A Deligne-Riemann-Roch isomorphism

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1 Introduction

This is the final paper on the algebraic side of a partial response to questions posed in [Del87]. It amounts to the study of a functorial version of Riemann-Roch theorems. This is supposed to be understood in the following sense, for which we refer to loc.cit. for the best introduction. A special case of the Grothendieck-Riemann-Roch theorem can be understood as the formula

$$\mathrm{c}_1(Rf_*E) = f_*(\mathrm{ch}(E)\,\mathrm{Td}(T_f))^{(1)} \tag{1}$$

for a projective smooth morphism of smooth varieties $f : X \to Y$ (cf. [FL85], chapter V, §7 or the book [GBI71] for a precise and more general formulation). The general question on functoriality becomes whether there are categorical replacements of all the objects and homomorphisms involved. This is an approach to obtain secondary information which gets lost when one quotients out with various equivalences. Deligne deduces (cf. see [Del87], Théorème 9.9) a unique, up to sign, isomorphism of line bundles

$$(\det Rf_*L)^{\otimes 12} \simeq \langle \omega, \omega \rangle \langle L, L\omega^{-1} \rangle^{\otimes 6} \tag{2}$$

for $f : C \to S$ a smooth family of proper curves and L a line bundle on C. This isomorphism is suggested by the same Grothendieck-Riemann-Roch theorem which says that the classes of the two line bundles are the same in the Picard group (if S is regular enough). In earlier articles ([Eria], [Erib]) I have established various properties such as rigidity results for virtual categories as well as a functorial excess intersection theorem. These will serve in this paper to construct a functorial Lefschetz- and Adams-Riemann-Roch theorem. We will also deduce some geometric consequences when applying the Adams-Riemann-Roch theorem to the case of curves and the theory of discriminants.
This article is in preliminary form. In particular the section of the Grothendieck-Riemann-Roch theorem in general form, and its consequences, will be improved upon.

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2 Review

This paper builds on two other papers, [Eria] and [Erib]. We will review the main results and contents of these in this section.

2.1 The virtual category

We review the definition of the virtual category $V(\mathcal{C})$ of an exact category $\mathcal{C}$. Recall first that a Picard category is a symmetric monoidal groupoid where the "tensor functor" induces an equivalence of categories when fixing one of the variables, or more informally a "categorical group", with sum and associativity-isomorphisms instead of identities (cf. [Del87], Section 4, which is also the main reference for virtual categories used here). Deligne's virtual category $V(\mathcal{C})$ is a Picard category, together with a functor, $[-] : (\mathcal{C}, iso) \to V(\mathcal{C})$ (here the first category is the subcategory of $\mathcal{C}$ consisting of the same objects and the morphisms are the isomorphisms of $\mathcal{C}$), with the following universal property: Suppose we have a functor $[-] : (\mathcal{C}, iso) \to P$ where P is a Picard category, satisfying

(a) Additivity on exact sequences, i.e. for an exact sequence $A \to B \to C$ we have an isomorphism $[B] \simeq [A] + [C]$, and compatibility with isomorphisms of exact sequences.

(b) A zero-object of $\mathcal{C}$ is mapped to a zero-object in P.

(c) The additivity on exact sequences is compatible with admissible filtrations.

(d) If $A \to B \to 0$ is an exact sequence, with the first map being an isomorphism f, then the induced isomorphism $[B] \simeq [A] + [0] \simeq [A]$ is $[f]$.

Then the conclusion is that the functor $[-] : (\mathcal{C}, iso) \to P$ factors uniquely up to unique isomorphism through $(\mathcal{C}, iso) \to V(\mathcal{C})$. For an algebraic stack X we denote by $V(X)$ the virtual category of vector bundles on X. In [Eria], Section 1.1 there is also a virtual category associated to Waldhausen categories, such as the category of complexes on an abelian category together with quasi-isomorphisms. In both cases they can be realized as the fundamental groupoid of the associated K-theory space. In particular the group of isomorphism classes of any object of the virtual category is the usual Grothendieck group, $K_0(X)$, of the category of vector bundles on X, and the automorphism group of any object is $K_1(X)$ so the virtual category interpolates between the two. From this description it follows that questions such as equivalences or faithfulness of virtual categories can often be read from bijectivity and injectivity on induced maps on K_1. Also, to fix notation, denote by $\otimes : V(X) \times V(X) \to V(X)$ the natural tensor product on the virtual category induced by the tensor product of vector bundles, and $\cap : V(X) \times C(X) \to C(X)$ the same tensor product but replacing one of the arguments with the virtual category of coherent sheaves. We will also denote by $+ : V(X) \times V(X) \to V(X)$ the natural direct sum functor of virtual vector bundles.

2.2 Chow categories and rigidity

In [Fra90] and [Fra91] Franke has constructed Chow categories and Chern functors, in short categorifications of classical constructions. The definition of Chow categories is done using codimension, but in this article we will use (relative) dimension. The necessary work was established in [Eria], and we review the definition here.

Definition 2.0.1. Let X be a scheme of finite type over a regular scheme, admitting an ample family of line bundles. The i-th (homological) Chow category is the following: The objects are dimension i-cycles and whose homomorphisms are given by $\mathrm{Hom}(z, z') := \{f \in E^1_{i-1,-i}(X), d_1(f) = z' - z\}/d_1 E^1_{i-2,-i}(X)$, where the groups and differentials involved are the natural ones coming from the niveau spectral sequence. This category localized at $\mathbb{Q}$ is the (rational and homological) Chow category and we denote it by $\mathcal{CH}_i(Z)$. It has a natural structure of Picard category.

We will furthermore suppose that all our schemes are of the above form, i.e. of finite type over a regular scheme and admit an ample family of line bundles. The definition of Chow categories in [Fra90] uses codimension instead of dimension in the definition, defining categories $\mathcal{CH}^k$ instead, and we denote in this article the same category localized at $\mathbb{Q}$. Moreover, in [Eria] we relate this to the virtual category of coherent sheaves on X, and use this to define Chern functors. Using [Gil81], both [Fra90] and [Eria] establish most of the natural properties one might ask for of a categorification of the Chow group. We recall the main properties in the following proposition:

Proposition 2.1. *Fix a positive integer k. Then we have*
Proper pushforward: Suppose $f : X \to Y$ is a proper morphism, then there is a functor $f_* : \mathcal{CH}_k(X) \to \mathcal{CH}_k(Y)$.
Gysin-type functors: Suppose $f : X \to Y$ is a projective local complete intersection morphism of constant relative dimension d. Then there is a natural functor $f^* : \mathcal{CH}_k(Y) \to \mathcal{CH}_{k+d}(X)$.
Topological invariance: Consider the closed embedding $Z_{red} \subseteq Z$. The induced proper pushforward induces an equivalence of categories.
Homotopical invariance: Suppose $T \to X$ is a torsor under a vector bundle E. The pullback morphism is an equivalence of categories.
Compatibility with Franke's construction: If X is of pure dimension n and bicatenary, i.e. for any closed subscheme Z, we have $\dim Z + codim_X Z = \dim X$, then there is an identification

$$\mathcal{CH}_k(X) = \mathcal{CH}^{n-k}(X).$$

Here the most nontrivial part is the establishment of the Gysin-type functor. In [Eria] we deduce it from the same type of functor on G-theory, which is possible in view of that we are working with rational coefficients.

Proposition 2.2. *Let E be a line bundle and k a positive integer, and α a k-cycle. Then there is a natural additive functor $\mathbf{c}_k(E)\cap : \mathcal{CH}_n(X) \to \mathcal{CH}_{n-k}(X)$ satisfying the following properties:*
Normalization: If $E = \mathcal{O}(D)$ is a line bundle given by a Cartier divisor D, then $\mathbf{c}_1(\mathcal{O}(D)) \cap [X] \simeq [D]$ and $\mathbf{c}_0(E) \cap \alpha \simeq \alpha$ in general.
Commutativity: If E and E' are both vector bundles and k and k' are positive integers, then there is a natural transformation of functors

$$\mathbf{c}_k(E) \cap (\mathbf{c}_{k'}(E')\cap) \to \mathbf{c}_{k'}(E') \cap (\mathbf{c}_k(E)\cap).$$

Whitney sum isomorphism: Let $\mathbf{c}(E) = 1 + \mathbf{c}_1(E) \cap + \mathbf{c}_2(E) \cap + \ldots$ denote the total Chern class-type functor and suppose we have an exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0.$$

Then there is an natural transformation of functors :

$$\mathbf{c}(E) \to \mathbf{c}(E')\mathbf{c}(E'').$$

Projection formula: Let $f : X \to Y$ be a proper morphism. Then there is a natural transformation

$$f_* \left(\mathbf{c}_k(f^*E) \cap -\right) \to \mathbf{c}_k(E) \cap f_*.$$

It should be remarked that there are analogous of the above propositions in [Fra91]. When it makes sense (i.e. in the context of the last point of Proposition 2.1) to compare the two constructions they coincide. We also use the rigidity theorem

Theorem 2.3 ([Eria], Theorem 4.5)**.** *Any endofunctor f of the contravariant functor K_0, on the category of regular schemes, has a canonical lifting to the virtual category by a contravariant functor F. Moreover, any automorphism of the functor F is 2-torsion, in particular trivial if we invert 2.*

3 Explicit construction of characteristic classes

Let $\mathcal{X}$ be an algebraic stack, which from [Erib] we view as an algebraic space together with descent data, and consider the virtual category $V(\mathcal{X})$. Consider the full subcategory $V(\mathcal{X})^*$ of $V(\mathcal{X})$ consisting of the elements whose image $[-]$ in $K_0(\mathcal{X})$ is invertible. It is clear that tensor-product on $V(\mathcal{X})$ satisfies the pentagonal and hexagonal axioms for a Picard category (see section 2.1 in [Eria]). Also, by construction, for a fixed object B in $V(\mathcal{X})^*$, $A \mapsto A \otimes B$ is essentially surjective, and fully faithful since it acts on the automorphism-group $K_1(\mathcal{X})$ of an object by $[B]$ which is an automorphism in view of the fact that $K_1(\mathcal{X})$ is a $K_0(\mathcal{X})$-module. It follows that the category $V(\mathcal{X})^*$ together with the tensor product is a Picard category and thus for any object B in $V(\mathcal{X})^*$ there is an element, B^{-1}, unique up to unique isomorphism such that $B \otimes B^{-1} = 1$. As with the category $V(\mathcal{X})$, $V(\mathcal{X})^*$ comes equipped with a plethora of sign-anomalies associated with the fact that they are not strictly commutative. We start by showing that certain characteristic classes are constructible in a quite general context whenever we ignore these signs.

Definition 3.0.1. Suppose $(P, \oplus)$ is a Picard category with a distributive functor $\otimes : P \times P \to P$ with associativity and commutativity-constraints satisfying the hexagonal and pentagonal axioms (cf. loc. cit.) so that $\otimes$ makes P into a (non-unital) monoidal category. We call $(P, \oplus, \otimes)$ a Picard ring and often omit reference to $\oplus$ and $\otimes$. It is said to be strictly commutative if the operations $\oplus$ and $\otimes$ are strictly commutative, i.e. the symmetry-isomorphism $X \oplus X \to X \oplus X$ and $Y \otimes Y \to Y \otimes Y$ is the identity. A ring functor of Picard rings is a functor of Picard rings which is monoidal for both operations $\oplus$ and $\otimes$.
We say that a category P fibered over a category $\mathcal{C}$ is a category fibered in Picard rings (resp. categories) over a category $\mathcal{C}$ if for any object X of $\mathcal{C}$, $P(X)$ is a Picard ring (resp. category), and such that for any morphism $f : X \to Y$ in $\mathcal{C}$, there is a ring (resp. additive) functor $f^* : P(Y) \to P(X)$ satisfying the natural associativity constraints.

Clearly the virtual category $V(\mathcal{X})$ is a Picard ring and V defines a category fibered in Picard rings over the category of algebraic stacks. In general, for a Picard ring P, we can consider the full subcategory of elements P^* whose isomorphism-class in $\pi_0(P)$ is invertible under the operation $\pi_0(\otimes)$. By the same argument as above, $(P^*, \otimes)$ forms a Picard category, and similarly for a category fibered in Picard rings P one obtains a category fibered in Picard categories P^*. Similarly for the cohomological virtual category $W(\mathcal{X})$.

For the next proposition, recall that by [Tot04], Theorem 1.1, Proposition 1.3, a normal separated Noetherian algebraic stack (over $\operatorname{Spec} \mathbb{Z}$) with affine geometric stabilizers has the resolution property if and only if it is of the form $[U/GL_d]$ for quasi-affine U. In particular, a regular algebraic stack with affine stabilizers is of the form $[U/GL_d]$ for regular quasi-affine U if and only if it has the resolution property. This is in the same spirit as the following result which we shall also quote often:

Theorem 3.1 (Jouanolou-Thomason, [Wei89], Proposition 4.4)**.** *Let X be a scheme admitting an ample family of line bundles. Then there is a vector bundle $\xi \to X$ and a ξ-torsor $f : T \to X$ such that T is affine.*

Proposition 3.2 (Multiplicative characteristic classes in cohomological virtual categories on quotient stacks)**.** *Consider the cohomological virtual category W considered as a Picard ring fibered over the category of regular algebraic stacks with the resolution property and finite affine stabilizers (by the above, necessarily of the form $[U/GL_d]$ for quasi-affine U). Suppose we are given a powerseries*

$$F = 1 + \sum_{i=1}^{\infty} a_i x^i \in 1 + x\mathbb{Q}[[x]].$$

There is then a unique functor $\Theta : V \to W$, up to unique isomorphism, such that:

(a) Θ is a determinant functor $V \to W^*$.

(b) For a line bundle L on $\mathcal{X}$ there exists an isomorphism $\Theta(L) = F(L-1)$, which is well-defined by virtue of point (7) of the rigidity theorem.

