \mathcal{U}(\mathbf{y}) \quad \text{for} \quad \theta \in \mathbb{R}.$$

Differentiating this relation with respect to θ yields

$$0 = \frac{d}{d\theta}\Big|_{\theta=0} \mathcal{U}(S_{\boldsymbol{\mu}}(\theta)\mathbf{y}) = \sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{\mathbf{k}} \, \nabla^{\mathbf{k}} \mathcal{U}(\mathbf{y}) \, \mathrm{d}x \, . \tag{46}$$

4.3 Almost-invariants of the modulation system

We now multiply (44) with $\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) y^{-\mathbf{k}}$, integrate over $[-\pi, \pi]$, and sum over $\mathbf{k}$ with $\|\mathbf{k}\| \leq K$. Thanks to (46) and a partial integration, we obtain

$$\sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(y^{-\mathbf{k}} \, \partial_t^2 y^{\mathbf{k}} + \partial_x y^{-\mathbf{k}} \, \partial_x y^{\mathbf{k}} + \rho y^{-\mathbf{k}} \, y^{\mathbf{k}}\right) \mathrm{d}x$$
$$= \sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, e^{\mathbf{k}} \, \mathrm{d}x \, .$$

Since the second and third terms under the left-hand integral cancel in the sum, the left-hand side simplifies to

$$\sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, \partial_t^2 y^{\mathbf{k}} \, \mathrm{d}x = -\frac{d}{dt} \mathcal{J}_{\boldsymbol{\mu}}(\mathbf{y}, \partial_t \mathbf{y})$$

with

$$\mathcal{J}_{\boldsymbol{\mu}}(\mathbf{y}, \partial_t \mathbf{y}) = -\sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, \partial_t y^{\mathbf{k}} \, \mathrm{d}x \, .$$

Hence, we see that $\mathcal{J}_{\boldsymbol{\mu}}$ is almost conserved:

$$\frac{d}{dt} \mathcal{J}_{\boldsymbol{\mu}}(\mathbf{y}, \partial_t \mathbf{y}) = -\sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, e^{\mathbf{k}} \, \mathrm{d}x \, , \tag{47}$$

where we recall the $\mathcal{O}(\varepsilon^{N+1})$-bound of $\mathbf{e} = (e^{\mathbf{k}})$ on the right-hand side. We consider in particular $\boldsymbol{\mu} = \langle \ell \rangle = (0, \ldots, 0, 1, 0, 0, \ldots)$ with the only entry at the ℓth position, and denote $\mathcal{J}_\ell = \mathcal{J}_{\langle \ell \rangle}$, or written out,

$$\mathcal{J}_\ell(\mathbf{y}, \partial_t \mathbf{y}) = -\sum_{\|\mathbf{k}\| \leq K} \mathrm{i} k_\ell \, \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, \partial_t y^{\mathbf{k}} \, \mathrm{d}x \, .$$

Theorem 4.1 *Under the conditions of Theorem 3.1,*

$$\sum_{\ell\ge 0} \omega_\ell^{2s+1} \Big| \frac{d}{dt} \mathcal{J}_\ell(\mathbf{y}(t), \partial_t \mathbf{y}(t)) \Big| \le C\,\varepsilon^{N+2} \qquad \textit{for} \quad t \le \varepsilon^{-1}.$$

Proof. (a) We introduce the following norm and note the corresponding bound for $t \le \varepsilon^{-1}$, which is a direct consequence of (27) and inequality (19):

$$|||\mathbf{z}|||_{(s+1)} := \Big(\sum_{\|\mathbf{k}\|\le K} \sum_{j=-\infty}^{\infty} \sum_{\ell\ge 0} |k_\ell|\, \omega_\ell^{2(s+1)}\, |z_j^{\mathbf{k}}|^2 \Big)^{1/2} \le C\varepsilon\,. \tag{48}$$

We need to estimate the defect $\mathbf{d} = (d_j^{\mathbf{k}})$ of the non-resonant modes in a norm of the same type, with s instead of $s+1$. By (19) and (34) we have

$$|||\mathbf{d}|||_{(s)}^2 \le C \sum_{\|\mathbf{k}\|\le K} \sum_{j=-\infty}^{\infty} \boldsymbol{\omega}^{2s|\mathbf{k}|}\, |d_j^{\mathbf{k}}|^2 \le C \sum_{\|\mathbf{k}\|\le K} \|\boldsymbol{\omega}^{s|\mathbf{k}|} d^{\mathbf{k}}\|_1^2 \le (C\varepsilon^{N+1})^2$$

so that, again for $t \le \varepsilon^{-1}$,

$$|||\mathbf{d}|||_{(s)} \le C\,\varepsilon^{N+1}\,. \tag{49}$$

(b) By Parseval's formula $\frac{1}{2\pi}\int_{-\pi}^{\pi} y^{-\mathbf{k}}\, e^{\mathbf{k}}\, \mathrm{d}x = \sum_j y_{-j}^{-\mathbf{k}}\, e_j^{\mathbf{k}}$ and the Cauchy-Schwarz inequality, equation (47) implies

