# BASE CHANGE CONDUCTOR FOR JACOBIANS

DENNIS ERIKSSON, LARS HALVARD HALLE AND JOHANNES NICAISE

## 1. Introduction

Let $C$ be a smooth, projective and geometrically connected curve, defined over the quotient field $K$ of a complete discrete valuation ring $R$. In the literature, one finds many numerical invariants attached to $C$, which measure various properties of $C$ connected to degeneration. In this paper, we investigate the relationship between two such invariants, the *base change conductor*, which is defined in terms of the Néron model of the Jacobian variety of $C$, and the *Artin conductor*, which can be defined for any regular model of $C$.

The base change conductor was introduced for tori by Chai and Yu [CY01] and in general for semi-abelian varieties by Chai [Ch00]. If $A/K$ is a semi-abelian variety with semi-abelian reduction after a finite extension $L/K$, the base change conductor $c(A) \in \mathbb{Q}$ yields a measure of how the Lie algebra of the Néron model $\mathscr{A}/R$ of $A$ differs from the Lie algebra of $\mathscr{A}_L/R_L$ of $A \times_K L$. In fact, it is known that $c(A)$ is zero if and only if $A$ has semi-abelian reduction over $R$. Thus, for a curve $C$, one can view $c(\mathrm{Jac}(C))$ as an obstruction for $C$ to have semi-stable reduction over $R$.

On the other hand, given a proper regular model $\mathscr{X}/R$ of $C$, the Artin conductor $\mathrm{Art}_{\mathscr{X}/R}$ is defined as the difference of the $\ell$-adic Euler characteristics of the generic and special fibers, with a correction term provided by the so called Swan conductor. The definition goes back at least to Bloch.

It is easy to see from the definitions that if $C$ has semi-stable reduction over $R$, but not good reduction, the Artin conductor is non-zero for every regular model of $C$. Thus, one should expect some kind of correction term when comparing the base change conductor with the Artin conductor of a regular model of $C$. In the main result of this paper, Theorem 5.1.4, we compute explicitly this correction term when $\mathscr{X}/R$ is a strict normal crossings model of $C$, under a certain tameness assumption of $C$. In fact, Theorem 5.1.4 yields a closed formula for the difference between the two invariants under consideration, expressed entirely in terms of the combinatorial data associated with the special fiber $\mathscr{X}_s$ of $\mathscr{X}$, i.e., the intersection graph, and the genus and multiplicity of each component.

1.1. **Notation.** Throughout the paper, we will let $R$ denote a complete discrete valuation ring, with quotient field $K$ and residue field $k$. We will assume that $k$ is algebraically closed, with characteristic exponent $p \geq 1$.

We fix a separable closure $K^{\mathrm{sep}}$ of $K$. For any integer $d \in \mathbb{N}$ prime to $p$, we let $K(d)$ denote the unique tamely ramified extension of $K$ in $K^{\mathrm{sep}}$

of degree $d$. We let $R(d)$ be the integral closure of $R$ in $K(d)$, it is again a complete discrete valuation ring with residue field $k$.

We will denote by $C$ a smooth, proper and geometrically connected $K$-curve of genus $g > 0$. We also assume that $C$ has index 1. The Jacobian variety of $C$ is denoted $J$.

For any flat scheme $\mathscr{Z}$ over $S = \mathrm{Spec}(R)$, we denote by $\mathscr{Z}_s := \mathscr{Z} \times_R k$ the special fiber of $\mathscr{Z}$. We moreover denote by $\mathscr{Z}_\eta = \mathscr{Z} \times_R K$ the generic fibre, and by $\mathscr{Z}_{\overline{\eta}} = \mathscr{Z} \times_R K^{\mathrm{sep}}$ the geometric generic fiber.

## 2. Preliminaries on models of curves and Jacobians

2.1. **Regular models of curves.** A model of $C/K$ is a flat and proper $R$-scheme $\mathscr{X}$, endowed with an isomorphism of $K$-schemes

$$\mathscr{X} \times_R K \cong C.$$

Of particular importance is the minimal regular model $\mathscr{X}_{\mathrm{min}}$, which is characterized by the property that there does *not* exist any smooth rational curve $E$ in the special fiber with $E^2 = -1$. In many situations however, it is more convenient to work with the so called minimal regular model with strict normal crossings $\mathscr{X}_{\mathrm{sncd}}$. It is minimal among all regular models $\mathscr{Z}$ of $C$ such that $\mathscr{Z}_s$ is a divisor with strict normal crossings (for short, we shall call any such model an *sncd*-model of $C$).

For the applications in this paper, it is crucial to be able to compare regular models of $C$ with regular models of $C \times_K K'$, where $K'/K$ is a finite separable field extension. If $K'/K$ is a tame extension, we shall frequently use the following procedure (for details and proofs we refer to [Ha10a]).

Let $\mathscr{C}$ be an *sncd*-model of $C$, and let $S' = \mathrm{Spec}(R')$, with $R'$ the integral closure of $R$ in $K'$. We denote by $\widetilde{\mathscr{C}}$ the normalization of $\mathscr{C} \times_S S'$. Then $\widetilde{\mathscr{C}}$ has at most tame cyclic quotient singularities, whose local analytic structure can be determined purely in terms of the combinatorial properties of the special fiber $\mathscr{C}_s$, together with the degree $e(K'/K)$. Now let

$$\rho : \mathscr{C}' \to \widetilde{\mathscr{C}}$$

be the minimal desingularization of $\widetilde{\mathscr{C}}$. Then $\mathscr{C}'$ is in fact an *sncd*-model of $C \times_K K'$ with strict normal crossings.

No such "explicit" procedure is known in case $K'/K$ is wild, a fact which often complicates matters substantially.

2.2. **Logarithmic differential forms.** If $\mathscr{C}$ is a model of $C$, we will denote by $\omega_{\mathscr{C}/R}(\log \mathscr{C}_s)$ the sheaf of logarithmic differential forms on $\mathscr{C}$ over $R$. More precisely, if we denote by $\mathscr{C}^+$ the scheme $\mathscr{C}$ endowed with the log structure induced by $\mathscr{C}_s$ and by $S^+$ the scheme $S = \mathrm{Spec}\, R$ with the log structure induced by the closed point $s$, then

$$\omega_{\mathscr{C}/R}(\log \mathscr{C}_s) = \Omega^1_{\mathscr{C}^+/S^+}.$$

This is a coherent sheaf on the scheme $\mathscr{C}$, whose restriction to $C$ is naturally isomorphic to the canonical bundle $\omega_{C/K}$.

Now assume that $\mathscr{C}^+$ is log smooth over $S^+$ (this is the case, for instance, if $\mathscr{C}$ is an *sncd*-model of $C$ and all the multiplicities of the components of $\mathscr{C}_s$ are prime to $p$). Then $\omega_{\mathscr{C}/R}(\log \mathscr{C}_s)$ is a line bundle. Let $K'$ be a finite extension of $K$. We denote by $R'$ the integral closure of $R$ in $K'$ and we set $S' = \mathrm{Spec}\,(R')$ and

$$\mathscr{D}^+ = \mathscr{C}^+ \times_{S^+} (S')^+$$

where the product is taken in the category of fine and saturated (fs) log schemes. Let $\mathscr{D}$ be the underlying scheme of $\mathscr{D}^+$. Then the log structure on $\mathscr{D}^+$ is the divisorial log structure induced by $\mathscr{D}_s$. Moreover, $\mathscr{D}$ is canonically equipped with a finite morphism

$$\mathscr{D} \to \mathscr{C} \times_R R'$$

which is an isomorphism on the generic fibers since there the log structure is trivial. We also know that $\mathscr{D}^+$ is log smooth over the log regular scheme $(S')^+$, because log smoothness is preserved by base change in the category of fs log schemes. Thus $\mathscr{D}^+$ is itself log regular [Ka94, 8.2], which implies that the underlying scheme $\mathscr{D}$ is normal [Ka94, 4.1]. Therefore,

$$\mathscr{D} \to \mathscr{C} \times_R R'$$

must be a normalization map. Since the sheaves of log differentials are stable under fs base change, we find that

$$\omega_{\mathscr{D}/R'}(\log \mathscr{D}_s)$$

is canonically isomorphic to the pullback of

$$\omega_{\mathscr{C}/R}(\log \mathscr{C}_s)$$

to the normalization $\mathscr{D}$ of $\mathscr{C} \times_R R'$.