Proof. As for existence, we first recall the formalism of Hirzebruch polynomials. Let R be a λ-ring and denote by γ the corresponding γ-structure (cf. [FL85], chapter III). Suppose that $\phi(x) \in 1 + xR[[x]]$. We can associate multiplicative maps $M_\phi(x) : R \to R$ as follows. First, for u a line element, we simply define

$$M_\phi(u) = \phi(u - 1).$$

If e is a sum of line elements u_i, we set

$$M_\phi(e) = \prod_i M_\phi(u_i).$$

If W_i are independent variables, we consider the power-series

$$M_\phi(W_i t) = \sum H_j^\phi(s_1, \ldots, s_j) t^j$$

for some degree j-homogenous polynomial H_j^ϕ in the elementary symmetric functions s_k in the W_i. Here the H_j^ϕ are the associated (multiplicative) Hirzebruch polynomials. Now, regular schemes have the resolution property so by [Tot04], Theorem 1.1 they are of the form prescribed. Let X be a regular scheme, $R = K_0(X)_\mathbb{Q}$ and $\phi = F$. The associated H_j^F and M_F define homomorphisms $K_0(X)_\mathbb{Q} \to K_0(X)_\mathbb{Q}$ functorial on the category of regular schemes. By rigidity they define functors, which we denote by $H_j : W \to W^{(j)}$ and $M : W \to W$, such that for a line bundle L on a regular algebraic stack, $H_j(L) = a_j(L-1)^j$ and $M(L) = 1 + \sum_{i=1}^{\infty} a_i(L-1)^k$. This sum is again well-defined by rigidity. Rigidity also implies that for a sum of virtual bundles $u+v$ on an algebraic stack $\mathcal{X}$, there is a canonical isomorphism $H_j(u+v) \to \sum H_i(u) \otimes H_{i-j}(v)$ in $W^{(j)}$. It follows that there is an isomorphism $M(u+v) \to M(u) \otimes M(v)$ in $W(\mathcal{X})^*$ and thus Θ defines a determinant functor by the composition $V(\mathcal{X}) \to W(\mathcal{X}) \to W(\mathcal{X})^*$ and it is functorial by construction and satisfies the conditions of the theorem. We are left to establish unicity. Suppose $\mathcal{X} = [U/GL_d]$ is a regular algebraic stack with quasi-affine U. By the splitting principle it is sufficient to verify that the object $\Theta(L)$ in $W(\mathcal{X})$ is uniquely determined. The trivial bundle on U is then GL_d-equivariantly ample and there exists a GL_d-equivariant locally split monomorphism $L \subset \mathcal{O}^r$ for big enough r. This defines a section $i : \mathcal{X} \to \mathbb{P}^n_{\mathcal{X}}$ to the natural projection $p : \mathbb{P}^n_{\mathcal{X}} \to \mathcal{X}$ and $L = i^*\mathcal{O}(1) = Li^*\mathcal{O}(1)$. Since the algebraic stacks have the resolution property we can apply the functor Ri_* and we then have an isomorphism

$$\begin{array}{ccccc} Ri_*\Theta(L) & \longrightarrow & Ri_*Li^*\Theta(\mathcal{O}(1)) & \longrightarrow & Ri_*(\mathcal{O}_{\mathcal{X}}) \otimes \Theta(\mathcal{O}(1)) \\ \downarrow & & \downarrow & & \downarrow \\ Ri_*F(L-1) & \longrightarrow & Ri_*Li^*F(\mathcal{O}(1)-1) & \longrightarrow & Ri_*(\mathcal{O}_{\mathcal{X}}) \otimes F(\mathcal{O}(1)-1) \end{array} .$$

Since $Rp_*Ri_* = \mathrm{id}$ the isomorphism $\Theta(L) \to F(L-1)$ is determined by the isomorphism $\Theta(\mathcal{O}(1)) \to F(\mathcal{O}(1)-1)$ on $\mathbb{P}^n_{\mathcal{X}}$. However, $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathcal{X}}$ is the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{Z}}$ via the unique (in general non-representable) morphism $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ and this isomorphism is tautologically rigidified since $W(\mathbb{P}^n_{\mathbb{Z}}) = V(\mathbb{P}^n_{\mathbb{Z}})_{\mathbb{Q}}$ doesn't have any non-trivial automorphisms. □

Any class constructed in the above fashion will be called the associated multiplicative class to the given data. We now harvest the following corollary of the previous proposition.

Corollary 3.3 (Bott's cannibalistic class)**.** *Let $\mathcal{X}$ be a regular algebraic stack with finite affine stabilizers of the form $[U/GL_d]$ for a quasi-affine U.*

- *For $k \geq 1$, there is then a unique Bott-element $\theta_k : V(\mathcal{X}) \to W(\mathcal{X})$ which is k times the associated multiplicative class of the polynomial $F(x) = k^{-1}(1 + (x+1) + (x+1)^2 + \ldots + (x+1)^{k-1}) \in 1 + \mathbb{Q}[x]$.*
- *Suppose that v is a virtual vector bundle of rank r. Then $\theta_2(-v)$ is equal to*

$$\frac{1}{2^r}\left(1 - \frac{v-r}{2} + \frac{(v-r)^2 - \gamma^2(v-r)}{4} - \frac{(v-r)^3 - 2\gamma^2(v-r)(v-r) + \gamma^3(v-r)}{8}\right)$$

modulo $F^{(4)}W(\mathcal{X})$ [1].

Proof. This is all contained in the preceding proposition, except for the last point which is a direct calculation of the relevant Hirzebruch polynomial. □

[1] Here taking a Picard category P modulo a full sub Picard category P', is to be understood as the category P'' defined as follows: objects are those of P with the equivalence relation that $a \sim b$ if $a - b$ is in the essential image of P' in P. Morphisms are described by removing automorphisms coming from P'. It is clearly also a Picard category.

Proposition 3.4. *There is a canonical determinant functor $\theta_k : V(X) \to V(X)\left[\frac{1}{k}\right]^*$ such that for a line bundle L*

$$\theta_k(L) \simeq 1 + L + L^{\otimes 2} + \ldots + L^{\otimes (k-1)}.$$

Proof. For a line bundle we use the definition of Corollary 3.3 and given a complete flag of a vector bundle E we define it by multiplicativity. We need to prove that this isomorphism doesn't depend on the choice of flag. $\square$

3.1 An explicit functorial Lefschetz formula for cyclic diagonal actions

In this section we recall and formulate the Lefschetz-Riemann-Roch theorem of [Tho92] (in particular, Théorème 3.5) for regular schemes with the action of cyclic diagonalizable group (see below), and make it functorial. Recall that a regular scheme is to be understood as Noetherian, separated regular scheme.
Let S be a connected separated Noetherian scheme, and $T = \operatorname{Spec} S[M]$ a diagonalizable group of finite type determined by an abelian group M. By [DG70], I.4.7.3, a T-representation E on S is equivalent to a grading of weights $\bigoplus_{\lambda \in M} E_\lambda$ and so the K-groups of T-equivariant locally free sheaves are given by $K_*(S,T) = K_*(S) \otimes \mathbb{Z}[M]$, and to any prime ideal ρ of $\mathbb{Z}[M]$ consider

$$K_\rho = \{\lambda \in M | 1 - [\lambda] \in \rho \subseteq \mathbb{Z}[M]\}$$

and associate to it the sub group-scheme $D_S(M/K_\rho) = T_\rho \subseteq T$. T_ρ is called the support of ρ and has the property that for any closed diagonalizable sub group-scheme $T' = D_S(M/K) \subseteq T$, ρ is an inverse image of $\mathbb{Z}[M] \to \mathbb{Z}[M/K]$ if and only if $T_\rho \subseteq T'$ (see loc.cit. Proposition 1.2). Given a T-equivariant S-scheme X, we denote by $i : X^\rho \to X$ the fixed-point scheme of X under T_ρ.

Theorem 3.5 ([Tho92])**.** *Keep the above assumptions and assume in addition that X is a regular scheme.*

(a) X^T is also a regular scheme.

(b) For any prime ideal ρ of $\mathbb{Z}[M]$, we have an isomorphism of localizations at ρ, $i_* : K_*(X^\rho, T)_{(\rho)} \simeq K_*(X,T)_{(\rho)}$.

(c) If N_i is the normal bundle to $i : X^\rho \to X$, the inverse to i_* is given by $(\lambda_{-1}N_i^\vee)^{-1} \otimes i^*$ (part of the statement is that $\lambda_{-1}(N_i^\vee)$ is invertible in $K_0(X^\rho, T)_{(\rho)}$).

(d) Suppose that Y is also a regular and T-equivariant S-scheme with $j : Y^\rho \to Y$ and that $f : X \to Y$ is a proper T-equivariant morphism with induced morphism $f' : X^\rho \to Y^\rho$, then we have the formula

$$Rf_*(\mathcal{F}) = Rf'_*\left((\lambda_{-1}f'^*N_j^\vee) \otimes (\lambda_{-1}N_i^\vee)^{-1} \otimes Li^*\mathcal{F}\right)$$

in $K_*(Y^\rho, T)_{(\rho)}$.

We digress for a short moment on the following case. Suppose $M = \mathbb{Z}^r \oplus \mathbb{Z}/n$ and $T = D_S(M) = \mathbb{G}_m^r \times \mu_n$, a "cyclic diagonalizable group" (compare [Tho92], Remarque 1.5 and [Seg68]) and let X be a finite-dimensional connected regular S-scheme with trivial T-action, and L a line-bundle on X with no trivial eigenvalues by the action of T. That is, L is given by a line bundle L_0 and a grading $\lambda \in M \setminus 0$. Let Φ_n be the n-th cyclotomic polynomial and $\rho = \rho_T : \ker[\mathbb{Z}[M] = \mathbb{Z}[T_0^{\pm 1}, T_1^{\pm 1}, \ldots, T_r^{\pm 1}]/(T_0^n - 1) \to \mathbb{Z}[T_0^{\pm 1}, T_1^{\pm 1}, \ldots, T_r^{\pm 1}]/(\Phi_n)]$ where the homomorphism is the canonical one. Then $K_\rho = \emptyset$ so $X^\rho = X^T$ and we can verify directly that the element $1 - L$ is invertible in $K_0(T, X)_{(\rho)}$. Indeed, first we see that when $L_0 = 1$ is the trivial bundle, $1 - \lambda$ is invertible since it is not zero in $\mathbb{Z}[M]_{(\rho)}/\rho$ which is just the field $\mathbb{Q}(\mu_n)(x_1, \ldots, x_r)$, the function field of the n-th cyclotomic field with r independent variables. Then, we calculate, in $K_0(X^T, T)_{(\rho)}$:

$$\begin{aligned} \frac{1}{1 - \lambda L_0} &= \frac{1}{1 - \lambda + \lambda - \lambda L_0} = \frac{1}{1 - \lambda} \frac{1}{1 - \lambda/(1 - \lambda) L_0} \\ &= \frac{1}{1 - \lambda} \sum \left(\frac{\lambda}{1 - \lambda} \right)^k (1 - L_0)^k. \end{aligned}$$

Since the rank 0-part of K-theory of a regular scheme is nilpotent, more precisely $(1 - L_0)^k = 0$ for $k > \dim X$, this sum is well-defined [2].

Definition 3.5.1. Fix a cyclic diagonalizable group-scheme T over $\operatorname{Spec} \mathbb{Z}$ and denote by $\mathfrak{R}_T$ the category whose objects are regular T-schemes and morphisms are T-equivariant morphisms of T-schemes. For a T-scheme, denote by $|X| = X^T$ and $V(X, T)$ the virtual category of T-equivariant vector bundles on X denote by $V(X, T)_{(\rho)}$ the localization of $V(X, T)$ at the prime ideal $\rho = \rho_T$ exhibited above and then at $\mathbb{Q}$ (defined in the naive way). Also, denote by α_X the virtual bundle $\lambda_{-1}(N^\vee_{|X|/X})$ in $V(|X|, T)$.

The following lemma gives an explicit construction of the class $\lambda_{-1}(N^\vee_{|X|/X})^{-1}$ appearing in Thomason's result in a special case.

Lemma 3.6 (Inverting λ_{-1})**.** *Let T be a cyclic diagonalizable group-scheme corresponding to a finitely generated abelian group $M = \mathbb{Z}^r \times \mathbb{Z}/n$. Let X be a regular scheme with a trivial T-action and E a vector bundle on X with no trivial eigenvalues for the action of T. There is then a unique way of expressing the inverse bundle $\lambda_{-1}(E)^{-1}$ as a power-series in $V(X, T)_{(\rho)}$ such that it stable under base change and compatible with exact sequences.*

Proof. First notice that there is an equivalence of categories $V(X, T)_{(\rho)} = (V(X)_\mathbb{Q} \otimes \mathbb{Q}[M])_{(\rho)}$ [3]. Let E be such a vector bundle. Since $|X| = X$ it is given by a grading $E = \oplus_{\lambda \in M, \lambda \neq 0} E_\lambda$. Then we propose that for a virtual bundle u_λ on X with pure weight $\lambda \neq 0$

$$\Lambda_{-1}(u_\lambda) = (1 - \lambda)^{\operatorname{rk} u} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\lambda - 1} \right)^k \gamma^k(u - \operatorname{rk} u).$$

[2] In [VV02], results similar to those of [Tho92] were obtained, but with a different choice of localization (cf. [VV02], Section 2). Clearly the results in this text can be reformulated with respect to such localizations.

[3] Here the tensor product $V \otimes M$ for a Picard category and a vector space M refers to the Picard category whose objects are formal finite sums $\sum v_i \otimes m_i$ with v_i an object of V and $m_i \in M$ and $\otimes$ is bilinear. Morphisms are determined by the condition $\operatorname{Hom}_{V \otimes M}((v \otimes m, v' \otimes m')) = \operatorname{Hom}(v, v')_V \otimes M$.