$$\sum_{\ell\ge 0} \omega_\ell^{2s+1} \Big| \frac{d}{dt} \mathcal{J}_\ell(\mathbf{y}, \partial_t \mathbf{y}) \Big| \le |||\mathbf{y}|||_{(s+1)}\, |||\mathbf{e}|||_{(s)}\,,$$

where the $e_j^{\mathbf{k}}$ for near-resonant modes $(j,\mathbf{k})$ with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$ may be set to 0, since they appear multiplied with the corresponding $y_{-j}^{-\mathbf{k}}$, which are 0 by construction. By (48), we have $|||\mathbf{y}|||_{(s+1)} = |||\mathbf{z}|||_{(s+1)} \le C\varepsilon$, and by (49), $|||\mathbf{e}|||_{(s)} = |||\mathbf{d}|||_{(s)} \le C\varepsilon^{N+1}$. □

In the following it will be more convenient to consider the almost-invariant $\mathcal{J}_\ell$ as a function of the modulation sequence $\mathbf{z}(\varepsilon t)$ rather than of $\mathbf{y}(t)$ defined by (41). With an obvious abuse of notation, we write $\mathcal{J}_\ell(\mathbf{z}, \dot{\mathbf{z}}) = \mathcal{J}_\ell(\mathbf{y}, \partial_t \mathbf{y})$. By Parseval's formula we have

$$\begin{aligned} \mathcal{J}_\ell(\mathbf{z}, \dot{\mathbf{z}}) &= -\sum_{\|\mathbf{k}\|\le K} \sum_{j=-\infty}^{\infty} \mathrm{i}k_\ell\, z_{-j}^{-\mathbf{k}} \Big(\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega}) z_j^{\mathbf{k}} + \varepsilon \dot z_j^{\mathbf{k}} \Big) \\ &= \sum_{\|\mathbf{k}\|\le K} \sum_{j=-\infty}^{\infty} k_\ell \Big((\mathbf{k}\cdot\boldsymbol{\omega}) |z_j^{\mathbf{k}}|^2 - \mathrm{i}\varepsilon\, z_{-j}^{-\mathbf{k}}\, \dot z_j^{\mathbf{k}} \Big)\,. \end{aligned} \tag{50}$$

4.4 Relationship of almost-invariants and actions

We now show that the almost-invariant $\mathcal{J}_\ell$ of the modulated Fourier expansion is close to the sum of the harmonic actions $I_\ell + I_{-\ell}$ of the solution of the nonlinear wave equation, where for $u, v \in L^2(\mathbb{T})$ with Fourier coefficients u_j, v_j,

$$J_\ell = I_\ell + I_{-\ell} \qquad \text{with} \quad I_j(u,v) = \frac{\omega_j}{2}\, |u_j|^2 + \frac{1}{2\omega_j}\, |v_j|^2\,.$$

Theorem 4.2 *Under the conditions of Theorem 3.1, along the solution $u(t) = u(\cdot, t)$ of Eq. (1) and the associated modulation sequence $\mathbf{z}(\varepsilon t)$, it holds that*

$$\mathcal{J}_\ell\big(\mathbf{z}(\varepsilon t), \dot{\mathbf{z}}(\varepsilon t)\big) = J_\ell\big(u(t), \partial_t u(t)\big) + \gamma_\ell(t)\,\varepsilon^3$$

for $t \le \varepsilon^{-1}$ and for all $\ell \ge 0$, with $\sum_{\ell \ge 0} \omega_\ell^{2s+1} \gamma_\ell(t) \le C$.

Proof. The bounds of Section 3.7 show that (50) is of the form

$$\mathcal{J}_\ell = \sum_{j=-\infty}^{\infty} \omega_\ell \Big(|z_j^{\langle \ell \rangle}|^2 + |z_j^{-\langle \ell \rangle}|^2 \Big) + \mathcal{O}_\ell(\varepsilon^3)$$

where $\mathcal{O}_\ell(\varepsilon^3)$ stands for a term $\alpha_\ell\,\varepsilon^3$ with $\sum_{\ell \ge 0} \omega_\ell^{2s+1} \alpha_\ell \le C$. The bounds of Section 3.7 further yield

$$\mathcal{J}_\ell = \omega_\ell \Big(|z_\ell^{\langle \ell \rangle}|^2 + |z_\ell^{-\langle \ell \rangle}|^2 \Big) + \omega_\ell \Big(|z_{-\ell}^{\langle \ell \rangle}|^2 + |z_{-\ell}^{-\langle \ell \rangle}|^2 \Big) + \mathcal{O}_\ell(\varepsilon^3)$$