If $\mathscr{C}$ is any regular model of $C$, then we can also consider the canonical sheaf $\omega_{\mathscr{C}/R}$, which is a line bundle on $\mathscr{C}$ that extends the canonical bundle $\omega_{C/K}$. Its relation with $\omega_{\mathscr{C}/R}(\log \mathscr{C}_s)$ is explained in the following proposition.

**Proposition 2.2.1.** *If $\mathscr{C}$ is a regular model of $C$ such that $\mathscr{C}^+$ is log smooth over $S^+$, then*

$$\omega_{\mathscr{C}/R}(\log \mathscr{C}_s) = \omega_{\mathscr{C}/R}((\mathscr{C}_s)_{\mathrm{red}} - \mathscr{C}_s)$$

*as subsheaves of $j_*\omega_{C/K}$, where $j$ denotes the open immersion $j : C \to \mathscr{C}$.*

*Proof.* The statement is local for the étale topology. By Kato's toroidal description of log smooth morphisms [Ka94, 3.5], we know that étale-locally at every double point of $(\mathscr{C}_s)_{\mathrm{red}}$, the model $\mathscr{C}$ is of the form

$$\mathscr{X} = \mathrm{Spec}\, R[x, y]/(\pi - x^a y^b)$$

where $\pi$ is a uniformizer in $R$ and $a, b$ are non-negative integers such that $a$ is prime to $p$ (see also [St05, 5.2]). The standard computation of the relative canonical sheaf (see for instance [Li02, 6.4.14]) shows that $\omega_{\mathscr{X}/R}$ is generated by $x^{1-a}y^{-b}dy$ at every point of $\mathscr{X}_s$. On the other hand, the sheaf of logarithmic differentials $\omega_{\mathscr{X}/R}(\log \mathscr{X}_s)$ is generated by $dy/y$. □

2.3. **Néron models of Jacobians.** Let $\mathscr{X}$ be a regular model of $C$, let $\mathscr{J}$ be the Néron model of $J$ and let $\mathscr{J}^0$ be its identity component. Since $C$, by assumption, has index 1, there is a natural isomorphism

$$\mathrm{Pic}^0_{\mathscr{X}/R} \cong \mathscr{J}^0$$

(cf. [BLR90]). Via this description of $\mathscr{J}^0$, it is possible to reduce many computations concerning Néron models to computations on regular models of curves, something which is often very useful. In particular, this is true in the case of base change conductors of Jacobians.

**2.4.** Let

$$f : \mathscr{X} \to S = \mathrm{Spec}\,(R)$$

be a regular model of $C$ with relative dualizing sheaf $\omega_{\mathscr{X}/S}$. Let $e_{\mathscr{J}} : S \to \mathscr{J}$ be the unit section of the Néron model of $J = \mathrm{Jac}(C)$. Recall that the module of invariant differentials

$$\omega_{\mathscr{J}/S} := e^*_{\mathscr{J}} \Omega^1_{\mathscr{J}/S}$$

is a locally free sheaf on $S$ of rank equal to $g$, the relative dimension of $\mathscr{J}/S$.

Let now $\mathscr{F}$ denote either of $\mathscr{J}$ and $\mathrm{Pic}_{\mathscr{X}/S}$. Following [LLR02], let $\mathcal{L}ie(\mathscr{F})$ be the $fppf$ Lie algebra sheaf on $S$ associated to $\mathscr{F}$. We can similarly speak of $\mathcal{L}ie(\mathscr{F}^0)$ and since the natural map

$$\mathcal{L}ie(\mathscr{F}^0) \to \mathcal{L}ie(\mathscr{F})$$

is an isomorphism by [LLR02, Prop. 1.1 (d)], these two sheaves will in what follows be identified. Finally, we write $\mathrm{Lie}(\mathscr{F}^0)$ for the restriction of $\mathcal{L}ie(\mathscr{F}^0)$ to the (usual) Zariski topology on $S$. By [LLR02, Prop. 1.1(b)] there is a canonical $\mathcal{O}_S$-module isomorphism

$$\mathrm{Lie}(\mathscr{J}^0) \to \omega^{\vee}_{\mathscr{J}/S}.$$

**Proposition 2.4.1.** *There is an $\mathcal{O}_S$-module isomorphism*

$$\alpha_{\mathscr{X}} : \omega_{\mathscr{J}/S} \to f_*\omega_{\mathscr{X}/S}.$$

*Proof.* By [LLR02, Prop. 1.3], there exists a canonical isomorphism

$$\underline{R}^1 f_* \mathcal{O}_{\mathscr{X}} \to \mathcal{L}ie(\mathrm{Pic}^0_{\mathscr{X}/S})$$

of *fppf*-sheaves of $\mathcal{O}_S$-modules. Restricting to the Zariski topology on $S$ we find an isomorphism

$$R^1 f_* \mathcal{O}_{\mathscr{X}} \to \mathrm{Lie}(\mathrm{Pic}^0_{\mathscr{X}/S}).$$

Composing with the isomorphisms

$$\mathrm{Lie}(\mathrm{Pic}^0_{\mathscr{X}/S}) \cong \mathrm{Lie}(\mathscr{J}^0) \cong \omega^{\vee}_{\mathscr{J}/S}$$

and dualizing, yields an isomorphism

$$\omega_{\mathscr{J}/S} \to \mathcal{H}om_{\mathcal{O}_S}(R^1 f_* \mathcal{O}_{\mathscr{X}}, \mathcal{O}_S).$$

Here we have identified $\omega_{\mathscr{J}/S}$ with its double dual. On the other hand, Grothendieck duality provides an isomorphism

$$\mathcal{H}om_{\mathcal{O}_S}(R^1 f_* \mathcal{O}_{\mathscr{X}}, \mathcal{O}_S) \to f_*\omega_{\mathscr{X}/S},$$

so by composition we get the desired map

$$\alpha_{\mathscr{X}} : \omega_{\mathscr{J}/S} \to f_*\omega_{\mathscr{X}/S}.$$

□

## 3. The base change conductor

**3.1. Definition of the base change conductor.** Let $A$ be an abelian $K$-variety and let $\mathscr{A}$ denote its Néron model over $R$. Let moreover $K'/K$ be a finite separable field extension of ramification index $e(K'/K)$. We let $\mathscr{A}'$ denote the Néron model of $A \times_K K'$ over $R'$, the integral closure of $R$ in $K'$. Since $\mathscr{A} \times_R R'$ is smooth and $\mathscr{A}'$ is a Néron model, there exists a unique morphism

$$h_{A,K'} : \mathscr{A} \times_R R' \to \mathscr{A}'$$

extending the canonical isomorphism of the generic fibers. We shall refer to this morphism as the *base change morphism*. For simplicity, we will usually denote it simply by $h$.

On the level of Lie algebras, the base change morphism induces an injective homorphism

$$\mathrm{Lie}(h) : \mathrm{Lie}(\mathscr{A}) \otimes_R R' \to \mathrm{Lie}(\mathscr{A}')$$

of free $R'$-modules of rank $g = \dim(A)$.