If $k > \dim X + 1$, by Corollary 4.12 of [Eria], there is a completely canonical trivialization $\gamma^k(u) = 0$ in $V(X)_{\mathbb{Q}}$ and by truncating the powerseries at such a k these isomorphisms glue together to an object. By the same corollary, for $u = L$ a line bundle, $\gamma^k(L-1) = 0$ for $k > 1$ so that $\Lambda_{-1}(u_\lambda) = (1-\lambda)(1 + \frac{\lambda}{1-\lambda}(L-1)) = 1 - \lambda L = \lambda_{-1}(\lambda L)$. By Corollary 4.11, [Eria], $\Lambda_{-1}((u+v)_\lambda) = \Lambda_{-1}(u_\lambda)\Lambda_{-1}(v_\lambda)$ so that $u \mapsto \lambda_{-1}(u_\lambda)$ defines an additive functor from $V(X)$ to the Picard category $V(X,T)^*_{(\rho)}$ of invertible elements and for a vector bundle $E = \oplus_{\lambda \in M, \lambda \neq 0} E_\lambda$ with no trivial eigenvalues for the action of T, $\Lambda_{-1}(E) = \otimes_{\lambda \in M, \lambda \neq 0} \Lambda_{-1}(E_\lambda) = \lambda_{-1}(E)$. Now, returning to the case of a virtual bundle $u = \oplus_{\lambda \in M, \lambda \neq 0} u_\lambda$, we put $\lambda_{-1}(u) = \otimes_{\lambda \in M, \lambda \neq 0} \lambda_{-1}(u_\lambda)$. And thus for the same vector bundle we propose the element $\lambda_{-1}(E)^{-1} = \Lambda_{-1}(-E)$.
Next, we show that this constructed class is unique. By the splitting principle we can suppose that E is a line bundle. The scheme X is regular and thus has an ample family of line bundles (cf. [GBI71], II 2.2.4). The argument of [Wei89], Proposition 4.4 then provides us with a T-equivariant torsor $\operatorname{Spec} R \to X$ under a T-equivariant vector bundle, which can be chosen with trivial T-action. Then $\operatorname{Spec} R$ is regular and $V(T,X)_{(\rho)} \to V(T, \operatorname{Spec} R)_{(\rho)}$ is an equivalence of categories so we can suppose X is affine regular. But then the trivial bundle $\mathcal{O}_X$ is equivariantly ample and choosing a surjection $\mathcal{O}^n \twoheadrightarrow L^\vee$ we obtain that $L = i^*(\mathcal{O}(1))$ for a section $i : X \to \mathbb{P}^n_X$ to the natural projection $p : \mathbb{P}^n_X \to X$. Then $Ri_*(\Lambda_{-1}(L_\lambda)) = Ri_* Li^* \Lambda_{-1}(L_\lambda) = Ri_* \mathcal{O}_X \otimes \Lambda_{-1}(\mathcal{O}(1)_\lambda)$ and Ri_* is faithful by virtue of it having a right inverse Rp_*. But $\Lambda_{-1}(\mathcal{O}(1)_\lambda)$ is the unique pullback of $\Lambda_{-1}(\mathcal{O}(1)_\lambda)$ on $\mathbb{P}^n_{\mathbb{Z}}$ and in this case $V(\mathbb{P}^n_{\mathbb{Z}}, T)_{(\rho)}$ has no nontrivial morphisms and the objects $\Lambda_{-1}(\pm\mathcal{O}(1)_\lambda)$ are uniquely determined. □

Let X be a regular T-scheme and denote by $X^T = |X|$ and suppose that $|X| \to X$ is a closed regular immersion. Assume in addition that the square

$$\begin{array}{ccc} |X| & \xrightarrow{i_X} & X \\ \downarrow f' & & \downarrow f \\ |Y| & \xrightarrow{i_Y} & Y \end{array}$$

is Cartesian. First, also suppose that f is a closed regular immersion. We obtain a surjection $N^\vee_{X/Y} \twoheadrightarrow N^\vee_{|X|/|Y|}$ and the kernel is the excess bundle E. By [Ful98], Example 6.3.2, we also have a surjection $N^\vee_{|Y|/Y} \twoheadrightarrow N^\vee_{|X|/X}$ whose kernel is also E. Hence we obtain the formula

$$\lambda_{-1}(N^\vee_{|Y|/Y}) = \lambda_{-1} N^\vee_{|X|/X} \otimes \lambda_{-1}(E)$$

and since $N^\vee_{|Y|/Y}$ and $N^\vee_{|X|/X}$ have no non-trivial eigenvalues for the action of T so that

$$\lambda_{-1}(E) = (\lambda_{-1} N^\vee_{|X|/X})^{-1} \otimes \lambda_{-1}(N^\vee_{|Y|/Y})$$

where the $(\lambda_{-1} N^\vee_{|X|/X})^{-1}$ is defined as above. Via the projection formula we immediately see that the Lefschetz-formula above takes the form of an excess intersection-formula, valid without any localization. Note that since X is regular it has the resolution-property, and by [Tho87b] it also has the T-equivariant resolution property so the excess-formula of [Erib] can be applied to stacks of the form $[X/T]$ and it is clearly valid after localization.

In the rest of this section we put together the already constructed isomorphisms to obtain a functorial Lefschetz-formula. Fix T a cyclic diagonalizable group of finite type (i.e. of the form $D_{\operatorname{Spec}\mathbb{Z}}(\mathbb{Z}^r \oplus \mathbb{Z}/n)$. Let X be a T-equivariant regular scheme, and denote by $|X|$ the fixed point set $i_X : |X| \to X$ of the action of T, and write α_X for the class $\lambda_{-1}(N_{|X|/X})$ in $V(X,T)_{(\rho)}$. Denote by $L_X : V(X,T)_{(\rho)} \to V(|X|,T)_{(\rho)}$ the functor $x \mapsto \alpha_X^{-1} \otimes Li_X^* x$ where α_X^{-1} is the class constructed above. Then

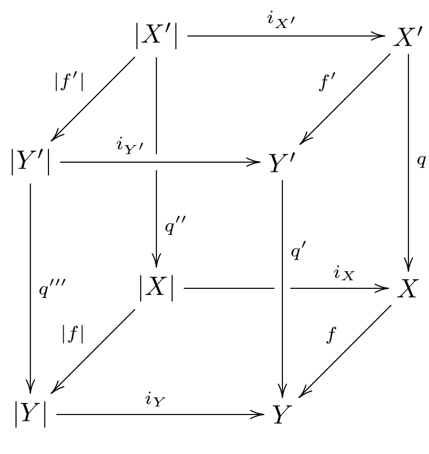
Lemma 3.7. *Let X be a regular T-scheme for a cyclic diagonalizable group. Then there are natural equivalences of functors $L_X Ri_* = \mathrm{id}$ and $Ri_* L_X = \mathrm{id}$. Moreover, for $q : X' \to X$ with induced morphism $|q| : |X'| \to |X|$ there is a natural isomorphism $L_{X'} Lq^* = \alpha_{X'/X} L_X |q|^*$ for $\alpha_{X'/X} = \lambda_{-1}(\ker[N^\vee_{|X|/X} \twoheadrightarrow N^\vee_{|X'|/X'}])$.*

Proof. By [Erib], there is a self-intersection formula $Li_X^* Ri_{X,*} = \alpha_X$ and thus naturally $\alpha_X^{-1} Li_X^* Ri_{X,*} = \mathrm{id}$. By Theorem 3.5, Ri_* induces a bijection on automorphism-groups and surjection on objects and is thus an equivalence of categories and to exhibit a natural isomorphism $Ri_{X,*} L_X = \mathrm{id}$ it suffices to establish $Ri_{X,*} L_X Ri_{X,*} = Ri_{X,*}$. We can construct such an isomorphism using the isomorphism $Li_X^* Ri_{X,*} = \alpha_X$ already established. Given $q : X' \to X$, then $\alpha_{X'/X}\alpha_X^{-1} = \alpha_{X'}^{-1}$ and we thus define the isomorphism in the second part of the lemma. □

Corollary 3.8. *Given a proper morphism $f : X \to Y$, there is a canonical isomorphism of functors $\Upsilon_f : R|f|_* L_X \to L_Y Rf_*$.*

Proof. Apply the above explicit equivalence of categories to the composition of functors $R|f|_* Ri_{Y,*} = Ri_{X,*} Rf_*$. □

Given a projective morphism $f : X \to Y$ and any morphism $q : Y' \to Y$, both in $\mathfrak{R}_T$, there is the question of how the just established isomorphism transforms under base change. Consider the cube



with commutative squares. Since the cube is not transversal in the sense of the main theorem of [Erib] we cannot directly apply the functorial excess-formula to calculate this. We proceed as follows. For a projective morphism $f : X \to Y$ in $\mathfrak{R}_T$ we can define the cotangent complex which is a two-term complex of equivariant vector bundles canonically determined up to canonical quasi-isomorphism, $L_{X/Y} = L_f$. If f is a closed immersion

$$L_{X/Y} = [N \to 0]$$

for N the conormal bundle and if f is smooth

$$L_{X/Y} = [0 \to \Omega_{X/Y}].$$

For a composition of projective local complete intersection morphism $X \xrightarrow{f} Y \xrightarrow{g} Z$ there is an exact triangle

$$L_{X/Y} \to L_{Y/Z} \to L_{X/Z} \to L_{X/Y}[1]$$

(cf. [GBI71], VIII, 2, the arguments are easily made equivariant). In the above setting, we obtain exact triangles

$$L_{|X|/X} \to L_{X/Y} \to L_{|X|/Y} \to L_{|X|/X}[1]$$

and

$$L_{|X|/|Y|} \to L_{|Y|/Y} \to L_{|X|/Y} \to L_{|X|/|Y|}[1].$$

Define E (resp. E') to be the homology of $[L_{X/Y} \to L_{X'/Y'}]$ (resp. $[L_{|X|/|Y|} \to L_{|X'|/|Y'|}]$. Then E is the excess-bundle of the Cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

whereas E' is represented by a complex whose $\lambda_{-1}(E') = \alpha_{E'}$ can be defined by $\Lambda_{-1}(L_{|X|/|Y|} - L_{|X'|/|Y'|})$ which is seen to be well-defined. Moreover, using excess and the isomorphism $i_{X,*}L_X = \mathrm{id}$ one constructs an isomorphism

$$R|f'|_*(\alpha_{E'} \otimes Lq'''^*) = Lq''^* R|f|_* \tag{3}$$

and by functoriality we have an isomorphism $\alpha_{Y'/Y} \otimes \lambda_{-1}(E) = \alpha_{X'/X} \otimes \lambda_{-1}(E')$. We then have two isomorphisms

$$Lq'''^* L_Y Rf_* = \alpha_{Y'/Y} L_{Y'} Lq'^* Rf_* = \alpha_{Y'/Y} L_{Y'} Rf'_*(\lambda_{-1}(E) \otimes Lq^*) \tag{4}$$

$$\begin{aligned} Lq'''^* R|f|_* L_X &= R|f'|_*(\alpha_{E'} \otimes Lq''^* L_X) \\ &= R|f'|_*(\alpha_{E'} \otimes \alpha_{X'/X} \otimes L_{X'} Lq^*) \\ &= R|f'|_*(\lambda_{-1}(E) \otimes \alpha_{Y'/Y} \otimes L_{X'} Lq^*) \\ &= \alpha_{Y'/Y} R|f'|_* L_{X'}(\lambda_{-1}(E) \otimes Lq^*)) \end{aligned} \tag{5}$$

From the definition it is not difficult to verify that these two isomorphisms are compatible with the isomorphisms $L_Y Rf_* = R|f|_* L_X$ and $L_{Y'} Rf'_* = R|f'|_* L_{X'}$. The theorem is that these properties essentially characterize the Lefschetz-isomorphism:

Theorem 3.9 (Functorial Lefschetz-Riemann-Roch for cyclic diagonalizable groups)**.** *Fix positive integers r and n and let $M = \mathbb{Z}^r \oplus \mathbb{Z}/n$ and let $T = \operatorname{Spec}\mathbb{Z}[M]$ be the associated diagonalizable group. Consider the category $\mathfrak{R}_T^p$ of T-equivariant regular schemes and morphisms given by $f : X \to Y$ a T-equivariant morphism of regular schemes which is equivariantly projective, i.e. factors equivariantly into a projective bundle $X \hookrightarrow \mathbb{P}(E) \to Y$ for a closed immersion $X \hookrightarrow \mathbb{P}(E)$ and an equivariant vectorbundle E on Y. Denote the induced morphism on fixed points $|f| : |X| \to |Y|$. Then there is a family, unique up to unique isomorphism, of functor-isomorphisms for f a morphism in $\mathfrak{R}_T^p$,*

$$\Upsilon_f : R|f|_* L_X \to L_Y Rf_*$$

satisfying the following compatibilities:

(a) Stability under composition: Given $X \stackrel{f}{\to} Y, Y \stackrel{g}{\to} Z$ in $\mathfrak{R}_T^p$, the composition

$$R(|g||f|)_* L_X \stackrel{\Upsilon_f}{\to} R|g|_* L_Y Rf_* \stackrel{\Upsilon_g}{\to} L_Z R(gf)_*$$

is Υ_{gf}.

(b) Stability under base change in $\mathfrak{R}_T$; if $q' : Y' \to Y$ is an equivariant morphism such that $q : X' \to Y'$ is also in $\mathfrak{R}_T$, the isomorphisms (4) and (5) intertangle to give a commutative diagram:

$$\begin{array}{ccc} Lq'''^* L_Y Rf_* & \xrightarrow{(4)} & \alpha_{Y'/Y} L_{Y'} Rf'_*(\lambda_{-1}(E) \otimes Lq^*) \\ \downarrow \Upsilon_f & & \downarrow \Upsilon_{f'} \\ Lq'''^* R|f|_* L_X & \xrightarrow{(5)} & \alpha_{Y'/Y} R|f'|_* L_{X'}(\lambda_{-1}(E) \otimes Lq^*) \end{array}$$

(c) Suppose Z is also in $\mathfrak{R}_T$, and $h : Z \to Y$ is closed regular immersion T-equivariant immersion and $f : X \to Y$ is a morphism $\mathfrak{R}_T$ whose image is disjoint with that of Z. Then both sides of Υ_f are canonically trivialized and we require that Υ respects these trivializations.

(d) The isomorphism is compatible with the projection-formula, i.e. the diagram

$$\begin{array}{ccc} R|f|_* L_X(u \otimes Lf^* v) & \longrightarrow & L_Y Rf_*(u \otimes Lf^* v) \\ \downarrow & & \downarrow \\ R|f|_* L_X u \otimes L|f|^* Li_Y^* v & & L_Y(Rf_* u \otimes v) \\ \downarrow & & \downarrow \\ R|f|_* L_X u \otimes Li_Y^* v & \longrightarrow & L_Y Rf_* u \otimes Li_Y^* v \end{array}$$

commutes.

Remark 3.9.1. The proof proceeds as in the case of the functorial excess-formula and also follows the corresponding proof for Grothendieck-Riemann-Roch in the unpublished manuscript [Fra], which uses a reduction to the arithmetic case.

This in turn is an adaption of the usual proof of [GBI71] to the functorial situation. Clearly the isomorphism exists in greater generality by Corollary 3.8, the stronger statement is the uniqueness-property. It is possible to establish a similar isomorphism for more general diagonalizable groups but one has to introduce a normalization-condition analogous to that of the rough excess-isomorphism in [Erib].