and, in terms of the modulated Fourier expansion $\widetilde{u}_j = \sum_{\|\mathbf{k}\| \le K} z_j^{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}$,

$$\begin{aligned}
\mathcal{J}_\ell &= \frac{\omega_\ell}{4} \Big(\big|\widetilde{u}_\ell + (\mathrm{i}\omega_\ell)^{-1} \partial_t \widetilde{u}_\ell\big|^2 + \big|\widetilde{u}_\ell - (\mathrm{i}\omega_\ell)^{-1} \partial_t \widetilde{u}_\ell\big|^2 \Big) \\
&\quad + \frac{\omega_\ell}{4} \Big(\big|\widetilde{u}_{-\ell} + (\mathrm{i}\omega_\ell)^{-1} \partial_t \widetilde{u}_{-\ell}\big|^2 + \big|\widetilde{u}_{-\ell} - (\mathrm{i}\omega_\ell)^{-1} \partial_t \widetilde{u}_{-\ell}\big|^2 \Big) + \mathcal{O}_\ell(\varepsilon^3) \\
&= J_\ell(\widetilde{u}, \partial_t \widetilde{u}) + \mathcal{O}_\ell(\varepsilon^3) \\
&= J_\ell(u, \partial_t u) + \mathcal{O}_\ell(\varepsilon^3),
\end{aligned}$$

where we have used the remainder bound of Theorem 3.1 in the last step. □

4.5 From short to long time intervals

We apply Theorem 4.1 repeatedly on intervals of length ε^{-1}, for modulated Fourier expansions corresponding to different starting values $(u(t_n), \partial_t u(t_n))$ at

$$t_n = n\varepsilon^{-1}$$

along the solution $u(t) = u(\cdot, t)$ of (1). As long as u satisfies the smallness condition (8) (with 2ε in place of ε), Theorem 3.1 gives a modulated Fourier expansion $\widetilde{u}^n(t)$ that corresponds to starting values $(u(t_n), \partial_t u(t_n))$. We denote the sequence of modulation functions of this expansion by $\mathbf{z}_n(\varepsilon t)$. We now show that

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \Big| \mathcal{J}_\ell(\mathbf{z}_n(1), \dot{\mathbf{z}}_n(1)) - \mathcal{J}_\ell(\mathbf{z}_{n+1}(0), \dot{\mathbf{z}}_{n+1}(0)) \Big| \le C\varepsilon^{N+2}. \tag{51}$$

This bound is obtained as follows: Theorem 3.1 shows that

$$\Big(\|\widetilde{u}^n(\varepsilon^{-1}) - u(t_{n+1})\|_{s+1}^2 + \|\partial_t \widetilde{u}^n(\varepsilon^{-1}) - \partial_t u(t_{n+1})\|_s^2 \Big)^{1/2} \le C\varepsilon^{N+1}.$$

By the Lipschitz continuity of Section 3.7 and inequality (19), this implies

$$|||\mathbf{z}_n(1) - \mathbf{z}_{n+1}(0)|||_{(s)} + |||\dot{\mathbf{z}}_n(1) - \dot{\mathbf{z}}_{n+1}(0)|||_{(s)} \le C\varepsilon^{N+1}.$$

Together with (48), this bound yields (51) by the same argument as in part (b) of the proof of Theorem 4.1.

The bound (51) and Theorem 4.1 now yield

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \Big| \mathcal{J}_\ell(\mathbf{z}_{n+1}(0), \dot{\mathbf{z}}_{n+1}(0)) - \mathcal{J}_\ell(\mathbf{z}_n(0), \dot{\mathbf{z}}_n(0)) \Big| \le C\varepsilon^{N+2}$$

and hence, for $\tau \le 1$,

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \Big| \mathcal{J}_\ell(\mathbf{z}_n(\tau), \dot{\mathbf{z}}_n(\tau)) - \mathcal{J}_\ell(\mathbf{z}_0(0), \dot{\mathbf{z}}_0(0)) \Big| \le C\, n\, \varepsilon^{N+2},$$

which is smaller than $C\varepsilon^3$ for $n \le \varepsilon^{-N+1}$, i.e., for $t_n = n\varepsilon^{-1} \le \varepsilon^{-N}$. By Theorem 4.2 and Theorem 3.1, this implies

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \Big| J_\ell\big(u(t), \partial_t u(t)\big) - J_\ell\big(u(0), \partial_t u(0)\big) \Big| \le C\varepsilon^3 \quad \text{for} \quad t \le \varepsilon^{-N}.$$

This is the estimate of Theorem 2.1. It also shows that the smallness condition (8) remains indeed satisfied (with 2ε instead of ε, say) at $t_0, t_1, t_2, \dots$ up to times $t \le \varepsilon^{-N}$, so that the construction of the modulated Fourier expansions on each of the subintervals of length ε^{-1} is indeed feasible with bounds that hold uniformly in n. The proof of Theorem 2.1 is thus complete.

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