**Definition 3.1.1.** *We call the rational number*

$$c(A, K') := \frac{1}{e(K'/K)} \cdot \mathrm{length}_{R'}(\mathrm{coker}\mathrm{Lie}(h))$$

*the $K'$-base change conductor associated to $A$.*

*If $A \times_K K'$ has semi-abelian reduction over $R'$, we simply write $c(A) := c(A, K')$, and call this value the base change conductor of $A$.*

It is easily checked that the definition of $c(A)$ is independent of choice of extension $K'/K$ over which $A$ has semi-abelian reduction. For our purposes, it is important to also discuss an alternative way in which one can compute the base change conductor.

Let $K'/K$ be a finite separable extension as above. Then, pulling back the canonical map

$$\Omega^1_{\mathscr{A}'/R'} \to \Omega^1_{\mathscr{A}/R} \otimes_R R'$$

through the unit section $e_{\mathscr{A}'}$ of $\mathscr{A}'$, one obtains an injective homomorphism

$$\kappa : \omega_{\mathscr{A}'/R'} \to \omega_{\mathscr{A}/R} \otimes_R R'.$$

Then we can also compute the base change conductor as

$$c(A, K') = \frac{1}{e(K'/K)} \cdot \mathrm{length}_{R'}(\mathrm{coker}(\kappa)).$$

**3.2. Edixhoven's filtration and $c_{\mathrm{tame}}(A)$.** Edixhoven [Ed92] constructed a descending filtration $\mathcal{F}^\alpha \mathscr{A}_k$ for $\alpha \in \mathbb{Z}_{(p)} \cap [0, 1[$, where $\mathcal{F}^0 \mathscr{A}_k = \mathscr{A}_k$ and $\mathcal{F}^\alpha \mathscr{A}_k$ is a smooth connected $k$-group for each $\alpha > 0$. This filtration jumps at finitely many values $j \in [0, 1[$, by the *multiplicity* $m(j)$ we mean the drop in dimension at the jump $j$.

**Definition 3.2.1.** *The tame base change conductor of $A$ is the value*

$$c_{\text{tame}}(A) = \sum_{j \in \mathcal{J}_A} m(j) \cdot j,$$

*where $\mathcal{J}_A$ denotes the (finite) set of jumps.*

We will use the following key properties of $c_{\text{tame}}(A)$ in this paper.

*Fact 1*: If $A$ acquires semi-abelian reduction over a tamely ramified extension of $K$, then the equality $c(A) = c_{\text{tame}}(A)$ holds.

*Fact 2*: If $A$ is the Jacobian of a curve $C$, the jumps and their multiplicities only depend on the combinatorial data of the special fiber of the minimal *sncd*-model of $C$. In particular, one finds that $j \in \mathbb{Q} \cap [0, 1[$ for every $j \in \mathcal{J}_A$.

## 4. Base change conductor and Riemann-Roch theorem with supports

**4.1.** Let $h : M \to M'$ be an injective homomorphism of free $R$-modules, which is an isomorphism when tensored with $K$. Then there is an induced map $\det M \subseteq \det M'$, and the length $n$ of $\text{coker}(h)$ is the same as the $k$-dimension of the cokernel $\det M' / \det M$. Alternatively, $\det M = \pi^n \det M'$. It seems reasonable to introduce the additive notation

$$\det M' + n = \det M.$$

Hence, to understand the base change conductor for the Jacobian of a curve $C/K$, it suffices to understand the difference between $\det R^1 f_* \mathcal{O}_{\mathscr{X}}$ and $\det R^1 f_* \mathcal{O}_{\mathscr{X}'}$, where $\mathscr{X}$, resp. $\mathscr{X}'$, is a regular model of $C$, resp. $C' = C \times_K K'$, and where $K'/K$ is chosen such that $C'$ has semi-stable reduction over $R'$, the integral closure of $R$ in $K'$. Recall that the last condition is equivalent to $\text{Jac}(C')$ having semi-abelian reduction over $R'$.

For a relative curve $f : \mathscr{C} \to S$ and a line bundle $\mathscr{L}$ on $\mathscr{C}$, denote by $\lambda(\mathscr{L}) = \det Rf_* \mathscr{L}$ the determinant of the perfect complex $Rf_* \mathscr{L}$ (cf. [?]). It has the property that for any point $s \in S$, the restriction $\lambda(\mathscr{L})_s = (\det f_* \mathscr{L}|_{\mathscr{C}_s}) \otimes (\det R^1 f_* \mathscr{L}|_{\mathscr{C}_s})^\vee$. In the particular case when $\mathscr{L} = \mathcal{O}_{\mathscr{C}}$ and $f_* \mathcal{O}_{\mathscr{C}} = \mathcal{O}_S$, then

$$\lambda(\mathcal{O}_{\mathscr{C}}) = (\det f_* \mathcal{O}_{\mathscr{C}}) \otimes (\det R^1 f_* \mathcal{O}_{\mathscr{C}})^\vee = (\det R^1 f_* \mathcal{O}_{\mathscr{C}})^\vee.$$

We recall the following well known result.

**Lemma 4.1.1** (SGA 7, Exp. X, Théorème 1.13)**.** *Suppose that the fibers of $f : \mathscr{C} \to S$ are geometrically connected, and that the greatest common divisor of the lengths of the local rings of the closed points of the geometric fibers is 1. This is in particular satisfied if $C/K$ admits a zero-cycle of degree 1. Then the natural morphism $\mathcal{O}_S \to f_* \mathcal{O}_{\mathscr{C}}$ is an isomorphism and commutes with arbitrary base change.*

For two line bundles $\mathscr{L}, \mathscr{M}$ on $\mathscr{C}$, denote by

$$\langle \mathscr{L}, \mathscr{M} \rangle (\mathscr{C}/S)$$

the Deligne brackets. This is a line bundle on $S$, and is locally generated by symbols $\langle \ell, m\rangle$ (modulo some relations), where $\ell$ (resp. $m$) is a rational section of $\mathscr{L}$ (resp. $\mathscr{M}$), such that their divisors have disjoint support. It defines a bimultiplicative functor from the category of line bundles on $\mathscr{C}$, to the category of line bundles on $S$ [?]. The interest in this formulation comes from [?], which states that if the family $\mathscr{C} \to S$ is smooth, there is an isomorphism

$$\lambda(\mathcal{O}_{\mathscr{C}})^{12} \simeq \langle \omega_{\mathscr{C}/S}, \omega_{\mathscr{C}/S}\rangle,$$

and if the family is only generically smooth and $\mathscr{C}$ is regular, then

$$\lambda(\mathcal{O}_{\mathscr{C}})^{12} \simeq \langle \omega_{\mathscr{C}/S}, \omega_{\mathscr{C}/S}\rangle - \mathrm{Art}_{\mathscr{C}/S}$$

(see Proposition 4.2, [Er13], for the formulation for non-regular models $\mathscr{C}$).