Proof. The isomorphism has already been constructed and the properties follow either from construction or from the discussion. We review its construction in the case of a closed immersion and a projective bundle projection to possibly clarify the situation.
Suppose $i : X \to Y$ is a closed immersion of regular T-schemes. Thus we can apply the excess-formula to algebraic stacks of the form $[X/T]$ which gives an isomorphism, by the arguments preceding the theorem,

$$\alpha_Y^{-1} \otimes Li_Y^* Rf_* = R|f|_* \left(\alpha_X^{-1} \otimes Li_X^*\right).$$

It moreover satisfies the given conditions by virtue of them being satisfied for the excess-isomorphism. This also gives a description of Lefschetz for closed immersions via a rough excess-argument.
Now, suppose $f : \mathbb{P}(N) \to Y$ is a projective bundle projection for N a T-equivariant vector bundle on Y, whose restriction $N|_{|Y|}$ is diagonalized to $\oplus_{\lambda \in M} N_\lambda$. Then $|\mathbb{P}_{|Y|}(\oplus N_\lambda)| = \coprod_{\lambda \in M} \mathbb{P}_{|Y|}(N_\lambda)$ (cf. [KR01], Proposition 2.9). Thus we are given a diagram

$$\begin{array}{ccccc} \coprod_{\lambda \in M} \mathbb{P}_{|Y|}(N_\lambda) & \longrightarrow & \mathbb{P}_{|Y|}(N|_{|Y|}) & \longrightarrow & \mathbb{P}_Y(N) \\ & \searrow & \downarrow & & \downarrow f \\ & & |Y| & \longrightarrow & Y \end{array}$$

with Cartesian square. We treat first the left triangle and suppose that $Y = |Y|$. Denote by $i_\lambda : \mathbb{P}_Y(N_\lambda) \to \mathbb{P}_Y(N)$ the closed immersion, $i = \coprod i_\lambda : \coprod \mathbb{P}_Y(N_\lambda) \to \mathbb{P}_Y(N)$, $|f|_\lambda = f i_\lambda$ and $|f| = fi$. For any virtual bundle x, we need to construct a functorial isomorphism

$$Rf_*(x) = \sum R|f|_{\lambda,*}(\lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda{}^* x).$$

We can assume x is of the form $\sum_{k=0}^{n-1} Lf^* a_k \otimes \mathcal{O}(-k)$ for $n = \operatorname{rk} N$. The right hand side is thus isomorphic to, via the projection formula,

$$\sum_{k,\lambda} R|f|_{\lambda,*}(\lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* \mathcal{O}(-k)) \otimes a_k). \tag{6}$$

By the excess isomorphism in [Erib], there is a canonical isomorphism of functors $Li_\lambda^* Ri_{\lambda,*}(-) = \lambda_{-1} N_\lambda \otimes (-)$ and thus $\lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* Ri_{\lambda,*} = \mathrm{id}$. Moreover, it is known that $Ri_{\lambda,*}$ is an equivalence of categories. Thus any x is of the form $Ri_{\lambda,*} y$ and we deduce the isomorphism

$$Ri_{\lambda,*} \lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* x = \lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* Ri_{\lambda,*} y = Ri_{\lambda,*} y = x.$$

Applying this isomorphism to (6) we obtain $\sum_k Rf_*(\mathcal{O}(-k)) \otimes a_k$. For $k > 0$, $Rf_*(\mathcal{O}(-k)) = 0$ and for $k = 0$, $Rf_*(\mathcal{O}) = \mathcal{O}$. In general we compose this with the excess-isomorphism for the Cartesian square. For a projective bundle-projection it is also true that $E = E' = 0$ for base changes. That this respects composition is done exactly as in the case of the excess-isomorphism.

We are left to show that the morphism is unique. We can clearly suppose Y is connected and treat the cases of a closed immersion and projective bundle projections separately. In the case of a closed immersion it is immediate to verify that all the schemes that arise in the case of a deformation to the normal cone are regular T-schemes so thus we stay in the correct category. The essential point is the trivialization condition (c) to exclude unwanted factors. Then a deformation to the normal cone-argument analogous to that of argument related to the excess-formula shows that we are reduced to the case of an embedding $i : X \to \mathbb{P}(N)$ for some equivariant vector bundle N of rank n defined by an inclusion $L \subset N$ for some line bundle N. Let $p : \mathbb{P}_X(N) \to X$ be the projection. Then $|\mathbb{P}_X(N)| = \coprod_{\lambda \in M} \mathbb{P}_{|X|}(N|_{|X|,\lambda})$ (loc.cit.). For a virtual bundle u the isomorphism $u = Li^*Lp^*u$ and compatibility with the projection formula shows that we are reduced to showing that $\Upsilon_i(\mathcal{O})$ is uniquely determined.
We apply the base change-property to the Cartesian diagram

$$\begin{array}{ccc} |X| & \longrightarrow & \coprod_{\lambda \in M} \mathbb{P}_{|X|}(N|_{|X|,\lambda}) \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_{P_X(N)}} \\ X & \longrightarrow & P_X(N) \end{array}$$

and see that $\Upsilon_i(\mathcal{O})$ is determined by the functor $\Upsilon_{|i|}$ and hence by $\Upsilon_{|i|}(\mathcal{O})$. Since $|\mathbb{P}_X(N)| = \coprod_{\lambda \in M} \mathbb{P}_{|X|}(N|_{|X|,\lambda})$ and we can assume Y to be connected we can assume furthermore that $Y = \mathbb{P}_{|X|}(N_\lambda)$ for some fixed $\lambda \in M$ and vector bundle $N = N_\lambda$ on $|X| = X$ with single grading λ. We need to verify that the Lefschetz-isomorphism $Rf_*\mathcal{O} \to Rf_*\mathcal{O}$ in this case necessarily is the identity. By 3.1 there exists a torsor $t : \operatorname{Spec} R \to X$ under some vector bundle on X which we endow with the trivial action. Then by the affine bundle theorem of [Tho87a] there are equivalences of categories $Lt^* : V(X,T) \to V(\operatorname{Spec} R, T)$ and thus $V(X,T)_{(\rho)} \to V(\operatorname{Spec} R, T)_{(\rho)}$ as well. Similarly there is an equivalence $V(\mathbb{P}_X(N), T)_{(\rho)} \to V(\mathbb{P}_{\operatorname{Spec} R}(t^*N), T)_{(\rho)}$ and by the base change-property we can assume that X is in fact affine. Then we can choose a surjection $\mathcal{O}^r \to N^\vee$ so that we have a flag $L \subset N \subset \mathcal{O}^r$ which is concentrated on the single grading λ. Consider the Grassmannian $\tau : G = \mathrm{Gr}_{1,n,r} \to X$ of flags $L' \subset N' \subset \mathcal{O}^r$ on X with L' (resp. N') have rank 1 (resp. n). Then G is regular, has trivial T-action and the flag $L \subset N \subset \mathcal{O}^r$ defines a section $j : X \to G$. If $\mathcal{L} \subset \mathcal{N} \subset \mathcal{O}^r$ is the universal flag on G, $p^*j^*\mathcal{L} \subset p^*j^*\mathcal{N}$ similarly define a section $j' : \mathbb{P}(N) \to \mathbb{P}(\mathcal{N})$ so that we have a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{P}(N) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ G & \xrightarrow{i'} & \mathbb{P}(\mathcal{N}) \end{array} .$$

Then compatibility with the projection-formula shows that

$$Rj'_*\Upsilon_i = Rj'_*(\mathcal{O}_X) \otimes \Upsilon_{i'}.$$

However, Rj'_* is faithful since $R\tau_* Rj'_* = \mathrm{id}$ so Υ_i is determined by $\Upsilon_{i'}$. The varieties G and $\mathbb{P}(\mathcal{N})$ are equivariantly defined over $\operatorname{Spec}\mathbb{Z}$. This shows that $\Upsilon_{i'}$ is obtained by base change from $X \to \operatorname{Spec}\mathbb{Z}$ so we can assume that $X = \mathbb{Z}$. In this case uniqueness is tautological since G is cellular so $K_1(G,T)_{(\rho),\mathbb{Q}}$ is a free $K_1(\mathbb{Z},T)_{(\rho),\mathbb{Q}}$-module and (cf. beginning of this section)

$$K_1(\mathbb{Z},T)_{(\rho),\mathbb{Q}} = K_1(\mathbb{Z}) \otimes_{\mathbb{Q}} \mathbb{Q}[M]_{(\rho)} = 0$$

since $K_1(\mathbb{Z}) = \pm 1$.

Now, consider the case of a projective bundle projection $p : \mathbb{P}_Y(N) \to Y$ for some T-equivariant vector bundle N on Y. Arguing as above, we reduce to the case of $Y = |Y|$ being an affine scheme and the case of a virtual bundle of the form $\mathcal{O}(-i)$ on $\mathbb{P}_Y(N)$. If $N = \oplus_{\lambda \in M} N_\lambda$ with N_λ of rank n_λ, choose a locally split injection $N \subset \mathcal{O}^r$ which restricts to $N_\lambda \subset \mathcal{O}_\lambda^{r_\lambda}$ on each grading and consider the Grassmannian $G = \mathrm{Gr}_{n,r}$ of flags $N' \subset \oplus_{\lambda \in M} \mathcal{O}_\lambda^{r_\lambda}$ with N' of rank n. In a way similar to the case of closed immersions we reduce to the case of the diagram

$$\begin{array}{ccc} \coprod_{\lambda \in M} \mathrm{Gr}_{n_\lambda, r_\lambda}(\mathcal{N}_\lambda) & \longrightarrow & \mathbb{P}_G(\mathcal{N}) \\ \downarrow & & \downarrow \\ \coprod_{\lambda \in M} \mathrm{Gr}_{n_\lambda, r_\lambda} & \longrightarrow & \mathrm{Gr}_{n,r} \end{array}$$

where $\mathcal{N}$ is the universal rank n subbundle of $\oplus_{\lambda \in M} \mathcal{O}_\lambda^{r_\lambda}$ on G. Again this diagram is equivariantly defined over $\operatorname{Spec}\mathbb{Z}$ and we conclude as before. □

3.2 An Adams-Deligne-Riemann-Roch formula for regular schemes

In this section we state and prove a functorial Adams-Riemann-Roch formula. We will continuously work in the category $V(X)_{\mathbb{Q}}$ of virtual vector bundles on a scheme X, which comes equipped with Adams operations and various other operations (cf. Proposition 3.8 in [Eria]). This coincides with the cohomological virtual category $W(X)$, and whenever X is regular it also comes equipped with various additional operations considered in section 4 of [Eria] which we shall use freely. Also recall that a regular scheme is a separated, Noetherian regular scheme.

We recall to the reader that one formulation of the Adams-Riemann-Roch formula is as follows (cf. [FL85], V, Theorem 7.6). Suppose that $f : X \to Y$ is a projective local complete intersection morphism of schemes and that Y has an ample family of line bundles so that any coherent sheaf is the quotient of a coherent locally free sheaf. Also, define Ω_f to be the class of the cotangent-bundle of f, and $\theta_{k,f} = \theta_k(\Omega_f)$ where θ_k is the unique multiplicative characteristic class in $K_0(X)$ such that for a line bundle L, $\theta_k(L) = 1 + L + \ldots + L^{k-1}$. Then for any $k \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\theta_{k,f}^{-1} \otimes \Psi^k} & K_0(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\ K_0(Y) & \xrightarrow{\Psi^k} & K_0(Y)_{\mathbb{Q}} \end{array} .$$

To formulate the functorial version of the Adams-Riemann-Roch formula, recall the following Lemma which is a corollary of Corollary 3.3 while noticing that for an regular scheme, $W(X) = V(X)_{\mathbb{Q}}$:

Lemma 3.10. *There is a unique family of functors, determined up to unique isomorphism, on the category of regular schemes, stable under base change, such that for a regular scheme X*

$$\theta_k : V(X) \to V(X)_{\mathbb{Q}}$$

such that θ_k is a determinant functor $\mathbf{P}(X) \to V(X)_{\mathbb{Q}}^*$ and for a line bundle L on X there exists an isomorphism

$$\theta_k(L) = 1 + L + \ldots + L^{k-1} = 1 + L + \ldots + L^{k-1} = \sum_{j=0}^{k-1} a_{j,k}(L-1)^j$$

where $a_{j,k} = \sum_{i=j}^{k-1} \binom{i}{j}$.

Now, given a projective morphism $f : X \to Y$ of regular schemes factoring as $X \overset{i}{\hookrightarrow} P \overset{p}{\to} Y$ for a closed immersion i and smooth morphism p, define $\theta_k^{-1}(\Omega_f)_{i,p}$ to be the virtual bundle $\theta_k(N_i^\vee - i^*\Omega_{P/Y})$. We analyze its properties before stating the functorial Adams-Riemann-Roch-theorem. The usual proof in [GBI71], VIII, Proposition 2.2, shows that the virtual bundle $N_i^\vee - i^*\Omega_{P/Y}$ glues together to a virtual bundle Ω_f which is independent of factorization . We define $\theta_{k,f}^{-1} := \theta_k^{-1}(\Omega_f)$ which is an object dependent only on f determined up to unique isomorphism. Moreover, suppose $q : Y' \to Y$ is any morphism such that $f' : X' = Y' \times_Y X \to Y'$ is also a projective morphism of regular schemes. Then it is clear from the definition that there is a canonical isomorphism $Lq^*\theta_{k,f}^{-1} = \theta_{k,f'}^{-1} \otimes \theta_k E$ with E the excess bundle of the diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}.$$

One deduces from the splitting principle an isomorphism

$$\theta_k E \otimes \lambda_{-1}(E) = \Psi^k(\lambda_{-1}(E)) \tag{7}$$

and thus an isomorphism

$$Lq^*\theta_{k,f}^{-1} \otimes \lambda_{-1}(E) = \theta_{k,f'}^{-1} \otimes \Psi^k(\lambda_{-1}(E)). \tag{8}$$

Finally, for a composition of projective morphisms of regular schemes, $X \overset{f}{\to} Y \overset{g}{\to} Z$, there is also a canonical isomorphism

$$\theta_{k,gf}^{-1} = \theta_{k,g}^{-1} \otimes Lf^*\theta_{k,g}^{-1} \tag{9}$$

(cf. [GBI71], VIII, Proposition 2.6).

