Karlsruher Institut für Technologie

Fakultät für Mathematik
Institut für Angewandte und
Numerische Mathematik

Dr. David Cohen
Dr. Julia Schweitzer

Geometric Numerical Integration, Serie 4 5.6.2012

Exercise 11: *Derivation of the Euler-Lagrange equations*

Consider the following problem:

Let $L(x, y, y') : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. The task is to find a function y, such that the functional

$$T(y) = \int_a^b L(x, y(x), y'(x))\,\mathrm{d}x$$

is minimal for all functions $y \in \mathcal{C}^1([a,b];\mathbb{R})$, which satisfy $y(a) = A$ and $y(b) = B$.

(a) Prove the following Lemma:

Let $y \in \mathcal{C}^1([a,b];\mathbb{R})$ be a function, that satisfies $y(a) = A$ and $y(b) = B$. Assume that $L(x, y(x), y'(x))$ is continuously differentiable in the neighbourhood of $(x, y(x), y'(x)) \in \mathbb{R}^3$ for all $x \in [a,b]$. If y is an extremum of $T(y)$, then

$$\int_a^b \left(\frac{\partial L}{\partial y}\Big(x, y(x), y'(x)\Big)h(x) + \frac{\partial L}{\partial y'}\Big(x, y(x), y'(x)\Big)h'(x) \right) \mathrm{d}x = 0 \qquad (\star)$$

holds for all $h \in \mathcal{C}^1([a,b];\mathbb{R})$ with $h(a) = h(b) = 0$. A function y, that fulfills $(\star)$, is called *extremal (stationary point)*.

(b) Prove the following Theorem:

Let $L(x, y, y') : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that all first partial derivatives exist and are continuous. A function y satisfying $y(a) = A$ and $y(b) = B$ is an extremal of the functional T if and only if

(i) $\frac{\partial L}{\partial y'}(x, y(x), y'(x))$ is continuously differentiable with respect to x and

(ii) $\frac{\partial L}{\partial y}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial y'}(x, y(x), y'(x)) = 0$ is satisfied.

Hint: In order to prove this theorem, make use of the Lemma of Du Bois-Reymond: Let $d : [a,b] \to \mathbb{R}$ be continuous and $\int_a^b d(x)h'(x)\,\mathrm{d}x = 0$ for all $h \in \mathcal{C}^1([a,b];\mathbb{R})$ that satisfy $h(a) = h(b) = 0$. Then $d(x)$ is constant.

Exercise 12:

Consider the Hénon mapping $\Psi_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\Psi_{a,b}(p,q) = \begin{pmatrix} p \\ 1 + bq + ap^2 \end{pmatrix} .$$

Show that $\Psi_{a,b}$ is symplectic, namely that

$$\left(\frac{\partial}{\partial(p,q)}\Psi_{a,b}\right)^T J \left(\frac{\partial}{\partial(p,q)}\Psi_{a,b}\right) = J\,, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

holds, if and only if $b = 1$.

Exercise 13:

Show that a linear mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ is symplectic if and only if $\det A = 1$.

Programming Exercise 5: *Perturbed Kepler problem*

We consider the perturbed Kepler problem with the Hamiltonian function

$$H(p,q) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}} - \frac{0.005}{2\sqrt{(q_1^2 + q_2^2)^3}}$$

and initial conditions

$$p_1(0) = 0\,, \quad p_2(0) = \sqrt{\frac{1+e}{1-e}}\,, \quad q_1(0) = 1 - e\,, \quad q_2(0) = 0\,,$$

where $e = 0.6$ denotes the eccentricity, on the time interval $[0, 200]$. From Programming Exercise 2 we know that the unperturbed Kepler problem conserves two quantities, namely the Hamiltonian function $H(p,q)$ itself an the angular momentum $L(p,q) = q_1 p_2 - q_2 p_1$. Show that the angular momentum is still an invariant for Hamiltonian systems of the form

$$H(p,q) = T(p_1^2 + p_2^2) + V(q_1^2 + q_2^2).$$

Solve the problem numerically with the explicit Euler method as well as the symplectic Euler method (implicit in p), both with time step size $h = 0.03$.

(a) Use the methods without any projection.

(b) Use the methods with projection onto the manifold given by $H(p,q) = H(p_0, q_0)$.

(c) Use the methods with projection onto the manifold given by $H(p,q) = H(p_0, q_0)$ as well as $L(p,q) = L(p_0, q_0)$.

For each numerical solutions plot the second position coordinate q_2 versus the first q_1.

Discussion in the exercise class on 14.6.2012.