**Theorem 4.1.2.** *Let $K'/K$ be a finite extension of fields, and let $\mathscr{X}$ be a regular model over $R$ of a curve $C$, and $\mathscr{X}'/S'$ be a regular model of $C'$, dominating $\mathscr{X} \times_S S'$. Denote by $\pi : \mathscr{X}' \to \mathscr{X}$ the natural morphism, and by $\Gamma = \omega_{\mathscr{X}'/S'} - \pi^*\omega_{\mathscr{X}/S}$ the discrepancy. Then the following formula holds:*

$$12\left(\lambda(\mathcal{O}_{\mathscr{X}'}) - R' \otimes_R \lambda(\mathcal{O}_{\mathscr{X}})\right) =$$

$$\Gamma^2 + 2\Gamma \cdot \pi^*\omega_{\mathscr{X}/S} - \mathrm{Art}_{\mathscr{X}'/S'} + [K' : K]\, \mathrm{Art}_{\mathscr{X}/S}.$$

*Proof.* It is immediate that the difference $12\left(\lambda(\mathcal{O}_{\mathscr{X}'}) - R' \otimes_R \lambda(\mathcal{O}_{\mathscr{X}})\right)$ is given by

$$\langle \omega_{\mathscr{X}'/S'}, \omega_{\mathscr{X}'/S'}\rangle - R' \otimes_R \langle \omega_{\mathscr{X}/S}, \omega_{\mathscr{X}/S}\rangle - \mathrm{Art}_{\mathscr{X}'/S'} + [K' : K]\, \mathrm{Art}_{\mathscr{X}/S}.$$

By functoriality of base change of the Deligne brackets, and the proof of Theorem 4.1, [Er13],

$$R' \otimes_R \langle \omega_{\mathscr{X}/S}, \omega_{\mathscr{X}/S}\rangle(\mathscr{X}/S) = \langle \pi^*\omega_{\mathscr{X}/S}, \pi^*\omega_{\mathscr{X}/S}\rangle(\mathscr{X}'/S').$$

By [Er13], if $D$ is a Cartier divisor supported on the special fiber of $\mathscr{X}$, and $\mathscr{L}$ is any line bundle on $\mathscr{X}$, then the order of the trivialization of the line bundle

$$\langle \mathcal{O}(D), \mathscr{L}\rangle(\mathscr{X}/S)$$

determined $\mathcal{O}(D)|_{\mathscr{X}_\eta} \simeq \mathcal{O}_{\mathscr{X}_\eta}$ is given by $D \cdot \mathscr{L} := \deg_D(\mathscr{L}|_D)$. The result follows. □

The above theorem reduces the problem of computing the base change conductor for Jacobians to that of understanding the Artin conductors and discrepancies, for some well chosen models.

**Proposition 4.1.3.** *Suppose that $\mathscr{C}/S$ is a regular normal crossings model of $C/K$, with no components in the special fiber with multiplicities divisible by $p$. Let $K'/K$ be an arbitrary finite separable field extension, with integer ring $S'$. Denote by $\widetilde{\mathscr{C}}$ the normalization of $\mathscr{C} \times_S S'$, and by $p : \widetilde{\mathscr{C}} \to \mathscr{C}$ the induced morphism. Then*

$$\omega_{\widetilde{\mathscr{C}}/S'} - p^*\omega_{\mathscr{C}/S} = p^*\mathscr{C}_{s,\mathrm{red}} - \widetilde{\mathscr{C}}_{s,\mathrm{red}}.$$

*Proof.* Under the assumption, the relative cotangent bundle of logarithmic differentials $\Omega_{\mathscr{C}/S}(\log/\log)$ is locally free. By [?], it is isomorphic to $\omega_{\mathscr{C}/S}(\mathscr{C}_{s,\mathrm{red}} - \mathscr{C}_s)$. The sheaf $\Omega_{\mathscr{C}/S}(\log/\log)$ commutes with base change, followed by normalization, to the effect that $p^*\Omega_{\mathscr{C}/S}(\log/\log) = \Omega_{\widetilde{\mathscr{C}}/S'}(\log/\log)$. The sheaf $\Omega_{\widetilde{\mathscr{C}}/S'}(\log/\log)$ is isomorphic to $\omega_{\widetilde{\mathscr{C}}/S'}(\widetilde{\mathscr{C}}_{s,\mathrm{red}} - \widetilde{\mathscr{C}}_s)$ for similar reasons. □

**Proposition 4.1.4.** *Keep the assumptions of the previous proposition. Then $\widetilde{\mathscr{C}}/S'$ has only canonical singularities. In particular, if*

$$\rho : \mathscr{C}' \to \widetilde{\mathscr{C}}$$

*denotes the minimal desingularization, then $\rho^*\omega_{\widetilde{\mathscr{C}}/S'} = \omega_{\mathscr{C}'/S'}$, and*

$$\omega_{\mathscr{C}'/S'} - \pi^*\omega_{\mathscr{C}/S} = \pi^*\mathscr{C}_{s,\mathrm{red}} - p^*\widetilde{\mathscr{C}}_{s,\mathrm{red}}.$$

*Proof.* Omitted. □

## 5. Relating the base change conductor and the Artin conductor

Let us denote by $\mathscr{X}/S$ a regular model of $C$ such that $\mathscr{X}_s = \sum_{i\in I} n_i E_i$ is a divisor with strict normal crossings. Recall that we have

$$\chi(\mathscr{X}_s) = \chi(\mathscr{X}_{s,red}) = \sum \chi(E_i) - \#\mathscr{X}_{s,red}^{sing} = \sum \chi(E_i) - \sum_{i<j} E_i \cdot E_j.$$

We use the notation $E_i^\circ = E_i \setminus \cup_{j\neq i} E_j$ for the open part of $E_i$ that does not meet the rest of the special fiber. The $\ell$-adic Euler characteristic of the generic fiber can be computed by the formula

$$\chi(\mathscr{X}_{\overline{\eta}}) = \sum_i n_i \chi(E_i^\circ).$$

This is a general fact for degenerations over discrete valuation rings, with normal crossing special fiber, and is essentially a consequence of the Lefschetz trace formula. Hence, the "tame" part of the Artin conductor

$$\mathrm{Art}_{\mathrm{tame}}(\mathscr{X}) = \chi(\mathscr{X}_{\overline{\eta}}) - \chi(\mathscr{X}_s)$$

can be computed entirely in terms of the special fiber.

### 5.1.

**Proposition 5.1.1.** *Let $C$ and $\mathscr{X}/S$ be as at the start of this section, and assume in addition that $(p, n_i) = 1$ for all $i \in I$. Then the following formula holds*

$$c(\mathrm{Jac}(C)) = -\frac{1}{4}\cdot \mathrm{Art}_{\mathscr{X}/S} - \frac{1}{12}\cdot \sum_{i<j} (E_i \cdot E_j)\frac{n_i^2 + n_j^2 + (n_i, n_j)^2}{n_i n_j}.$$

*Proof.* By the assumptions made, $C$ is tamely ramified. We fix an extension $K'/K$ (of sufficiently large degree) that realizes semi-stable reduction of $C$.

By the above proposition, $\Gamma = \pi^*(\mathscr{X}_{s,red} - \mathscr{X}_s)$, since $(\widetilde{\mathscr{X}_s})_{red} = \widetilde{\mathscr{X}_s}$. It follows that

$$\Gamma^2 = (\pi^* \mathscr{X}_{s,red})^2 = [K' : K]\mathscr{X}_{s,red}^2$$

and

$$\Gamma \cdot \pi^* \omega_{\mathscr{X}/S} = [K' : K]\mathscr{X}_{s,red} \cdot \omega_{\mathscr{X}/S} + [K' : K]\chi(\mathscr{X}_{\overline{\eta}}).$$

Since the component $E_i$ of $\mathscr{X}_{s,red}$ is smooth, by adjunction (cf. [?], Exp. X, Proposition 1.11) the $\ell$-adic Euler characteristic satisfies $\chi(E_i) = -E_i \cdot (E_i + \omega_{\mathscr{X}/S}) = -E_i^2 - E_i \cdot \omega_{\mathscr{X}/S}$, so

$$\mathscr{X}_{s,red} \cdot \omega_{\mathscr{X}/S} = \sum_i \left(-\chi(E_i) - E_i^2\right).$$

It follows that

$$[K' : K]^{-1} \left(\Gamma^2 + 2\Gamma \cdot \pi^* \omega_{\mathscr{X}/S}\right) =$$
$$2\sum_{i<j} E_i \cdot E_j - \sum_i E_i^2 - 2\sum_i \chi(E_i) + 2\chi(\mathscr{X}_{\overline{\eta}}).$$

It is not difficult to see that

$$E_i^2 = -\sum_{i \neq j} \frac{n_j}{n_i} E_i \cdot E_j$$

so that

$$\sum E_i^2 = -\sum_{i<j} \frac{n_i^2 + n_j^2}{n_i n_j} E_i \cdot E_j.$$

Now we compute $\mathrm{Art}_{\mathscr{X}'/S'}$. In order to do this, we use the explicit description provided in [Ha10a] of the natural map $p : \widetilde{\mathscr{X}} \to \mathscr{X}$ and the minimal desingularization $\rho : \mathscr{X}' \to \widetilde{\mathscr{X}}$.