Theorem 3.11 (Functorial Adams-Riemann-Roch)**.** *Suppose $f : X \to Y$ is a projective morphism of regular schemes (automatically a local complete intersection), and $k \geq 1$. There is a unique family of functorial isomorphisms*

$$\psi_{k,f} : \Psi^k Rf_* \to Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k) \tag{10}$$

characterized further by the following properties:

(a) Stability under composition of projective local complete intersection morphisms: That is, for a composition of projective morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

the isomorphism

$$\begin{array}{rcl} R(gf)_*(\theta_{k,gf}^{-1} \otimes \Psi^k(u)) & \overset{(9)}{=} & R(gf)_*(\theta_{k,g}^{-1} \otimes Lf^*\theta_{k,g}^{-1} \otimes \Psi^k(u)) \\ & = & Rg_*(\theta_{k,g}^{-1} \otimes Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u))) \\ & \overset{\psi_{k,f}}{=} & Rg_*(\theta_{k,g}^{-1}\Psi^k Rf_*(u) \\ & \overset{\psi_{k,g}}{=} & \Psi^k Rg_* Rf_*(u) \\ & = & \Psi^k R(gf)_* u \end{array}$$

is $\psi_{k,gf}$.

(b) Stability under the projection-formula: That is, the diagram

$$\begin{array}{ccc} \Psi^k Rf_*(u \otimes Lf^*v) & \longrightarrow & Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u \otimes Lf^*v)) \\ \downarrow & & \downarrow \\ \Psi^k Rf_*(u) \otimes \Psi^k(v) & \longrightarrow & Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u)) \otimes \Psi^k(v) \end{array}$$

commutes where the horizontal isomorphisms are given by $\Psi_{k,f}$ and the vertical isomorphisms are given by the projection-formula.

(c) Compatibility with base change and excess: Suppose $q : Y' \to Y$ is a morphism such that the induced morphism $f' : X' = Y' \times_X Y \to X$ is also a projective morphism of regular schemes, and denote by $q' : X' \to X$ the morphism obtained by base change, and denote by E the associated excess bundle. Then the diagram

$$\begin{array}{ccc} Lq^*\Psi^k Rf_*(u) & \xrightarrow{Lq^*\psi_{k,f}} & Lq^* Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u)) \\ \downarrow{\scriptstyle Excess,[Erib]} & & \downarrow{\scriptstyle Excess,[Erib]} \\ \Psi^k Rf'_*(\lambda_{-1}(E) \otimes Lq'^* u) & & Rf'_*(\lambda_{-1}(E) \otimes Lq'^*(\theta_{k,f}^{-1} \otimes \Psi^k(u)) \\ \downarrow{\scriptstyle \psi_{k,f'}} & & \downarrow \\ Rf'_*(\theta_{f',k}^{-1} \otimes \Psi^k(\lambda_{-1}(E) \otimes Lq'^* u)) & & Rf'_*(\lambda_{-1}(E) \otimes Lq'^*\theta_{k,f}^{-1} \otimes \Psi^k Lq'^* u) \\ \downarrow & \nearrow{\scriptstyle (8)} & \\ Rf'_*(\theta_{f',k}^{-1} \otimes \Psi^k(\lambda_{-1}(E)) \otimes \Psi^k Lq'^* u) & & \end{array}$$

commutes, where the diagonal morphism is deduced from the isomorphism (8). In particular, for $k = 1$ this reduces to the excess-isomorphism of [Erib], and if q is flat the isomorphism strictly commutes with pullback.

(d) Suppose we are given a closed immersion $h : Z \to Y$ whose image in Y doesn't intersect that of X. Then both $Rh_*(\mathcal{O}_Z) \otimes \Psi^k Rf_*(u)$ and $Rh_*(\mathcal{O}_Z) \otimes Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u))$ are canonically trivialized. We demand that the isomorphism $\psi_{k,f}$ interchanges these trivializations. We don't require Z to be regular.

Proof. The proof proceeds as in the case of the functorial excess-formula and also closely follows the corresponding proof for Grothendieck-Riemann-Roch in the unpublished article [Fra], which uses a reduction to the arithmetic case. We indicate the necessary changes from the case of the excess-isomorphism. Suppose $i : X \to Y$ is a regular closed immersion of regular schemes. Given a Koszul resolution built out of $s : N^\vee \to \mathcal{O}_Y$ of $\mathcal{O}_X$, one first defines a rough Adams-Riemann-Roch-isomorphism for closed immersions as follows: Let L be a line bundle on X and $\mathcal{L}$ an extension of X to a line bundle on Y. As in (7) we have a natural isomorphism $\theta_k(N^\vee) \otimes \lambda_{-1}(N^\vee) \simeq \Psi^k(\lambda_{-1}(N^\vee))$. Then we have an isomorphism

$$\Psi^k i_*(L) \simeq \Psi^k(\lambda_{-1}N^\vee \otimes \mathcal{L}) \simeq \theta_k(N^\vee) \otimes \lambda_{-1}N^\vee \otimes \mathcal{L}^{\otimes k}.$$

This is, by another projection-formula-argument, isomorphic to $i_*(\theta_{i,k}(N_i^\vee) \otimes \Psi^k L)$.
One needs to verify that the deformation to the normal cone-argument can be used to reduce to this case. The same proof goes through with the remark that if $X \to Y$ is a closed regular immersion of regular schemes, then the blow-up of Y in X is a regular scheme with regular exceptional divisor. Indeed, the exceptional divisor is simply $\mathbb{P}_X(N)$ so thus regular. It is a regular Cartier divisor in $Bl_X Y$ and so forces $Bl_X Y$ to be regular (see [DG67], 19.1.1). Thus we stay in the correct category of regular schemes while deforming.
We are left to show that the morphism is unique. A deformation to the normal cone-argument (which is justified by the above reasoning) shows that we are reduced to the case of an embedding $i : X \to \mathbb{P}(N)$ for some vector bundle N of rank d defined by an inclusion $L \subset N$ for some line bundle L. Let $p : \mathbb{P}(N) \to X$ be the projection. By stability under the projection formula we have a commutative diagram

$$\begin{array}{ccccc} Ri_*(\Psi^k(E) \otimes \theta_{i,k}) & \longrightarrow & Ri_*(\Psi^k(Li^* Lp^* E) \otimes \theta_{i,k}) & \longrightarrow & \Psi^k(Lp^* E) \otimes Ri_*(\theta_{i,k}) \\ \downarrow & & \downarrow & & \downarrow \\ \Psi^k(Ri_* E) & \longrightarrow & \Psi^k(Ri_* Li^* Lp^* E) & \longrightarrow & \Psi^k(Lp^* E) \otimes \Psi^k(Ri_* \mathcal{O}_X) \end{array} .$$

and thus we can assume that $E = \mathcal{O}_X$. By Theorem 3.1, there exists an affine torsor $T \to Y$ under some vector bundle E on Y, and base change to this variety is an equivalence of virtual categories so we can assume X and Y are affine. We loose the assumption that $Y = \mathbb{P}(N)$ but gain that X is affine. By another deformation to the normal cone argument we can again assume $Y = \mathbb{P}(N)$. Since X is affine $N^\vee$ is generated by global sections $\mathcal{O}_X^n \to N^\vee$ for some n and we have an injection $N \subset \mathcal{O}_X^n$. Consider the flag variety $G = \mathrm{Gr}_{\mathrm{n,d,1,X}}$ of locally split flags $L' \subset N' \subset \mathcal{O}^d$, with L' and N' of rank 1 and d respectively. The flag $L \subset N \subset \mathcal{O}^d$ defines a section $s : X \to G$ to $r : G \to X$. If $\mathcal{L} \to \mathcal{N} \to \mathcal{O}^d$ denotes the universal flag on G we have the following commutative diagram

$$\begin{array}{ccc}
G = \mathrm{Gr}_{n,d,1} & \xrightarrow{i'} & \mathbb{P}_G(\mathcal{N}) \\
s \uparrow \downarrow r & & s' \uparrow \downarrow r' \\
X & \xrightarrow{i} & \mathbb{P}(N) = \mathbb{P}(s^*\mathcal{N})
\end{array}$$

with the section s' is defined by the flag $r^*L \subset r^*N$ on G. Then $Rr_*\mathcal{O}_G = \mathcal{O}_X$ and $Rr'_*\mathcal{O}_{\mathbb{P}_G(\mathcal{N})} = \mathcal{O}_{\mathbb{P}(N)}$. Then there is no excess for the Cartesian diagram of closed immersions and so $Ls^*\theta_{k,i'}^{-1} = \theta_{k,i}^{-1}$ and $Ls^*Ri_* = Ls'^*Ri'_*$. We have the commutative diagrams

$$\begin{array}{ccc}
Rs'_*(Ri_*(\theta_{k,f}^{-1})) & \xrightarrow{Rs'_*(\psi_{k,i})} & Rs'_*(\Psi^k Ri_* 1) \\
\downarrow & & \downarrow \\
Rs'_* Ls'^* Ri'_*(\theta_{k,i'}^{-1}) & \longrightarrow & Rs'_* Ls'^*(\Psi^k Ri'_* 1) \\
\downarrow & & \downarrow \\
Ri'_*(\theta_{k,i'}^{-1}) \otimes Rs'_*(1) & \xrightarrow{\psi_{k,i'} \otimes Rs'_*(1)} & \Psi^k Ri'_*(1) \otimes Rs'_*(1)
\end{array}$$

where the upper square is the base change-property and the lower square is the natural transformation associated to the projection-formula. Since $Rr'_*Rs'_* = \mathrm{id}$, Rs'_* is faithful and thus $\psi_{k,i}$ is determined by $\psi_{k,i'} \otimes Rs'_*(1)$ which is determined by $\psi_{k,i'}$. It has a morphism to $\operatorname{Spec}\mathbb{Z}$ and we may assume by base change that X is $\operatorname{Spec}\mathbb{Z}$. In this case the virtual category under consideration doesn't have any non-trivial automorphisms, since $K_1(\mathbb{Z}) \pm 1$ and we tensor with $\mathbb{Q}$. Thus the isomorphism in question is uniquely rigidified in the case of a closed immersion.

Suppose now that $f : \mathbb{P}(N) = X \to Y$ is a projective bundle projection for some vector bundle N on Y of rank d. By the projective bundle formula for K-theory in [Eria] we can assume $u = \sum_{i=0}^{d-1} Lf^*u_i \otimes \mathcal{O}(-i)$ for virtual bundles u_i on Y. By the projection-formula and the multiplicative property of the Adams operations (cf. Corollary 3.9, [Eria]) we only need to define the isomorphism for bundles of the type $u = \mathcal{O}(-i)$. We calculate both sides. First, $\Psi^k Rf_*(\mathcal{O}(-i)) = 0$ if $i > 0$ and isomorphic to 1 if $i = 0$. On the other hand, there is a universal exact sequence

$$0 \to \Omega_{X/Y} \to f^*N^\vee \otimes \mathcal{O}(1) \to \mathcal{O}_X \to 0$$

on X. Then we obtain the isomorphism $\theta_k(f^*N^\vee \otimes \mathcal{O}(1)) = \theta_k(\mathcal{O}_X) \otimes \theta_k(\Omega_{X/Y}) = k\theta_k(\Omega_{X/Y})$. Thus we want to construct isomorphisms

$$Rf_*(\theta_k(f^*N^\vee \otimes \mathcal{O}(1)) \otimes \mathcal{O}(-ik)) = \begin{array}{ll} 1/k & \text{if } i = 0 \\ 0 & \text{if } i = 1, \ldots, d-1 \end{array}.$$

We need the following lemma:

Lemma 3.12 ([FL85], II, Lemma 3.3). *Let R be a commutative ring in which k is invertible. For $a, b \in R$, define $a \oplus b = (1+a)(1+b) - 1$ and for an integer j define $[j]a = a \oplus a \oplus \ldots \oplus a$ taken j times. Let $a_1, \ldots, a_d, Z$ be independent variables and define*

$$R[[a_1, \ldots, a_d, Z]] \ni F_{n,k}(Z) = (1+Z)^{nk} \prod_{j=1}^{d} \frac{Z \oplus a_j}{[k](Z \oplus a_j)}.$$

There exists unique elements $b_0^{i,k}, \ldots, b_d^{i,k} \in R[[s_1, \ldots, s_d]]$ (where s_j are the elementary symmetric functions in the a_j) such that

$$F_{i,k}(Z) \equiv b_0^{i,k} + b_1^{i,k} Z + \ldots + b_{d-1}^{i,k} Z^{d-1} \ mod \prod_{j=1}^{d} Z \oplus a_j$$

and we have

$$\sum_{v=0}^{d-1} (-1)^v b_v^{i,k} = \begin{matrix} 1/k & \text{if } n = 0 \\ 0 & \text{if } i = 1, \ldots, d-1 \end{matrix} .$$

In particular, this result holds as an identity on $R = K_0(\mathbb{P}(N)_X)_{\mathbb{Q}}$ whenever inserting $Z = (\mathcal{O}(-1) - 1)$ and $s_j = \gamma^j(N - d)$ and by ridigity this lifts to the virtual category. Also, by rigidity there is a canonical isomorphism $F_{i,k}(Z)$ and $\theta_k(f^* N^\vee \otimes \mathcal{O}(1)) \otimes \mathcal{O}(-ik)$ and we define the Adams-Riemann-Roch-isomorphism as the isomorphism interchanging the two calculations we have done above. By functoriality of the rigidity-construction this isomorphism clearly satisfies all the proposed properties, except possibly the one concerning compatibility of composition of morphisms.

We now show uniqueness for $f : X = \mathbb{P}(N) \to Y$ a projective bundle projection for some vector bundle N on Y of rank d. By Theorem the projection formula for K-theory in [Eria] we can assume $u = \sum_{i=0}^{d-1} f^* u_i \otimes \mathcal{O}(-i)$ for virtual bundles u_i on Y. By additivity and compatibility with the projection formula we can assume that $u = \mathcal{O}(-i)$ for some $i, 0 \leq -i < d$. Again as in the case of a closed immersion we can assume that Y is affine and that we have an injection $N \subset \mathcal{O}_X^n$ and consider the Grassmannian $\mathrm{Gr}_{d,n,Y}$ of locally split flags $N' \subset \mathcal{O}^n$ with N' a rank d vector bundle with universal flag $\mathcal{N} \subset \mathcal{O}^n$. Again, arguing as above one reduces to the case of $\mathbb{P}(\mathcal{N}) \to \mathrm{Gr}_{d,n,Y}$ and then $Y = \operatorname{Spec} \mathbb{Z}$ where the statement is tautological.

