Envariabelanalys 1, HT-2012: Exercises (summary)

Section P4

- You should know the definition of the *domain* and *range* of a function.
- A function f is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is *even* if $f(-x) = f(x)$ for all x in the domain of f.
- You should be able to *sketch the graph* of simple functions.

Section P5

- You should know what is the *sum/difference*, *product/division* and *composition* of two given functions. You should also know what is the domain of these new functions.

Chapter 1

- The *average velocity* of an object moving from x_1 to x_2 over a time interval $[t_1, t_2]$ is given by the quantity
$$\frac{\Delta x}{\Delta t},$$
where $\Delta x = x_2 - x_1$ is the change in the distance and $\Delta t = t_2 - t_1$ is the length of the time interval.
- You should be able to compute *limits* of the form
$$\lim_{x \to a} f(x) \quad \text{or} \quad \lim_{x \to a+} g(x) \quad \text{or} \quad \lim_{x \to \infty} h(x) \quad \text{or} \quad \lim_{x \to a} k(x) = \infty.$$
For this, multiplication by the conjugate; finding a common factor; the rules for calculating limits; squeezing arguments; or the formula $u^2 - v^2 = (u - v)(u + v)$ could be useful.
- You should know the definition of *continuity* and *left and right continuity.*
- The Max-Min Theorem (page 82) and the Intermediate-Value Theorem (page 84) are useful results.
- You should know the *formal definition of limit* with ε and δ (page 88).

Chapter 2

- The *tangent line* to the graph of a function f at the point $(x_0, f(x_0))$ has equation
$$y = f'(x_0)(x - x_0) + f(x_0),$$
where the *slope* of f at x_0 is defined as
$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
- The *slope of the normal* is $\frac{-1}{\text{slope of the tangent}}$.
- You should be able to use the definition of the *derivative* of a function f:
$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
and the notation of *differentials*
$$\mathrm{d}y = f'(x)\mathrm{d}x.$$
- The *rules of differentiation* (sums, constant multiples, products, quotients, power rule, chain rule) are useful tools to compute derivatives.
- It is useful to know the derivatives of some classical functions (polynomials, trigonometric, logarithmic, exponential, hyperbolic, ...).
- The Mean-Value Theorem (page 136) is an important result.
- Theorem 12 (page 139) gives you conditions to assert whether a function is *increasing* or *decreasing*.
- Knowing the derivative of a function can be useful for applications: *antiderivatives* $\int F(x)\,\mathrm{d}x$; *initial-value problems* $y'(x) = F(y)$, $y(0) = a$ like exponential growth models, falling object; *optimisation* problems like maximisation of an area/volume; *rates of change* of a quantity $\Delta y \approx y'\Delta x$; *extreme-value problems*; etc.

Chapter 3

- You should be able to compute the *inverse* of a function f, i.e., the function f^{-1} such that
$$y = f^{-1}(x) \quad \Longleftrightarrow \quad x = f(y)$$
with domain of f^{-1} is the range of f and range of f^{-1} is the domain of f.
- You should be able to do some computations with exponential and logarithmic functions. For this, the laws of exponents (page 170), the laws of logarithms (page 171), properties of *ln* (page 175) and of the exponential function (page 177) are useful. The fact that (page 183)

  In a struggle between a power and an exponential, the exponential wins
  In a struggle between a power and a logarithm, the power wins

  is also important.

Chapter 4

- The *method of Newton*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

  is a powerful tool to find a numerical approximation of a *zero of a function*, i.e. a value x such that $f(x) = 0$. Here, one starts the above iteration with an initial guess x_0.

- The *rules of l'Hospital* (page 228 and page 230) permit to evaluate indeterminate forms of type $[0/0]$ or $[\infty/\infty]$.

- In order to *sketch the graph* of a function f one can use the following informations:

  1. Domain of f, *asymptotes*, symmetries, *intercepts points*
  2. *Critical points*, i.e points such that $f'(x) = 0$, in order to find *extrema* of the function f
  3. Intervals where f' is positive, resp. negative, in order to show that f is increasing, resp. decreasing
  4. *Inflection points*
  5. Look at the positivity of f'' in order to show that f is *concave up (or down)* on an interval.

- If a function f is too complicated one can use a *linear approximation* of f around a value a

$$L(x) = f(a) + f'(a)(x - a)$$

  to understand the main behaviour of f. The error of this approximation is $E(x) = \frac{f''(s)}{2}(x-a)^2$ for some number s between a and x.

- A better approximation of a function f is given by *Taylor polynomials of degree n*

$$f(x) \approx P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

  The error of this approximation is given by $E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1}$ for some number s between a and x.