For each $E_i$, let us denote by $\widetilde{E_i}^\circ$ the inverse image of $E_i^\circ$ under $p$. Then the induced map

$$\widetilde{E_i}^\circ \to E_i^\circ$$

is étale of degree $n_i$, hence

$$\chi(\widetilde{E_i}^\circ) = n_i \chi(E_i^\circ).$$

Consider now a point $x \in E_i \cap E_j$, where $i \neq j$. The inverse image of $x$ under $p$ consists of $(n_i, n_j)$ distinct points, each of them being a transversal intersection of distinct branches of $\widetilde{\mathscr{X}_s}$. Moreover, the formal structure of $\widetilde{\mathscr{X}}$ at any of these points is that of an $A_{n_x}$ singularity, where

$$n_x = \frac{[K' : K](n_i, n_j)}{n_i n_j}.$$

The exceptional locus of this singularity consists of a chain of $n_x - 1$ smooth rational curves $F_k$. In particular, we find that $\chi(F_k^\circ) = 0$ and that each of the $(n_i, n_j)$ preimages of $x$ give rise to $n_x$ singular points in the special fiber of $\mathscr{X}'$. Then, using the additivity of the Euler characteristic with respect to disjoint unions, we compute that

$$\chi(\widetilde{\mathscr{X}'_s}) = \sum_{i'} \chi(E'_{i'}) - |(\widetilde{\mathscr{X}'_s})^{sing}| =$$

$$\sum_i \chi(\widetilde{E_i^\circ}) + \sum_{i<j}(E_i \cdot E_j)\frac{[K':K](n_i,n_j)^2}{n_i n_j} =$$

$$\sum_i n_i \chi(E_i^\circ) + \sum_{i<j}(E_i \cdot E_j)\frac{[K':K](n_i,n_j)^2}{n_i n_j}.$$

Here $E'_{i'}$ denotes the components of $\widetilde{\mathscr{X}'_s}$. This finally allows us to conclude that

$$\mathrm{Art}_{\mathscr{X}'/S'} = -[K':K]\sum_{i<j}(E_i \cdot E_j)\frac{(n_i,n_j)^2}{n_i n_j}.$$

Now we use Theorem 4.1.2. After dividing by $[K':K]$, we find that

$$-12 \cdot c(\mathrm{Jac}(C)) =$$

$$2\sum_{i<j} E_i \cdot E_j - \sum_i E_i^2 - 2\sum_i \chi(E_i) + 2\chi(\mathscr{X}_{\overline{\eta}}) +$$

$$\sum_{i<j}(E_i \cdot E_j)\frac{(n_i,n_j)^2}{n_i n_j} + \chi(\mathscr{X}_{\overline{\eta}}) - \chi(\mathscr{X}_s) =$$

$$3\left(\chi(\mathscr{X}_{\overline{\eta}}) - \chi(\mathscr{X}_s)\right) + \sum_{i<j}(E_i \cdot E_j)\frac{n_i^2 + n_j^2 + (n_i,n_j)^2}{n_i n_j},$$

which easily yields the desired formula.

□

**Remark 5.1.2.** Observe that if $C$ has good reduction over $R$, both sides of the equation are zero. Moreover, the correction term is now expressed in terms of the singular points only, which is what one would expect in case $C$ has semi-stable reduction over $R$, but not good reduction.

We will now show that the above formula in fact holds for *any* curve $C/K$ and *any* strict normal crossings model $\mathscr{X}/S$ of $C$, provided that we replace $c(\mathrm{Jac}(C))$ and $\mathrm{Art}_{\mathscr{X}/S}$ by their tame counterparts $c_{\mathrm{tame}}(\mathrm{Jac}(C))$ and $\mathrm{Art}_{\mathrm{tame}}(\mathscr{X})$. We would like to point out that, already in the case where $C$ is tamely ramified, this does not follow directly from Proposition 5.1.1, because it may very well happen that $p$ divides $n_i$ for some $i$. (In fact, it can happen that this is the case for every *sncd*-model of $C$!) Instead, we will show that the formula can be "transported" from characteristic zero.

**Definition 5.1.3.** *For any curve $C$ and any regular model $\mathscr{X}/S$ such that the special fiber $\mathscr{X}_s$ is a normal crossings divisor, we define the* virtual number of nodes *to be the value*

$$R(\mathscr{X}) = \frac{1}{3} \cdot \sum_{x \in \mathscr{X}^{\mathrm{sing}}_{s,\mathrm{red}}} \frac{n_x^2 + n_x'^2 + (n_x, n'_x)^2}{n_x n'_x},$$

*where $n_x$ and $n'_x$ denote the multiplicities of the two formal branches of the special fiber crossing transversally at $x$.*

There are several reasons why one should think of $R(\mathscr{X})$ as a virtual number of nodes. First, $R(\mathscr{X})$ in fact frequently coincides with the number of nodes, one obvious case being the semi-stable case. However, it turns out to also be true in less obvious cases; we show in 5.2 that this property holds when $C$ has potential purely multiplicative reduction. On the other hand, it is not hard to give examples $R(\mathscr{X})$ is not the number of nodes, or even an integer. We provide several examples of this in Example 5.2.4.

Second, $R(\mathscr{X})$ behaves like the number of nodes with respects to blow ups. More precisely, if $\mathscr{X}' \to \mathscr{X}$ is the blow up of $\mathscr{X}$ in a point in $x \in \mathscr{X}_s$, it is straightforward to check that $R(\mathscr{X}') = R(\mathscr{X}) + 1$.

**Theorem 5.1.4.** *Let $C/K$ be a curve, and let $\mathscr{X}/S$ be a model of $C$ such that $\mathscr{X}_s = \sum_i n_i E_i$ is divisor with strict normal crossings. Then*

$$c_{\text{tame}}(\text{Jac}(C)) = -\frac{1}{4} \cdot (\text{Art}_{\text{tame}}(\mathscr{X}) + R(\mathscr{X})) .$$

*Proof.* Let $\Gamma(\mathscr{X})$ be the dual graph of the special fiber $\mathscr{X}_s$, and denote by $\Gamma_{\text{lab}}(\mathscr{X})$ the *labelled* dual graph, where each vertex $[E_i]$ has been labelled by the numerical data $(g(E_i), n_i)$. By [Ha10b] and [HN11], $c_{\text{tame}}(\text{Jac}(C))$ only depends on $\Gamma_{\text{lab}}(\mathscr{X})$, and we have already observed that the same is true for $\text{Art}_{\text{tame}}(\mathscr{X})$.