Now, given a factorization $f : X \xrightarrow{i} \mathbb{P}(N) \xrightarrow{p} Y$ one defines $\psi_{k,f,i,p}$ via $\psi_{k,p}\psi_{k,i}$ which is defined by requiring that condition 1. holds. One needs to go over the same steps as in the case of the excess formula to establish that it is independent of factorization and satisfies the conditions of the theorem. They are proved similarly, with one exception:

Lemma 3.13. *Suppose we are given a Cartesian square*

$$\begin{array}{ccc} \mathbb{P}_X(N_X) & \xrightarrow{i'} & \mathbb{P}_Y(N) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{i} & Y \end{array}$$

of morphisms in of regular schemes with i, i' closed immersions, N a vector bundle on Y, p the natural projection and $N_X = N|_X$. Then $\psi_{k,p}\psi_{k,i'} = \psi_{k,i}\psi_{k,p'}$.

Proof. This can be done by direct calculation, but we show how to reduce to the arithmetic situation. All functors are compatible with the projection formula so we only need to show that $\psi_{k,p}\psi_{k,i'}(u) = \psi_{k,i}\psi_{k,p'}(u)$ for virtual bundles $u = \mathcal{O}(-i), i = 0, 1, \ldots, d-1$ with d being the rank of N. By Theorem 3.1 and the properties already established we may assume that Y, and thus X, is affine. Also, a deformation to the normal cone-argument shows that we can suppose $Y = \mathbb{P}_X(M)$ for a vector bundle M on X and a Grassmannian argument as in the proof of the main result of section 3.1 shows we can reduce to the case of a diagram of Grassmannians and reduces to the case of Grassmannians over $\operatorname{Spec}\mathbb{Z}$ in which case the isomorphisms are rigidified. □

The rest of the proof is just like in the proof of the excess formula. We conclude the proof of the Adams-Riemann-Roch theorem. □

Recall that the relative dimension of a local complete intersection morphism $f : X \to Y$ is the rank of the virtual bundle Ω_f defined at the beginning of this section. This is locally constant on X. Also, denote by $F^iV(X) = F^iW(X)$ and $V(X)^{(i)} = W(X)^{(i)}$ where $F^iW(X)$ is part of the filtration of the cohomological virtual category exhibited in section 4 of [Eria]. This is motivated by the equivalence of categories $V(X)_{\mathbb{Q}} = W(X)$.

Corollary 3.14. *Let $f : X \to Y$ be a projective local complete intersection morphism of regular schemes of constant relative dimension n. Then the morphism $Rf_* : V(X) \to V(Y)$ restricts canonically to a morphism $Rf_* : F^iV(X) \to F^{i-n}V(Y)$. In other words, the essential image of $F^iV(X)$ in $V(Y)$ lies in (the essential image of) $F^{i-n}V(Y)$.*

Proof. Denote by p_j the composition of $V \to V^{(j)} \to V$. Then, by the Adams decomposition in Theorem 4.7 (g) of [Eria], any object $v \in F^iV(X)$ is equivalent to a sum of the form $\sum_{j\geq i} v_j$ for $v_j \in V(X)^{(j)}$. Let $P(t)$ be the multiplicative characteristic class associated to $(t-1)/\log(t)$ which exists and is unique by Proposition 3.2 and the arguments of Lemma 3.10. An application of the splitting principle establishes that for a virtual vector bundle v of rank r, there is a canonical isomorphism $\Psi^k P(v)\theta_k^{-1}(v) = k^{-r}P(v)$ stable under arbitrary base change of regular schemes. An application of the above Riemann-Roch theorem to the virtual bundle $v = P(\Omega_f) \otimes P(\Omega_f)^{-1} \otimes v \in F^iV$ establishes an isomorphism, putting $v_j = p_j[P(\Omega_f)^{-1} \otimes v]$

$$\begin{aligned}\Psi^k Rf_*(P(\Omega_f) \otimes v_j) &= Rf_*(\theta_k(\Omega_f)^{-1} \otimes \Psi^k P(\Omega_f) \otimes v_j) \\ &= k^{j-n} Rf_*(P(\Omega_f) \otimes v_j).\end{aligned}$$

Because the Adams-Riemann-Roch isomorphism is functorial we obtain, by already cited Theorem 4.7 (g) of [Eria], a functorial projection of $Rf_*(v)$ onto $F^{i-n}V(Y)$. □

3.3 A general Grothendieck-Riemann-Roch formula for schemes

We expand on the previous section by proving a Grothendieck-Riemann-Roch-transformation for general schemes, in the sense of [Ful98]. Henceforth, a scheme will be supposed to be a separated scheme admitting an ample family of line bundles. We will denote by S an affine regular scheme.

Theorem 3.15. *There is a canonical equivalence of categories* $\tau : VC(Y) \to \mathcal{CH}_*(Y)$ *such that*

(a) *If* E *is a complex of vector bundles on* Y*, then* $\tau(E \cap \mathcal{F}) = \boldsymbol{ch}(E) \cap \tau(\mathcal{F})$.

(b) *Suppose* $f : X \to Y$ *is a projective morphism. Then there is a canonical isomorphism* $\tau(Rf_*\mathcal{F}) = f_*(\tau\mathcal{F})$.

(c) *Suppose* $f : X \to Y$ *is a flat or local complete intersection morphism, then* $f^*\tau(\alpha) = \tau(Lf^*\alpha) \cap \boldsymbol{Td}(f)$.

(d) *If* $f : X \to Y$ *is an projective morphism, then there is a canonical isomorphism*

$$\tau_Y(Rf_*(\alpha)) = f_*\tau_X(\alpha).$$

(e) *If* $g : S' \to S$ *is a morphism of affine regular schemes, and if* X *is an* S*-scheme such that* $X' = X \times_S S'$ *is a transversal intersection, then* $g^*\tau_X$.

(f) *Suppose that* E *is a complex of vector bundles whose support is disjoint from that of* $\mathcal{F}$*, then both sides of* (a) *are canonically trivialized, we demand they respect this trivialization.*

(g) *If* V *is a closed subvariety of* X *with* $\dim V = n$*, then there is a canonical isomorphism*

$$\tau_X(\mathcal{O}_V) = [\mathcal{O}_V] + \textit{terms of lower dimension.}$$

Moreover, this isomorphism is characterized by (c).

Proof. We start by remarking that if X both and Y are smooth over a regular basescheme S and $f : X \to Y$ is an local complete intersection morphism, this is basically the usual Grothendieck-Riemann-Roch theorem and the proof carries over. In particular one can mimic the proof of the functorial Adams-Riemann-Roch theorem for the above version. We leave this to the reader.
For the general case, suppose that X admits an embedding into a scheme M smooth over a regular scheme S. Then we define, tentatively,

$$\tau_X^M(\alpha) = \mathbf{ch}_X^M(\alpha) \cap (\mathbf{Td}(T_{M/S}) \cap [M]).$$

We need to show that different choices of M glue together to a single functor τ_X such that moreover we have an isomorphism verifying (d). This is done by the following three lemmas, which are proven as in the classical case in [Ful98], chapter 18.3.

Lemma 3.16. *Suppose that* $X \to M \to M'$ *are closed immersions with* M *and* M' *smooth over* S*. Then there is a canonical functor isomorphism*

$$\tau_X^M \simeq \tau_X^{M'}.$$

Lemma 3.17. *Suppose that* $X \xrightarrow{i} Y \to M'$ *are closed immersions with* M *smooth over* S*. Then there is a canonical functor isomorphism*

$$i_*\tau_X^M \simeq \tau_Y^M i_*.$$

Lemma 3.18. *Suppose that $X = \mathbb{P}(E)_Y$ for a scheme Y, $Y \to M$ an immersion with M smooth over S, and that E is the pullback of a vector bundle E' on M. Then there is a functor isomorphism $\tau_Y^M f_* \simeq f_* \tau_{\mathbb{P}(E)}^{\mathbb{P}(E')}$.*

It follows from the lemmas 3.16, 3.17, 3.18 that the functors τ_X^M for various smooth M glue together to one single functor $\tau_X : VC(X) \to \mathcal{CH}_*(X)$. Moreover,

We verify unicity. In case we have two different τ, τ' verifying the above axioms, then one has that $T = \tau'^{-1} \circ \tau$ is an additive functor $T : VC \to VC$ (recall that τ is an equivalence of categories by construction). If X is smooth over S, $VC(X)$ is equivalent to the virtual category of vector bundles $V(X)$ and by the universal property we have to prove that T is necessarily uniquely fixed on vector bundles. By Jounalou-Thomason and (c) we can assume that X is moreover affine. We can then use the reduction to the arithmetic situation with flag varieties to conclude that T is uniquely trivialized.

□

4 Some geometric consequences

4.1 Case of finite morphisms: classical discriminants

Let $f : D \to S$ be a finite flat generically tale morphism of regular schemes. Then as in [Gro] I, Proposition 4.10 there is a canonical rational section $\delta_{S'/S}$ of the linebundle $\det f_*(\mathcal{O}_{S'})^{\otimes 2}$, called the discriminant section, with divisor $\Delta_{S'/S}$. Also recall there is a canonical isomorphism [Del87], section 7, $\det f_*(L - 1) = N_{D/S}(L)$, where the latter is the norm of a line bundle. By loc. cit. we also have for a general virtual bundle v of rank r on S', $\det f_*(v - r) \simeq N_{S'/S}(\det v)$. Rigidity provides us with, for a virtual bundle u of rank m, a canonical and functorial isomorphism

$$\Psi^2(u) - m = (u - m)^{\otimes 2} + 2(u - m) - 2\gamma^2(u - m). \tag{11}$$

.

Proposition 4.1. *The discriminant section $\Delta_{S'/S}$ is given by the Riemann-Roch isomorphism, and for any virtual bundle v we have a canonical isomorphism*

$$(\det f_*(v))^{\otimes 2} \simeq N_{S'/S}(\det v)^{\otimes 2} \otimes \mathcal{O}(\Delta_{S'/S}).$$

Proof. We can suppose that S is a discrete valuation ring and thus that f is projective. The difference between the isomorphism determined by $\Delta_{S'/S}$ and the above constructed isomorphism defines an element $c(v) \in H^0(S, \mathbb{G}_m)_{\mathbb{Q}}$ which is additive and stable under pullback. To prove it is trivial we can assume Y is the spectrum of a field so that S' is the union of spectra of separable field extensions given by irreducible polynomials in n and that v is trivial. We can assume this is irreducible and given by f of degree n with non-trivial discriminant. There is a universal family over $\mathbb{A}_k^n \setminus \Delta$ where Δ is the divisor given by the usual discriminant, $\Delta(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2$, with the α_i are the roots of f and the construction extends over all of $\mathbb{A}_k^n$. This can be pulled back from $\operatorname{Spec} \mathbb{Z}$ and we can suppose that we are in the universal situation of families over $\mathbb{A}_{\mathbb{Z}}^n$ and the isomorphism of line bundles on this scheme extends to all of

$\mathbb{A}^n_{\mathbb{Z}}$ and the statement becomes that, since $H^0(\mathbb{A}^n_{\mathbb{Z}}, \mathbb{G}_m)_{\mathbb{Q}} = 0$, that the order of vanishing of $\det f_*(\Omega_{S'/S})$ along the discriminant locus is actually $\Delta_{S'/S}$, which is the classical discriminant-different relation. □

4.2 Case of curves: Mumford's isomorphism and comparison with Deligne's isomorphism

In this section we deduce some geometric consequences of the above theory. In particular we apply it to the case of families of curves and obtain a unified approach to results of D. Mumford, P. Deligne and T. Saito in Theorem 4.9.
Let $f : C \to S$ be an flat local complete intersection generically smooth proper morphism with geometrically connected fibers of dimension 1, with S any connected normal Noetherian locally factorial scheme. Given a virtual bundle v on C, denote by $\lambda_f(v) = \lambda(v) = \det Rf_*v$ for $\omega = \omega_{C/S}$ being the relative dualizing sheaf, also write $\lambda_n = \det Rf_*\omega^{\otimes n}_{C/S}$. Let $f : \overline{\mathcal{C}_g} \to \overline{\mathcal{M}_g}$ (resp. $\mathcal{C}_g \to \mathcal{M}_g$) be the universal stable curve of genus g (resp. universal smooth curve of genus g) and let $\Delta_g = \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ the discriminant locus of singular curves and write $\delta = \mathcal{O}_S(\Delta_g)$ and $\mu = \lambda_1^{\otimes 12} \otimes \delta^{-1}$. Then we have an isomorphism

$$\lambda_n = \mu^{n(n-1)/2} \otimes \lambda_1 \tag{12}$$

on $\overline{\mathcal{M}_g}$ which is unique up to sign (see [Mum77], Theorem 5.10). One deduces the same formula over a general base for a stable curve. In the case C is regular the corresponding factor δ^{-1} is related to the conductor of the curve (see [Sai88a]). In the case C is non-regular (with the same hypothesis on S) an unpublished result of J. Franke as a consequence of his functorial Riemann-Roch in [Fra] establish a formula of the "discriminant" as a localized Chern class.
The classical proof (see [Mum77], loc.cit.) in the stable case is a simple calculation using Grothendieck-Riemann-Roch and the facts that for any $g \in \mathbb{N}$, we have:

(a) the Picard-group of the moduli-functor of stable curves is torsion-free.

(b) $H^0(\mathcal{M}_{g,\mathbb{Z}}, \mathbb{G}_m) = \pm 1$ (cf. [MB89], Lemme 2.2.3).

We show that our formalism and Adams-Riemann-Roch theorem restricted to dimension 1 implies a version of these results. It should also be noted that the context is greatly simplified by the assumption that we tensor with $\mathbb{Q}$. In particular, inverting 2 eliminates sign-considerations which are without a doubt the greatest obstacle to obtaining integral functorial isomorphisms.

Definition 4.1.1. Henceforth "$f : C \to S$ is a curve" is to be as above, with the additional hypothesis that C and S are regular.

Definition 4.1.2. (Compare with [Fal84]) Given a scheme X, define $\mathfrak{Pic}(X)_{\mathbb{Q}}$ to be the Picard category of line bundles on X with isomorphisms, $\mathfrak{Pic}(X)$, localized at $\mathbb{Q}$. This category can be described as follows. The objects are (L, l) with L a line bundle and l a positive integer. Moreover, we have

$$\mathrm{Hom}_{\mathfrak{Pic}(X)_{\mathbb{Q}}}((L, l), (M, m)) = \lim_{n \to \infty} \mathrm{Hom}_{\mathfrak{Pic}(X)}(L^{\otimes nm}, M^{\otimes nl})$$

where the limit is taken over integers ordered by divisibility. Given two line bundles L, M on a scheme X, a $\mathbb{Q}$-morphism $f : L \to M$ i.e. for big enough n, there exists a morphism $L^{\otimes n} \to M^{\otimes n}$, up to obvious equivalence.

