Summary: Chapter 2

- We consider the following problem

$$\begin{aligned} \dot{y} &= f(t,y), \\ y(t_0) &= y_0. \end{aligned}$$

Let $s \geq 1$ be a given integer, $b_i, a_{ij} \in \mathbb{R}$ for $i,j = 1,\ldots,s$ and $c_i = \sum_{j=1}^{s} a_{ij}$. The numerical method

$$\begin{cases} k_i = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j) \quad i = 1,2,\ldots,s \\ y_1 = y_0 + h \sum_{j=1}^{s} b_j k_j \end{cases}$$

is called an *s-stage Runge-Kutta method*. Notation: $\begin{array}{c|c} c & a \\ \hline & b \end{array}$.

Examples: Explicit Euler scheme; Implicit Euler scheme; Midpoint rule; etc.

- Let $0 \leq c_1 < c_2 < \ldots < c_s \leq 1$ be real numbers. The *collocation polynomial* $u(x)$ of degree s is defined by

$$\begin{cases} u(t_0) = y_0 \\ u'(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h)) \quad i = 1,\ldots,s. \end{cases}$$

The *collocation method* is then defined by

$$y_1 = u(t_0 + h).$$

A collocation method is an s-stage Runge-Kutta method with

$$a_{ij} = \int_0^{c_i} \ell_i(\tau)\, \mathrm{d}\tau \quad \text{and} \quad b_i = \int_0^1 \ell_i(\tau)\, \mathrm{d}\tau,$$

where

$$\ell_i(\tau) = \prod_{k \neq i} \frac{\tau - c_i}{c_k - c_i}$$

is the Lagrange polynomial of degree $s-1$.

A collocation method has the same order as the underlying quadrature formula $(b_i, c_i)_{i=1}^{s}$.

Examples: *Gauß methods*: Let c_i, $i = 1,\ldots,s$ be the zeros of the Legendre polynomial of degree s. The Gauß (collocation) scheme has thus the order $p = 2s$. For $s = 1$, one gets the midpoint rule.

- We have seen the order conditions for Runge-Kutta schemes with the help of trees and B-series.

- A one-step numerical scheme $y_1 = \Phi_h(y_0)$ is *symmetric*, if $\Phi_h \circ \Phi_{-h} = Id$ or $\Phi_h = \Phi_{-h}^{-1}$. The numerical scheme $\Phi_h^* := \Phi_{-h}^{-1}$ is called the *adjoint method*.

  A Runge-Kutta scheme is symmetric, if

  $$a_{i,j} + a_{s+1-i,s+1-j} = b_{s+1-j} \quad \text{and} \quad b_i = b_{s+1-i}.$$

  Example: The midpoint rule is symmetric.

- Let us consider the following problem

  $$\begin{aligned} \dot{p} &= f(p,q), \\ \dot{q} &= g(p,q). \end{aligned}$$

  Let (b_i, a_{ij}) and $(\hat{b}_i, \hat{a}_{ij})$ be the coefficients of two Runge-Kutta methods. An *s-stage partitioned Runge-Kutta method* reads

  $$\begin{cases} k_i &= f(p_0 + h\sum_{j=1}^{s} a_{ij}k_j, q_0 + h\sum_{j=1}^{s} \hat{a}_{ij}\ell_j) \quad i = 1,2,\ldots,s \\ \ell_i &= g(p_0 + h\sum_{j=1}^{s} a_{ij}k_j, q_0 + h\sum_{j=1}^{s} \hat{a}_{ij}\ell_j) \quad i = 1,2,\ldots,s \\ p_1 &= p_0 + h\sum_{j=1}^{s} b_j k_j \\ q_1 &= q_0 + h\sum_{j=1}^{s} \hat{b}_j \ell_j. \end{cases}$$

  Examples: Symplectic Euler method; Störmer-Verlet scheme; etc.