By a result of Winters [Wi74], we can find a smooth geometrically connected curve $B/\mathbb{C}((t))$ and a regular model $\mathscr{Y}$ of $B$ with $\mathscr{Y}_s$ a strict normal crossings divisor, such that

$$\Gamma_{\text{lab}}(\mathscr{X}) = \Gamma_{\text{lab}}(\mathscr{Y}).$$

This implies that $c_{\text{tame}}(\text{Jac}(C)) = c_{\text{tame}}(\text{Jac}(B)) = c(\text{Jac}(B))$, and that $\text{Art}_{\text{tame}}(\mathscr{X}) = \text{Art}_{\text{tame}}(\mathscr{Y}) = \text{Art}_{\mathscr{Y}/\mathbb{C}[[t]]}$. Obviously, we also have that $R(\mathscr{X}) = R(\mathscr{Y})$. By Proposition 5.1.1, we have that

$$c(\text{Jac}(B)) = -\frac{1}{4} \cdot \left(\text{Art}_{\mathscr{Y}/\mathbb{C}[[t]]} + R(\mathscr{Y})\right),$$

and the theorem follows immediately from this. □

It may also be interesting to note the following alternative version of the formula.

**Corollary 5.1.5.** *Let $u$ denote the unipotent rank of* $\text{Jac}(C)$*. Then*

$$c_{\text{tame}}(\text{Jac}(C)) =$$

$$-\frac{1}{12} \cdot \left( \text{Art}_{\text{tame}}(\mathscr{X}) - 4u - \mathscr{X}_{s,red}^2 + \sum_{i<j} (E_i \cdot E_j) \frac{(n_i, n_j)^2}{n_i n_j} \right).$$

**5.2. Curves with potentially purely multiplicative reduction.** It is natural to ask if the formula in Theorem 5.1.4 can be established also between the base change conductor $c(\text{Jac}(C))$ and the Artin conductor $\text{Art}_{\mathscr{X}/S}$ in the wildly ramified case. Next we show that this is indeed the case, when we assume that $C$ has potentially purely multiplicative reduction. As a corollary, we obtain an interesting interpretation of the correction term $R(\mathscr{X})$.

Throughout this section, $C/K$ denotes a curve with potentially purely multiplicative reduction, and $\mathscr{X}/S$ denotes an *sncd*-model of $C$, with special fiber $\mathscr{X}_s = \sum_i n_i E_i$. For simplicity, we write $J = \mathrm{Jac}(C)$. We denote by $T_\ell(J)$ the $\ell$-adic Tate module associated to $J$, and we put $V_\ell(J) = T_\ell(J) \otimes \mathbb{Q}_\ell$. The inertia group $I = \mathrm{Gal}(K^{sep}/K)$ acts on $V_\ell(J)$, we write $V$ for the semi-simplification of this representation.

**Theorem 5.2.1.** *Let $C$ and $\mathscr{X}$ be as above. Then*

$$c(\mathrm{Jac}(C)) = -\frac{1}{4} \cdot \left(\mathrm{Art}_{\mathscr{X}/S} + R(\mathscr{X})\right).$$

*Proof.* The Artin conductor of the Galois representation $V$ can be written

$$\mathrm{Art}(V) = \dim(V) - \dim(V^I) + \mathrm{Sw}(V).$$

Since $C$ has potentially multiplicative reduction, Chai's formula (add ref.) states that

$$c(J) = \frac{1}{4} \cdot \mathrm{Art}(V).$$

Rewriting slightly, we find that

$$c(J) = \frac{1}{4} \cdot (2g - (2a + 2t) + \mathrm{Sw}(H^1(C \times_K K^{\mathrm{sep}}, \mathbb{Q}_\ell)).$$

(In fact, $a = 0$ under our assumptions.) Moreover, since the toric rank $t$ only depends on the combinatorial data, it is immediate that

$$c_{\mathrm{tame}}(J) = \frac{1}{4} \cdot (2g - (2a + 2t)),$$

so that $c(J) = c_{\mathrm{tame}}(J) + \frac{1}{4} \cdot \mathrm{Sw}(H^1(C \times_K K^{\mathrm{sep}}, \mathbb{Q}_\ell))$. On the other hand, our formula above states that

$$c_{\mathrm{tame}}(J) = -\frac{1}{4} \cdot (\mathrm{Art}_{\mathrm{tame}}(\mathscr{X}) + R(\mathscr{X})),$$

and we arrive at the desired formula, since

$$\mathrm{Art}_{\mathscr{X}/S} = \mathrm{Art}_{\mathrm{tame}}(\mathscr{X}) - \mathrm{Sw}(H^1(C \times_K K^{\mathrm{sep}}, \mathbb{Q}_\ell)).$$

□

**Corollary 5.2.2.** *Let $C$ and $\mathscr{X}$ be as above. Then*

$$c(\mathrm{Jac}(C)) = -\frac{1}{4} \cdot (\mathrm{Art}_{\mathscr{X}/S} + |\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}|).$$

*In particular, $R(\mathscr{X}) = |\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}|$.*

*Proof.* We will once again use Chai's formula. Observe first that, since $t$ equals the first Betti number of $\Gamma(\mathscr{X}_s)$, we have an equality

$$t = -|I| + \sum_{i<j} (E_i \cdot E_j) + 1.$$

We can write

$$\chi(\mathscr{X}_s) = \sum_i \chi(E_i) - \sum_{i<j} (E_i \cdot E_j) = 2|I| - \sum_{i<j} (E_i \cdot E_j),$$

since $g(E_i) = 0$ for all $i \in I$, by our assumption of potential multiplicative reduction.

Now Chai's formula yields

$$\frac{1}{4}\cdot(2g-2(-|I|+\sum_{i<j}(E_i\cdot E_j)+1))=$$

$$-\frac{1}{4}\cdot\left(2-2g-(2|I|-\sum_{i<j}(E_i\cdot E_j))+R(\mathscr{X})\right).$$

After performing some cancellations, this expression reduces to

$$|\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}|=\sum_{i<j}(E_i\cdot E_j)=R(\mathscr{X}),$$

and the formula in the assertion is established.

□

Note that Corollary 5.2.2 yields an interesting necessary condition for a curve $C$ to have potential purely multiplicative reduction: For any *sncd*-model $\mathscr{X}$, $R(\mathscr{X})$ should equal the number of nodes of $\mathscr{X}_{s,\mathrm{red}}$.

**Remark 5.2.3.** If we also assume that $C$ is tamely ramified in the above corollary, it is not hard to see that the formula implies that $c(J)=u/2$, with $u$ being the unipotent rank of $J$.

The following example shows that, even in the tamely ramified case, one should not expect that the formula in Corollary 5.2.2 holds without the assumption that $C$ has potentially purely multiplicative reduction.

**Example 5.2.4.** In this example, we assume that $C$ is a tamely ramified elliptic curve, and that we are in one of the following cases: $C$ has reduction type (a) $II^*$, resp. (b) $III^*$, resp. (c) $IV^*$. In each of these cases, the minimal regular model is an *sncd*-model, we denote it by $\mathscr{X}$. Then it is straightforward to compute that (a) $R(\mathscr{X})=6+2/3$ and $|\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}|=8$, resp. (b) $R(\mathscr{X})=6$ and $|\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}|=7$, resp. (c) $R(\mathscr{X})=5+1/3$ and $|\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}|=6$.

In all these examples, we see that $R(\mathscr{X})$ does not equal the number of nodes in the special fiber, moreover, it need not even be an integer.

**5.3. Relation to Saito's minimal discriminant.** Let $C/K$ be a curve, and denote by $\mathscr{C}/S$ its minimal regular model. Following Saito [Sa88], the correct notion of minimal discriminant for $C$ is

$$\Delta(C)_{\min}=-\operatorname{Art}_{\mathscr{C}/S}.$$

Our results allow us compare the base change conductor and the minimal discriminant, at least in the case where $\mathscr{C}/S$ is a model with normal crossings.