First a preliminary calculation showing that we obtain a version of (12).

Lemma 4.2. *Let $f : C \to S$ be a smooth curve. Then there is a unique canonical $\mathbb{Q}$-isomorphism*

$$\Delta_n : \lambda_n = \det Rf_* \omega_{C/S}^{\otimes n} \simeq \lambda_1^{\otimes(6n^2-6n+1)}$$

stable under base change $S' \to S$.

Proof. Uniqueness follows from descent and the preceding remarks adding that $\mathcal{M}_g$ is smooth over $\operatorname{Spec} \mathbb{Z}$ and so regular. We can also assume that S is the spectrum of a discrete valuation-ring since isomorphisms would then glue together to a global one by virtue of them being canonical. In this case f is automatically projective by [Lic68], Section 23. Let $\omega_f = \Omega_{C/S}$ be the relative dualizing bundle of f. Applying the Adams-Riemann-Roch theorem to the case $(1 - \omega_f)$, we obtain the "Grothendieck-Serre duality"-isomorphism

$$(\lambda_0 \otimes \lambda_1^{-1})^{\otimes(k-1)} \simeq 1$$

and in particular for $k = 2$ one has a canonical $\mathbb{Q}$-isomorphism $\lambda_0 \simeq \lambda_1$. Consider the cannibalistic Bott-class

$$\theta_{f,2}^{-1} \quad := \quad \frac{1-\omega_f}{1-\omega_f^2} \quad = \quad \frac{1}{2}\left(1 + \frac{1-\omega_f}{2} + \left(\frac{1-\omega_f}{2}\right)^{\otimes 2}\right) + F^3V(X).$$

The truncation is sufficient for our purposes since the relative dimension $C \to S$ is 1 and Corollary 3.14, so that $F^3V(C)$ has image in $F^2V(S)$ and the determinant functor is trivial on this category. It is moreover stable under base change by functoriality of the Adams-Riemann-Roch theorem. For $k = 2$, inserting this into the Adams-Riemann-Roch-theorem for the trivial line bundle and applying Grothendieck-Serre-duality this reduces to the expression

$$\lambda_1^{16} = \lambda_0^{16} = \lambda_0^7 \otimes \lambda_1^{-4} \otimes \lambda_2 = \lambda_1^3 \otimes \lambda_2$$

so that $\lambda_1^{13} = \lambda_2$. Repeatedly applying the theorem to the case of $1 - \omega_f^2$ one proceeds by induction on n to establish the general formula for λ_n. □

Thus for any curve $f : C \to S$ we obtain a canonical rational $\mathbb{Q}$-morphism $\Delta : \lambda(\omega^{\otimes 2}) \to \lambda(\omega)^{\otimes 13}$ which restricts to the above one over the smooth locus. This is the usual discriminant morphism considered in [Sai88a], for example.
We intend to compare our Adams-Riemann-Roch-isomorphism with that of Deligne (see [Del87], Théorème 9.9). Let's just first recall the main ingredients. Let $C \to S$ be a local complete intersection projective morphism of schemes with geometrically connected fibers of dimension 1. Given two line bundles L and M on C, by [Del77], XVIII, 1.3.11, [Elk89], one can form the line bundle $\langle L, M\rangle$ on S. The symbol $\langle \cdot, \cdot \rangle$ satisfies bimultiplicativity with respect to tensor product and has a cohomological description: if u and v are virtual vector bundles of rank 0 on C, then (see [Del87], 7.3.1);

$$\langle \det u, \det v \rangle = \lambda(u \otimes v) \tag{13}$$

and so in particular

$$\langle L, M \rangle = \lambda(L \otimes M)\lambda(L)^{-1}\lambda(M)^{-1}\lambda(\mathcal{O}_C).$$

Using this the discriminant section Δ is equivalent to a rational section $\langle\omega,\omega\rangle \to \lambda(\omega)^{\otimes 12}$ which we shall also call the discriminant section. Given any virtual bundle v on C, define $R(v) = (v - \mathcal{O}_C^{\operatorname{rk} v}) - (\det v - \mathcal{O}_C)$, one defines $I_{C/S}C_2(v) = \lambda(-R(v))$. The unicity statement of this functor in [Del87], Proposition 9.4 easily proves:

Proposition 4.3. *Let $f : C \to S$ be a curve. Then there exists a unique canonical $\mathbb{Q}$-isomorphism*

$$\Xi : \lambda(\gamma^2(u - \operatorname{rk} u)) = I_{C/S}C_2(u)$$

functorial on virtual bundles u on C, such that the isomorphism is compatible with the trivializations for u a line bundle and such that the following condition Λ holds: for an isomorphism $u = v + w$ the isomorphism

$$\lambda(\gamma^2(u - \operatorname{rk} u)) \to \lambda(\gamma^2(v - \operatorname{rk} v)) \otimes \lambda((v - \operatorname{rk} v)) \otimes (w - \operatorname{rk} w))) \otimes \lambda(\gamma^2(w - \operatorname{rk} w))$$

is compatible with the isomorphism

$$I_{C/S}C_2(u) \to I_{C/S}C_2(v) \otimes \langle \det v, \det w\rangle \otimes I_{C/S}C_2(w)$$

via Ξ and (13).

Proposition 4.4. *Suppose $C \to \operatorname{Spec} R$ is a local complete intersection curve (cf. 4.1.1, it is automatically projective by [Lic68], Section 23) where R be the spectrum of discrete valuation ring with special point s and generic point η. Let $\Omega_{C/S}$ be the coherent sheaf of relative differentials and $\omega_{C/S} = \omega$ the relative dualizing sheaf. The bundle $IC_2(\Omega_{C/S})$ is canonically trivialized over the generic point and the order of the trivialization at η is equal to the order of the discriminant.*

Proof. By the general theory $\Omega_{C/S}$ comes equipped with a natural morphism $\Omega_{C/S} \to \omega$ inducing an isomorphism $\det \Omega_{C/S} = \omega$. The Adams-Riemann-Roch theorem provides us with a canonical $\mathbb{Q}$-isomorphism

$$\lambda(1)^{\otimes 16} = \lambda(1 - 2(\Omega_{C/S} - 1) + (\Omega_{C/S} - 1)^{\otimes 2} - \gamma^2(\Omega_{C/S} - 1)).$$

By the cohomological description of the Deligne-pairing in (13), we have canonical isomorphisms $\lambda((\Omega_{C/S} - 1)^{\otimes 2}) = \langle\omega,\omega\rangle$ and $-(\Omega_{C/S} - 1) = -R(\Omega_{C/S}) - (\omega - 1)$ so that $\lambda(-(\Omega_{C/S} - 1)) = I_{C/S}C_2(\Omega_{C/S}) \otimes \lambda(\omega - 1) = I_{C/S}C_2(\Omega_{C/S})$ where the last isomorphism is by Lemma 4.2. Thus we obtain a canonical isomorphism

$$\lambda(\omega)^{\otimes 12} = \langle\omega,\omega\rangle \otimes I_{C/S}C_2(\Omega_{C/S})$$

which restricts to $\lambda(\omega)^{\otimes 12} = \langle\omega,\omega\rangle$ over the generic fiber via the trivialization $I_{C/S}C_2(\Omega_{C/S})$ over the generic fiber defined by the trivialization $I_{C/S}C_2(\omega_{C_\eta/\operatorname{Spec}\eta}) = 1$. Thus the order of the generic trivialization $1 \to IC_2(\Omega_{C/S})$ is the discriminant. □

Definition 4.4.1. Let R be a discrete valuation ring, and $C \to \operatorname{Spec} R$ a curve with special point s and generic point η as above. Let (u,t) be a couple with u a virtual bundle on C with a trivialization $t : \det u_{|C_\eta} \to 1$ on the generic fiber. Then the bundle $\langle u, v\rangle$ has a canonical trivialization by t on R over the generic point via the isomorphism $\langle u, v\rangle = \langle \det u, \det v\rangle = \lambda((\det u - 1)\otimes(\det v - 1))$. Then for another virtual bundle v on C, we define $c_1^D(u,t).c_1(v)$ to be the order of this trivialization. In a similar vein, suppose that (u, s) is a couple with u a virtual bundle on C with an isomorphism $s : u|_{|C_\eta} \to L$ on the generic fiber with L a line bundle. Then $IC_2(u)$ has a canonical trivialization by s on R over the generic point, cf. [Del87], Proposition 9.4 (ii) or above definition, and we define $c_2^D(u,s)$ to be the order of this trivialization.

The following is a slight extension of Lemma 2 in [Sai88a], to which we refer the reader for an idea of the proof.

Lemma 4.5. *Let X be a regular scheme and Z an effective divisor of X with complement U. Suppose we two strict perfect complexes E, F on X, and a have a quasi-isomorphism $t : E|_U \to F|_U$ over U. Denote by $\det t$ the corresponding rational section of the line bundle $\mathrm{Hom}_{\mathcal{O}_X}(\det E, \det F)$, and $\mathrm{div}\, t$ its divisor. Then the bivariant class $c_{1,Z}^X(E \to F)\cap$ acts as the intersection class $\mathrm{CH}_i(X) \to \mathrm{CH}_{i-1}(Z)$ given by simply restricting along $\mathrm{div}\, t$.*

Corollary 4.6. *The class $c_1^D(u,t).c_1(v)$ defined above coincides with $c_{1,C_s}^C(\det u, \det t).c_1(\det v)\cap[X]$.*

Proof. This follows from the above description and an application of Riemann-Roch for singular curves (on the special fiber) as in [Ful98], Example 18.3.4. ☐

Lemma 4.7. *Keep the assumptions and notations of the above definition. The cotangent sheaf $\Omega_{C/S}$ is a line bundle on the generic fiber and thus $IC_2(\Omega_{C/S})$ is canonically trivialized over the generic fiber. With this trivialization we have $c_2^D(\Omega_{C/S}) = c_{2C_s}^C(\Omega_{C/S})$, the associated localized Chern class (cf. [Blo87], section 1).*

Proof. First of all, $\gamma^2(\Omega_{C/S} - 1) = \lambda^2\Omega_{C/S}$, where λ^2 denotes the λ-operation on the virtual category, so that $\lambda^2\Omega_{C/S}$ is trivialized over the generic fiber. By [Sai88b], Proposition 2.3 the alternating lengths of the cohomology of $\lambda^2(\Omega_{C/S})$ is $c_{2C_s}^C(\Omega_{C/S})$. It follows that the order of the above induced trivialization of $\det Rf_*(\gamma^2(\Omega_{C/S} - 1)) = \det Rf_*(\lambda^2\Omega_{C/S})$ is $c_{2C_s}^C(\Omega_{C/S})$. We conclude by Proposition 4.3. ☐

Corollary 4.8 (Conductor-Discriminant formula by T. Saito)**.** *With the above assumptions, the order of the discriminant rational section Δ of the line bundle $\mathrm{Hom}_{\mathcal{O}_S}(\langle\omega,\omega\rangle, \lambda(\omega)^{\otimes 12})$ is equal to minus the Artin conductor of $C \to S$ (cf. [Sai88a]).*

Proof. Combine Proposition 4.3 and Corollary 4.7 and the main result of [Blo87] which identifies the localized Chern class with the Artin conductor. ☐

Remark 4.8.1. The proof of Corollary 4.8 is essentially different from that of [Sai88a] in that it does not use any kind of semi-stable reduction techniques. It does however share similarity to that of [Fra].

Next we deduce the following Deligne-Riemann-Roch isomorphism:

Theorem 4.9. *Let $f : C \to S$ be a curve over a regular stack S, i.e. a curve over any smooth presentation of S. Then there is a unique canonical $\mathbb{Q}$-isomorphism*

$$\lambda(\theta_f^{-2} \otimes \Psi^2 u) \simeq \langle \omega, \omega \rangle^{\operatorname{rk} u} \otimes IC_2(\Omega_{C/S})^{\operatorname{rk} u} \otimes IC_2(u)^{-12} \langle \det u, \det u \otimes \omega^{-1} \rangle^6$$

which is compatible with the Deligne-isomorphism up to torsion.

Remark 4.9.1. Although one might guess that the above procedure directly defines a discriminant in all dimensions this does not seem to be true.

Proof. By descent we can suppose that S is a regular scheme. The isomorphism in (11) also holds in this situation. We can multiply this out together with the inverse of the Bott class and use that there is a functorial product on the filtration $F^i V(C)$ to cancel out all the terms that are in $F^3 V(C)$. This provides us with a choice of canonical isomorphism with the right-hand side of the Deligne-Riemann-Roch theorem, compatible with base change and sums.
We verify that it is unique. This is an argument given in [Fra] which we reconsider here (more or less verbatim). Given any functorial isomorphism as above the lack of compatibility with the Deligne-isomorphism for a virtual bundle u is given by an element $c_{X/Y}(u) \in H^0(S, \mathbb{G}_m)_{\mathbb{Q}}$ which is stable under pullback of smooth curves, as well as isomorphisms and sums of virtual bundles. Locally on the base S the virtual bundle u is a sum of line bundles, so we can assume u is a line bundle. To show that $c_{X/Y}(L) = 1$ we can assume S is the spectrum of an algebraically closed field. Given a line bundle of degree d it is the pullback of the universal degree d-bundle $\mathcal{P}_d$ of $\mathcal{C}_{g,d} \to P_{g,d}$, moduli of genus g-curves with a given degree d bundle, thus constant. We remark that $P_{g,d}$ is smooth over $\mathcal{M}_g$ and thus regular since $\mathcal{M}_g$ is smooth over $\operatorname{Spec} \mathbb{Z}$.
Thus we are given universal constants $(c_{d,g}) \in H^0(\mathcal{M}_g, \mathbb{G}_m)_{\mathbb{Q}}$ on $\mathcal{M}_g$ which are 1 by virtue of the fact that $H^0(\mathcal{M}_g, \mathbb{G}_m) = \pm 1$. □

Remark 4.9.2. A word of caution is in place. We haven't actually constructed classes θ_f^{-2} etc for the virtual category of vector bundles on any stack. The bundles $\lambda(\theta_f^{-2} \otimes \Psi^2 u)$ etc. in question refer to the bundles one obtains by smooth descent from the case of a regular scheme, in which case it does make sense.

