Summary: Chapter 4

- The equations of motion of a general mechanical system with d degree of freedom and position coordinates $q = (q_1, \dots, q_d)^T$, is given by the *Euler-Lagrange equations*

  $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q},$$

  with the (given) *Lagrange function* $\mathcal{L} := \mathcal{L}(q, \dot{q}) := T(q, \dot{q}) - U(q)$. Here, $T(q, \dot{q})$ is the *kinetic energy* and $U(q)$ is the *potential energy.*

  Hamilton had the idea to define a new variable

  $$p_k := \frac{\partial \mathcal{L}}{\partial \dot{q}_k}, \quad k = 1, \dots, d,$$

  the *momentum* and to consider the *Hamiltonian*

  $$H(p, q) := p^T \dot{q} - \mathcal{L}(q, \dot{q}).$$

  This leads to *Hamilton's equations*

  $$\begin{aligned} \dot{p} &= -\nabla_q H(p, q), \\ \dot{q} &= \nabla_p H(p, q). \end{aligned}$$

  These equations are equivalent to the Euler-Lagrange equations.

  In the important case where $T(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$ with a symmetric positive definite matrix $M(q)$, the Hamiltonian function is the *total energy* $H = T + U$ of our problem.

- A differentiable map $g : U \longrightarrow \mathbb{R}^{2d}$, with $U \subset \mathbb{R}^{2d}$ an open set, is called *symplectic*, if for all $(p, q) \in U$ the map $g'(p, q)$ is a *symplectic linear map*, i.e.

  $$g'(p, q)^T J g'(p, q) = J \;\; \forall \; (p, q) \in U,$$

  with $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$.

  We have the following results:

  - The flow of a Hamiltonian system is symplectic.
  - If the flow of a differential equation $\dot{y} = f(y), y(0) = y_0$ is symplectic for all time (in a neighbourhood of y_0), then the differential equation is *locally Hamiltonian*, i.e., locally one has $f(y) = J^{-1}\nabla H(y)$.