**Corollary 5.3.1.** *Assume that the minimal regular model $\mathscr{C}/S$ of $C$ has normal crossings. Assume moreover that either $C$ is tamely ramified, or that $C$ has potential purely multiplicative reduction. Then*

$$c(J)=\frac{1}{4}\cdot(\Delta(C)_{\min}-R(\mathscr{C})).$$

*Proof.* Let $\mathscr{X}$ the minimal *sncd*-model of $C$, and consider the unique morphism $\mathscr{X} \to \mathscr{C}$ which blows up the "internal" nodes in the special fiber of $\mathscr{C}$ (i.e., nodes belonging to a unique irreducible component). Denote the number of internal nodes by $\delta$. Then $R(\mathscr{X}) = R(\mathscr{C}) + \delta$, and since the base change conductor is invariant with respect to the choice of regular model, it only remains to compute the difference $\mathrm{Art}_{\mathscr{C}/S} - \mathrm{Art}_{\mathscr{X}/S}$, or, what amounts to the same, the difference $\chi(\mathscr{X}_s) - \chi(\mathscr{C}_s)$. It is easily checked that this value equals $\delta$.

□

**5.4. Curves with potential good reduction.** Throughout this section we assume that $C$ is a tamely ramified curve with potential good reduction.

**5.4.1.** *The quotient construction.* Let $L/K$ be a tame extension such that $C \times_K L$ has good reduction over $R_L$. Then $G = \mathrm{Gal}(L/K)$ acts on the smooth model $\mathscr{Y}/R_L$, and the quotient $\mathscr{Z} := \mathscr{Y}/G$ is a normal $R$-model of $C$. Moreover, $\mathscr{Z}$ has tame cyclic quotient singularities at each of its (finitely many) singular points, which we denote by $Q_1, \ldots, Q_r$. These singularities can be resolved explicitly. Let

$$\rho : \widetilde{\mathscr{Z}} \to \mathscr{Z}$$

denote the minimal desingularization. Then $\widetilde{\mathscr{Z}}$ is an *sncd*-model of $C$, and the special fiber can be described as follows. Let $F_0$ denote the strict transform of $\mathscr{Z}_s$ (which is irreducible). For each $Q_j$, the exceptional locus $\rho^{-1}(Q_j)$ is a chain $\mathcal{F}_j$ of smooth rational curves $F_1^j, \ldots, F_{l_j}^j$, with $F_1^j$ intersecting $F_0$ transversely in a unique point.

In combinatorial terms, one can formulate this by saying that $\Gamma(\widetilde{\mathscr{Z}})$ is *star shaped*, i.e. it has a unique node, corresponding to the irreducible component $F_0$, and otherwise terminal chains attached to that node.

We make a simple observation:

**Lemma 5.4.2.** *Let us assume that $g = g(C) > 0$. Then $\widetilde{\mathscr{Z}}$ coincides with the minimal sncd-model $\mathscr{X}$ of $C$.*

*Proof.* By minimality of $\rho$, all exceptional components have self intersection $\leq -2$, hence are not contractible. By the assumption that $g = g(C) > 0$, $F_0$ must be a principal component [Ha10a], i.e., either $g(F_0) > 0$ or $r > 2$. In the former case, $F_0$ can never be contracted, and in the latter case, if $F_0$ is contractible on $\widetilde{\mathscr{Z}}$, the contracted scheme is not an *sncd*-model. □

Let us write $\mathscr{X}_s = \sum_{i \in I} n_i E_i$ for the special fiber of $\mathscr{X} (\cong \widetilde{\mathscr{Z}})$, and let $\mathscr{X}'$ be the normalization of $\mathscr{X} \times_R R_L$. Previously, we computed that

$$[L:K]^{-1} \cdot \mathrm{Art}_{\mathscr{X}'/R_L} = -\sum_{i<j} (E_i \cdot E_j) \frac{(n_i, n_j)^2}{n_i n_j}.$$

Under the assumption of tame potential good reduction, we will now provide another interpretation of this formula, in terms of intersection theory on $\mathscr{X}$.

**5.4.3.** Let us fix one of the chains $\mathcal{F}_j$. For simplicity, we drop reference to the index $j$. The chain has components $F_1, \ldots, F_l$, we'll denote by $m_i$ the multiplicity of $F_i$. It is easy to check that $(m_{n-1}, m_n) = m_l$ for all $1 \leq n \leq l$.

For each $1 \leq n \leq l$ we put

$$t_n := \sum_{i=1}^{n} \frac{1}{m_{i-1}m_i},$$

and define a divisor (with rational coefficients)

$$D_{\mathcal{F}} = \frac{m_l}{m_0} \cdot \sum_{n=1}^{l} m_n t_n F_n.$$

By the results in [BL02] (there is a sign error in their paper), computing the self intersection of this element, we find that

$$D_{\mathcal{F}} \cdot D_{\mathcal{F}} = -m_l^2 \cdot t_l = -m_l^2 \cdot (\frac{1}{m_0 m_1} + \ldots + \frac{1}{m_{l-1}m_l}).$$

For notational reasons, we put $t_{\mathcal{F}} = t_l$, so that $D_{\mathcal{F}} \cdot D_{\mathcal{F}} = -m_l^2 \cdot t_{\mathcal{F}}$.

**5.4.4.** Let $\mathbb{Z}^{|I|}$ be the free module with generators corresponding to the irreducible components $E_i$ of $\mathscr{X}_s$. Let $\mathcal{M} = (E_i \cdot E_j)$ be the intersection matrix, and let $\mathcal{R}^t = (\ldots, n_i, \ldots)$ be the multiplicity vector. In particular,

$$\Phi(J) = \mathrm{Ker}(\mathcal{R}^t)/\mathrm{Im}(\mathcal{M}).$$

For each chain $\mathcal{F}$, consider the vector

$$E(F_0, F_l) := (0, \ldots, 1, 0, \ldots, 0, \frac{-m_0}{m_l}, 0, \ldots, 0),$$

where the first non-zero coefficient is in the position corresponding to $F_0$, and the second in the position corresponding to $F_l$. Then $E(F_0, F_l)$ belongs to the kernel of multiplication by $\mathcal{R}^t$, and descends to give an element in the component group $\Phi(J)$. We denote this element by $\gamma_{\mathcal{F}}$. We also denote by

$$<;>: \Phi(J) \times \Phi(J) \to \mathbb{Q}/\mathbb{Z}$$

Grothendieck's canonical pairing. It is proved in [BL02, Prop. 5.1] that, modulo $\mathbb{Z}$, the equality

$$< \gamma_{\mathcal{F}}; \gamma_{\mathcal{F}} >= -m_0^2 \cdot t_{\mathcal{F}}$$

holds (note that there is a sign error in their paper).

Combined with our previous results, this discussion gives an interesting relation between the base change conductor and data concerning the component group. Before we state our result, a word on notation. We write $\gamma_j$ instead of $\gamma_{\mathcal{F}_j}$.

**Proposition 5.4.5.** *With notation as above, the formula*

$$12c(J) = \sum_j (m_{l_j}/m_0)^2 < \gamma_j; \gamma_j >$$

*holds modulo* $\mathbb{Z}$. *Here* $j$ *runs over the number of terminal chains.*

*Proof.* This is clear when one combines the discussion above with the formula in Corollary 5.1.5. □

**5.4.6.** Now let's return to the the divisors $D_{\mathcal{F}_j}$ we constructed above. Since every two distinct chains $\mathcal{F}_j$ and $\mathcal{F}_{j'}$ are disjoint, we find that

$$D_{\mathcal{F}_j} \cdot D_{\mathcal{F}_{j'}} = 0$$

whenever $j \neq j'$. Therefore, if we define $D := D_{\mathcal{F}_1} + \ldots + D_{\mathcal{F}_r}$, it follows that

$$D \cdot D = -\sum_{i<j} (E_i \cdot E_j) \frac{(n_i, n_j)^2}{n_i n_j}.$$

Using the formula in Corollary 5.1.5, we find that

$$c(J) = -\frac{1}{12} \cdot \left( \mathrm{Art}_{\mathscr{X}/S} - 4u - \mathscr{X}_{s,red}^2 - D \cdot D \right).$$

**5.4.7.** We make one further assumption, called $(*)$, in order to reduce somewhat the complexity of our problem. (Hopefully this assumption can be removed or weakened at a later point.)