4.3 The refined Riemann-Roch formula of T. Saito

Let X be a scheme and $i : Z \to X$ be a reduced closed subscheme with $j : U = X \setminus Z \to X$ the open complement. The Picard category kernel $j^* : \mathcal{CH}_*(X) \to \mathcal{CH}(U)$ is described as follows: Objects are objects A of $\mathcal{CH}_*(X)$ together with a trivialization $j^* A \simeq 0$. Isomorphisms of objects are isomorphisms in $\mathcal{CH}_*(X)$ respecting the trivializations. Using the long exact localization sequence one sees that the isomorphism classes of this category is $\mathrm{CH}_*(Z)$.

Indeed, if A is representative of a class in $\mathcal{CH}_*(X)$ of an element in $\mathrm{CH}_*(Z)$, any two trivializations $j^* A \simeq 0$ differ by an element of $\mathrm{Aut}0_U$, whose image in $\mathrm{CH}_*(Z)$ is 0 and thus comes from a global isomorphism. Conversely, any element of the kernel is isomorphic in $\mathcal{CH}_*(X)$ to some element $i_* B$ for B a cycle on Z. The default of this to be an isomorphism defines an element in

$B' \in \mathrm{Aut0}_U$ and $B - B'$ is then an element such that A is globally isomorphic to $B - B'$. It is easy to see that this sets up the required isomorphism.

Now, let $\mathcal{E}$ be an acyclic complex on U such that the vector bundles of the complex extends to all of X, but not necessarily the diffentials. Also let $\mathbf{T}$ be a characteristic class stable under open immersions. For simplicity we suppose that $\mathbf{T}$ is $\mathbf{ch}$ or $\mathbf{Td}$, the general argument carries through. Then the object $\mathbf{ch}(\mathcal{E}) \cap [X]$ of $\mathcal{CH}_*(X)$ is canonically trivialized on U and thus by the above defines an element in $\mathrm{CH}_*(Z)$.

Proposition 4.10. *This element coincides with the element* $\mathrm{ch}_X^Z(\mathcal{E}) \cap [X]$ *in* $\mathrm{CH}_*(Z)$.

Proof. By the arguments of [Ive76], it is enough to verify that if $\mathcal{E}$ is an acyclic complex on X, this element is 0, and that if $f : X \to Y$ is a proper morphism of schemes, then there is a projection formula isomorphism: $f_*(\mathbf{ch}(f^*\mathcal{E}) \cap [X]) = (\mathbf{ch}(\mathcal{E}) \cap f_*[X])$. But the first is trivial and the second statement follows from the projection formula for the first class. □

Theorem 4.11. *Suppose* $\pi : X \to Y$ *is a projective birational morphism of schemes, and* $f : Y \to T$ *is a projective local complete intersection morphism such that* $g = f \circ \pi$ *is also a projective local complete intersection . We suppose that there is an open subscheme* U *in* T *such that* π *restricted to* U *is an isomorphism, and write* $Z = X \setminus U$. *Denote by* L *the contangent complex of a morphism. Then if* $\mathcal{E}$ *is any complex of vector bundles on* Y, *adjunction defines a morphism* $Rf_*\mathcal{E} \to Rg_*\pi^*\mathcal{E}$ *which is a quasi-isomorphism on* U. *Similarly, there is a canonical isomorphism* $L_\pi = [L_g \to \pi^* L_f]$ *which defines a localized class* $\mathrm{Td}(L_\pi) - 1$. *Then we have:*

$$\mathrm{ch}_T^Z(Rf_*(\mathcal{E}) - Rg_*\pi^*\mathcal{E}) \cap [X] = g|_{Z,*}\pi^*(\mathrm{ch}(\mathcal{E}) \cap \mathrm{Td}(L_f)) \cap (\mathrm{Td}(L_\pi) - 1) \cap [X]$$

in $\mathrm{CH}_*(Z)$.

Proof. Applying the functorial Riemann-Roch theorem gives an isomorphism

$$\mathbf{ch}(Rf_*(\mathcal{E}) - Rg_*\pi^*\mathcal{E}) \cap [T] = g_*(\pi^*(\mathbf{ch}(\mathcal{E}) \cap \mathbf{Td}(L_f)) \cap (\mathbf{Td}(L_\pi) - 1) \cap [X])$$

which are both trivialized when restricting to U and the Riemann-Roch-isomorphism respect these trivializations by construction. By Proposition 4.10 these trivializations define the localized classes the theorem asks for. □

4.4 A Knudsen-Mumford expansion

Theorem 4.12. *Let* $f : X \to S$ *be a flat projective morphism of relative dimension* r *such that* S *is equidimensional and locally factorial, and* X *is regular. Let* $\mathcal{F}$ *be a complex of vector bundles on* X *and let* L *be line bundle on* X. *Defining* $\mathcal{F}(k) = \mathcal{F} \otimes L^k$, *there is a canonical* $\mathbb{Q}$*-isomorphism*

$$\lambda(\mathcal{F}(n)) = \det R\pi_*(\mathcal{F}(n)) \simeq \otimes_{k=0}^{r+1} \lambda_k^{k^n} \simeq \otimes_{n=0}^{r+1} \mathcal{M}_n^{\binom{n}{k}}$$

for some (rational) line bundles $\lambda_i, \mathcal{M}_i, i = 0, \ldots, r+1$ *on* S. *Here*

$$\lambda_n = \pi_*(\mathbf{c}_1(L)^n \cap (\mathbf{ch}(\mathcal{F}) \cap \mathbf{Td}(T_f))^{(r-n)}/n! \cap [X])^{(1)}$$

where $A^{(?)}$ *denotes the degree ?-part and in particular*

$$\mathcal{M}_{r+1} \simeq \langle L, \dots, L \rangle^{\operatorname{rk} \mathcal{F}}$$

and if $\mathcal{F} = \mathcal{O}_X$,

$$\mathcal{M}_r^2 \simeq \langle L^r K^{-1}, L, \dots, L \rangle.$$

In particular if L is relatively ample and k is big enough $\det R\pi_* L^k = \det \pi_* L^k$ and we recover part of the results of [KM76].

Proof. We can assume that S is the spectrum of a discrete valuation ring in which case f is a local complete intersection morphism. Then the above is an application of the functorial Riemann-Roch-theorem. □

4.5 A conjecture of Köck for the determinant of the cohomology

We recall the setting of [Köc98] in the special case of K_0. Let S be an separated Noetherian scheme and G a flat separated finite type group-scheme over S. Suppose we are given a G-projective local complete intersection morphism $f : X \to Y$ of G-equivariant schemes such that on Y any G-coherent module is the quotient of a locally free G-module. Denote by $K(X, G)$ the group $K_0(X, G)_{\mathbb{Q}}$ of G-equivariant K-theory of vector bundles of X tensor $\mathbb{Q}$. There is a natural pushforward $Rf_* : K(X, G) \to K(Y, G)$.

Definition 4.12.1. Fix a G-equivariant factorization $f : X \overset{i}{\hookrightarrow} \mathbb{P}_Y(\mathcal{E}) \overset{\pi}{\to} Y$ for some vector bundle $\mathcal{E}$ on Y and denote by Ω the sheaf of relative differentials of $\mathbb{P}(\mathcal{E}) \to Y$. Let d be the rank of $\mathcal{E}$ and denote by $\hat{K}(Y, G)$ the ring $K(Y, G)_{\mathbb{Q}}$ completed at the ideal generated by elements of the form $\lambda^1(\mathcal{E}) - d, \lambda^2(\mathcal{E}) - \binom{d}{2}, \dots, \lambda^d(\mathcal{E}) - 1$.

Then the following is proved:

Proposition 4.13 ([Köc98], Theorem 4.5)**.** *The Bott element* ${\theta'}_k^{-1} := \theta_k(N_i - i^*\Omega)$ *(defined as before) defines an element in* $K(X, G) \otimes_{K(Y,G)} \hat{K}(Y, G)$ *and for any* $k \geq 2$ *there is a commutative diagram*

$$\begin{array}{ccc} K(X,G) & \xrightarrow{{\theta'}_k^{-1} \otimes \Psi^k} & K(X,G) \otimes_{K(Y,G)} \hat{K}(Y,G) \\ \Big\downarrow{\scriptstyle Rf_*} & & \Big\downarrow{\scriptstyle \hat{R}f_*} \\ K(Y,G) & \xrightarrow{\Psi^k} & \hat{K}(Y,G) \end{array}.$$

We recast it in the following way to provide a formula for the first Chern-class of the cohomology, or equivalently, the determinant of the cohomology. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable proper local complete intersection morphism of separated Noetherian algebraic stacks with the resolution property and with smooth groupoid representations $[p, q : R \rightrightarrows X]$ and $[p', q' : R' \rightrightarrows Y]$ respectively, with induced morphisms $g : X \to Y, h : R \to R'$, and consider the following two "determinant of cohomology"-functors. First of all, given a vector bundle E on $\mathcal{X}$, one pushforwards to obtain a perfect complex Rf_*E on $\mathcal{Y}$, and then apply the determinant to obtain the determinant of the cohomology, $\lambda_1(E)$, considered as a linebundle on $\mathcal{Y}$. On the level of K-groups this corresponds to the homomorphism $K(\mathcal{X}) \stackrel{Rf_*}{\to} K(\mathcal{Y}) \stackrel{\det}{\to} \mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$, where the pushforward is the one exhibited in [Köc98] for quotient-stacks, and the last homomorphism is the determinant homomorphism. In a different vein, consider the same vector bundle E, and consider the perfect complex Rg_*E on Y. Since one has the relation $\det Lq^* = q^* \det$, the base change-isomorphism equips the determinant $\det Rg_*E$ with descent-data with respect to $R' \rightrightarrows Y$, thus we obtain another linebundle $\lambda_2(E)$ on $\mathcal{Y}$. The main observation of this section is that in the above setting, descent commutes with pushforward:

Lemma 4.14. *There is a natural equivalence of determinant functors $\lambda_1 \simeq \lambda_2$, and we denote both by $\lambda := \det Rf_*$.*

Proof. The proof is just unwinding the definitions. The definition of λ_1 is obtained by choosing an R-equivariant g_*-acyclic resolution $E \to F$, and then applying g_*, and finally the determinant functor. The descent-data for g_*F for acyclic F is given by the base change-isomorphism, so that the following diagram is commutative

$$\begin{array}{ccccccc} Lp'^*Rg_*E & \longrightarrow & Rh_*Lp^*E & \longrightarrow & Rh_*Lq^*E & \longrightarrow & Lq'^*Rg_*E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ p'^*g_*F & \longrightarrow & h_*p^*F & \longrightarrow & h_*q^*F & \longrightarrow & q'^*g_*F \end{array}$$

in the derived category of perfect complexes on R'. The upper line is given by a quasi-isomorphism composed by smooth base change and descent-data, whereas the lower one is an isomorphism given by ordinary smooth base change and the vertical maps are the natural quasi-isomorphisms. Applying the determinant functor to the perfect complex Rg_*E transforms quasi-isomorphisms to isomorphisms and derived pullbacks to pullbacks and thus provides us with descent-data of Rg_*E. This is the definition of λ_2 and provides us with the requested equivalence of functors. □

Notice that λ_1 defines a determinant functor from the category of virtual vector bundles on $\mathcal{X}$ admitting a f_*-acyclic resolution, whereas λ_2 is defined on the category of virtual vector bundles on $\mathcal{X}$ admitting g_*-acyclic resolutions on X.

Theorem 4.15. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable projective local complete intersection morphism of regular stacks. Then we have equalities*

$$\det(Rf_*E)^{\otimes k} = \det(Rf_*(\theta_{f,k}^{-1} \otimes \psi^k E))$$

in $\mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$.

Proof. By Theorem 3.11, for a R-equivariant vector bundle E the Adams-Riemann-Roch isomorphism $\Psi^k Rg_*E = Rf_*(\theta_{g,k}^{-1} \otimes \psi^k E)$ associated to $g : X \to Y$ is stable under smooth base change and thus defines descent-data of the isomorphism of $\mathbb{Q}$-line bundles

$$\det(Rg_*E)^{\otimes k} = \det(\Psi^k Rg_*E) = \det Rf_*(\theta_{g,k}^{-1} \otimes \psi^k E).$$

By The class in Pic of left hand side coincides with the map $K(\mathcal{X}) \stackrel{Rf_*}{\longrightarrow} K(\mathcal{Y}) \stackrel{\det}{\to} \mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$ and the right hand side $\det Rf_*(\theta_{g,k}^{-1} \otimes \psi^k E)$ coincides with $K(\mathcal{X}) \stackrel{\theta_k^{-1}\otimes\Psi^k}{\longrightarrow} K(\mathcal{Y}) \stackrel{\det}{\to} \mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$. □

Thus we obtain a non-completed version of Kck's Adams-Riemann-Roch. Define an action of $a \in K(Y,G)$ on $(r,L) \in \mathbb{Z} \oplus \mathrm{Pic}(Y,G)$ by

$$a.(r,L) = (r \cdot \mathrm{rk}\, a, \det(a)^r \otimes b^{\mathrm{rk}\, a}).$$

Then $\mathbb{Z} \oplus \mathrm{Pic}(Y,G)$ is a $K(Y,G)$-module and we put $\hat{\mathrm{Pic}}(Y,G)$ to be the quotient of $(\mathbb{Z} \oplus \mathrm{Pic}(Y,G)) \otimes_{K(Y,G)} \hat{K}(Y,G)$ by $\mathbb{Z} \otimes_{K(Y,G)} \hat{K}(Y,G)$. It follows from Lemma 4.14 that the image of the right side in $\hat{\mathrm{Pic}}(Y,G)$ of the above theorem necessarily coincides with the image under $Rf_*(\theta'^{-1}_k \otimes \Psi^k E)$ in $\hat{K}(Y,G) \stackrel{\det}{\longrightarrow} \hat{\mathrm{Pic}}(Y,G)$. Also, by the usual equivalence of the Adams-Riemann-Roch and Grothendieck-Riemann-Roch theorem one obtains expressions for the Chern-classes and the corresponding equivariant Grothendieck-Riemann-Roch theorem for the first Chern-class. This is example 5.11 of loc.cit. which is only known under the condition that f is continuous with respect to the γ-filtration on the K-groups, i.e. if $F^n K$ denotes the γ-filtration on K, then for any n we require that there is an m such that $Rf_* F^m K(X,G) \subset F^n K(Y,G)$ (cf. [Köc98], section 5).

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