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Exercise 1:

Show, that the flow of $y'(t) = \lambda y(t)$ for $\lambda \in \mathbb{C}$ is a commutative group of mappings acting on arbitrary initial conditions $z \in \mathbb{C}$ in the state space with respect to the composition operation.

Exercise 2:

Show, that the collocation method with collocation points $0 \leq c_1 < \ldots < c_s \leq 1$ is equivalent to the implicit Runge-Kutta method

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

with coefficients

$$a_{ij} = \int_0^{c_i} l_j(t)\, dt\,, \quad b_i = \int_0^1 l_i(t)\, dt\,, \quad \text{where} \quad l_i(t) = \prod_{k=1,k\neq i}^{s} \frac{t - c_k}{c_i - c_k}\,.$$

Proceed as follows:

(a) Inquire the degree of the Lagrange polynomials $l_i(t)$ and compute $l_i(c_j)$.

(b) Use the Lagrange polynomials to represent $u'(t_0 + th)$ as an interpolation polynomial through the points, where the collocation condition is fulfilled.

(c) Integrate the polynomial $u'(t_0 + th)$ to determine $u(t_0 + th)$. Make sure, that the initial condition is chosen correctly.

(d) Choose $Y_i = u(t_0 + c_i h)$ and $Y_i' = u'(t_0 + c_i h)$ to conclude the claim.

Exercise 3:

Compute all collocation methods with $s = 2$ for arbitrary $0 \leq c_1 < c_2 \leq 1$. Analyze the order of the methods. For which collocation points is the method of order 1, 2 or 3? Give an example of points, for which the methods is of order 4.

Programming Exercise 1: *Pendulum*

Consider the Hamiltonian

$$H(p, q) = \tfrac{1}{2}p^2 - \cos(q)\,.$$

Write down the Hamiltonian equations of motion. For the initial condition $(p(0), q(0)) = (0, 1)$, plot the position of the pendulum $q(t)$ over the time interval $[0, 50]$ computed with the

(a) Explicit Euler method

(b) Implicit Euler method

(c) Symplectic Euler method

(d) Störmer-Verlet method.

Employ the time step size $h = 0.2$ for each of them. In addition, plot the energy $H(p(t), q(t))$ for all methods over the time interval $[0, 200]$ using the same time step size $h = 0.2$.

Hint: For this particular problem, which yields a partitioned system of equations, the symplectic Euler methods turns out to be explicit, which is not the case in general.

Programming Exercise 2: *Kepler problem*

Consider the Hamiltonian

$$H(p,q) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}} .$$

Write down the corresponding Hamiltonian equations of motion and consider the initial data

$$q_1(0) = 1 - e , \quad q_2(0) = 0 , \quad p_1(0) = 0 , \quad p_2(0) = \sqrt{\frac{1+e}{1-e}} ,$$

where $e = 0.6$. In this case, we obtain elliptic trajectories with eccentricity e and periodic motion with period $T = 2\pi$.

(a) Employ the explicit Euler method, the midpoint rule and the symplectic Euler method with time step size $h = 2 \cdot 10^{-3}$ for the numerical solution. Plot the position of the first body $q_1(t)$ over the time interval $[0, 20T]$.

(b) Plot the energy $H(p(t), q(t))$ versus the time t. Use the time step size $h = 2 \cdot 10^{-3}$ for all three methods. Make an additional plot for the energy obtained with the symplectic Euler method and time step sizes $h = 10^{-3}$ and $5 \cdot 10^{-4}$. What do you observe?

(c) Show that the angular momentum

$$L(p,q) = q_1 p_2 - q_2 p_1$$

is conserved along the exact solution. What behavior do the numerical solutions show?

Note: Matlab templates for the programming exercises can be downloaded from `https://na.math.kit.edu/cohen/`.

Discussion in the exercise class on 3.5.2012.