$(*)$ *We assume that all exceptional components $E$ of the minimal desingularization $\rho$ have self intersection $E^2 = -2$.*

**Lemma 5.4.8.** *Assume that $(*)$ holds. Then $\widetilde{\mathscr{X}}$ coincides also with the minimal regular model $\mathscr{X}_{\min}$ of $C$.*

*Proof.* Let $\mathcal{E}$ be one of the exceptional chains, with components $E_1, \ldots, E_l$, where $E_l$ intersects $F$. We denote by $N_i$ the multiplicity of $E_i$, and to get easy notation, we put $E_{l+1} := F$. Let us first observe the easy fact that the sequence of multiplicities $N_1, \ldots, N_l, N_{l+1}$ is strictly increasing. Indeed, if we put $b_i = -E_i^2$ for $1 \leq i \leq l$, we first find that $b_1 N_1 = N_2$ (recall also that $b_i = 2$ for all $j$ by assumption). This gives $N_{i+1} = b_i N_i - N_{i-1} > N_i$, since, by induction $N_i > N_{i-1}$.

In particular, let us write $N$ for the multiplicity of $F$. Then, for every $1 \leq j \leq r$, we observe that the bound

$$N_{l_j} < N = N_{l_j+1} < 2N_{l_j}$$

holds. Intersecting $F$ with the special fiber yields the formula

$$-F^2 = \frac{1}{N} \sum_{j=1}^{r} N_{l_j}.$$

Let us choose $j_0$ so that $N_0 := N_{l_{j_0}}$ is minimal. Then we find that

$$-F^2 \geq \frac{1}{N} r N_0 \geq \frac{1}{N} 3N_0 > \frac{3N_0}{2N_0} = \frac{3}{2}.$$

Consequently, $-F^2 \geq 2$, and thus $F$ is not contractible. This finishes the proof. $\square$

**5.4.9.** *The correction term.* We will now compute the correction term

$$\mathcal{E}rr := -\sum_i E_i^2 + \sum_{i<j}(E_i \cdot E_j)\frac{(n_i,n_j)^2}{n_i n_j}.$$

In fact, we will compute a contribution $\mathcal{E}rr(\mathcal{E}_j)$ for each chain $\mathcal{E}_j$, so that

$$\mathcal{E}rr = -F^2 + \sum_{j=1}^{r} \mathcal{E}rr(\mathcal{E}_j).$$

Let $\mathcal{E}$ be a chain, with components $E_1, \ldots, E_l$, and write $E_{l+1} = F$. In this situation, we put

$$\mathcal{E}rr(\mathcal{E}) = -\sum_{i=1}^{l} E_i^2 + \sum_{i=1}^{l} \frac{(N_i, N_{i+1})^2}{N_i N_{i+1}}.$$

Observe that $-\sum_{i=1}^{l} E_i^2 = 2l$, so the difficult part is to compute the other term. For this, we heave the following lemma.

**Lemma 5.4.10.** *The error term associated to $\mathcal{E}$ equals*

$$\mathcal{E}rr(\mathcal{E}) = 2l + \frac{l}{l+1}.$$

*Proof.* It is easily seen that $(N_i, N_{i+1}) = N_1$ for all $1 \leq i \leq l$. Moreover, one also checks easily that $N_i = iN_1$. It then follows that

$$\sum_{i=1}^{l} \frac{(N_i, N_{i+1})^2}{N_i N_{i+1}} = N_1^2 \sum_{i=1}^{l} \frac{1}{iN_1(i+1)N_1} = \sum_{i=1}^{l} \frac{1}{i(i+1)}.$$

Then use

$$\sum_{i=1}^{l} \frac{1}{i(i+1)} = \frac{l}{l+1}.$$

□

**Proposition 5.4.11.** *The correction term equals*

$$\mathcal{E}rr = -2F^2 + 2|\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}|$$

*Proof.* For each $1 \leq j \leq r$, we write $E_{l_j}^{(j)}$ for the component of $\mathcal{E}_j$ intersecting $F$ and we denote by $N_{l_j}^{(j)}$ its multiplicity. By what we have seen, we can then write

$$\frac{N_{l_j}^{(j)}}{N} = \frac{l_j}{l_j+1}.$$

Now, intersecting $F$ with the special fiber yields the equality

$$-F^2 = \frac{1}{N}\sum_{j=1}^{r} N_{l_j}^{(j)},$$

hence we get that

$$-F^2 = \sum_{j=1}^{r} \frac{l_j}{l_j+1}.$$

Summing up everything, we find

$$\mathcal{E}rr = -F^2 + \sum_{j=1}^{r}(2l_j + \frac{l_j}{l_j+1}) =$$

$$-F^2 + 2|\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}| - F^2,$$

which gives the formula we are after. □

Combining everything, we arrive at the following result.

**Theorem 5.4.12.** *Let $C$ be a curve with potential good reduction, and assume that $(*)$ holds. Then the base change conductor and the Artin conductor are related by the formula*

$$c(J) = -\frac{1}{4}\cdot(\mathrm{Art}_{\mathscr{X}/S} + \frac{2}{3}(|\mathscr{X}_{s,\mathrm{red}}^{\mathrm{sing}}| - F^2)).$$

## References


[BL02] S. Bosch, D. Lorenzini. Grothendieck's pairing on component groups of jacobians. *Invent. Math.*, 148 (2002), 353-396.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models.* Volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1990.

[Ch00] C.-L. Chai. Néron models for semiabelian varieties: congruence and change of base field. *Asian J. Math.*, 4(4):715–736, 2000.

[CY01] C.-L. Chai and J.-K. Yu. Congruences of Néron models for tori and the Artin conductor (with an appendix by E. de Shalit). *Ann. Math. (2)*, 154:347–382, 2001.

[Ed92] B. Edixhoven. Néron models and tame ramification. *Compos. Math.*, 81:291–306, 1992.

[Er13] D. Eriksson: Discriminants and Artin conductors, preprint, http://www.math.chalmers.se/ dener/Multiplicity-of-discriminants.pdf

[Ha10a] L.H. Halle. Stable reduction of curves and tame ramification. *Math. Zeit.*, 265(3): 529–550, 2010.

[Ha10b] L.H. Halle. Galois actions on Néron models of Jacobians. *Ann. Inst. Fourier*, 60(3):853–903, 2010.

[HN11] L.H. Halle and J. Nicaise. Motivic zeta functions of abelian varieties, and the monodromy conjecture. *Adv. Math.*, 227:610–653, 2011.

[Ka94] K. Kato. Toric singularities. *Am. J. Math.*, 116(5):1073–1099, 1994.

[Li02] Q. Liu. *Algebraic geometry and arithmetic curves.* Volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, 2002.

[LLR02] Q. Liu, D. Lorenzini, and M. Raynaud. Néron models, Lie algebras, and reduction of curves of genus one. *Invent. Math.*, 157(3):455–518, 2004.

[Lo00] D. Lorenzini. Reduction of points in the group of components of the Néron model of a jacobian. *J. reine angew. Math.*, 527:117–149, 2000.

[Sa88] T. Saito. Conductor, discriminant, and the Noether formula of arithmetic surfaces. *Duke Math. J.*, 57(1):151-173, 1988.

[St05] J. Stix. A logarithmic view towards semistable reduction. *J. Algebraic Geom.*, 14(1):119–136, 2005.

[Wi74] G. B. Winters. On the existence of certain families of curves. *Am. J. Math.*, 96:215–228, 1974.