Discriminants and Artin conductors

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Abstract

We study questions of multiplicities of discriminants for degenerations coming from projective duality over discrete valuation rings. The main observation is a type of discriminant-different formula in the sense of classical algebraic number theory, and we relate it to Artin conductors via Bloch's conjecture. In the case of discriminants of planar curves we can calculate the different precisely. In general these multiplicities encode topological invariants of the singular fibers and in the case of characteristic p, also wild ramification data in the form of Swan conductors. This builds upon results of T. Saito.

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1 Introduction and statements of results

The theory of discriminants is an old field which was recently reanimated by the beautiful work of Gelfand-Zelevinsky-Kapranov (cf. [GKZ94]). This article concerns questions about the multiplicities of these discriminants along discrete valuation rings.
A motivating example of this article is the following. Let E be an elliptic curve over a discretely valued field K with ring of integers R which is henselian with algebraically closed residue field given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The discriminant of the equation is, for $a_1 = a_3 = a_2 = 0$, given by $\Delta = -16(4{a_4}^3 + 27{a_6}^2)$. Given a minimal Weierstrass model $\mathcal{W}$, over R so that $a_i \in R$, a famous formula of Ogg ([Ogg67], also see [Sai88] for a more general result and whose proof also repairs a gap in mixed characteristic $(0, 2)$ in the original article) states that for this minimal Weierstrass model we have

$$\operatorname{ord} \Delta = \deg c_{2,\mathcal{E}}^{\mathcal{E}_s}(\Omega_{\mathcal{E}/S}) = -\operatorname{Art}_{\mathcal{E}/S} = m_E - 1 + f_E$$

where m_E is the number of irreducible components in the Néron model $\mathcal{E}$ (resp. f_E the exponent of the conductor) of E over R. See below for the other two terms. From the point of view of computing the conductor, this formula is very powerful since Tate's algorithm can be implemented on a computer and actually allows us to find the minimal Weierstrass model. In [Tat74], p. 192, Tate asks about Ogg's formula: "It would be interesting to know what is behind this mysterious equality". Another famous formula is the Führer-Diskriminanten-produkt formula in classical number theory which relates the discriminant of a finite extension of local fields to the conductor. They are related through the different by:

$$\operatorname{ord} \Delta_{L/K} = \operatorname{ord} \operatorname{Norm}_{L/K}(\delta_{L/K}) = \operatorname{Art}_{L/K}$$

Both of these formulas relate different ways of measuring singularities, the discriminant is somewhat of a geometric object, whereas the conductor is an object built out of monodromy. The different in turn is an object constructed out of the Kähler differentials (in the classical context, this is [Ser80], III Proposition 14).
It seems that the first one to consider this connection in higher dimension was Deligne in [SGA7-2], Exposé XVI, where he conjectures an

equality between the Milnor number of a point and the (total) dimension of the vanishing cycles in the same point. In [Blo87], Bloch then conjectured a relation between a localized Chern class (see next section for the definition) and the Artin conductor, for regular schemes over a discrete valuation ring, which would correspond to the total dimension of the vanishing cycles and the different in classical number theory. He proved it in relative dimension one. The statement is that if X is regular and $X \to S$ is a flat generically smooth projective morphism of relative dimension n with, S is the spectrum of a discrete valuation ring with generic (resp. special) point η (resp. s, with perfect residue field $k(s)$ of characteristic $p \geq 0$), then

$$\deg c_{n+1,X}^{X_s}(\Omega_{X/S}) = (-1)^n \operatorname{Art}_{X/S}$$

where

$$\operatorname{Art}_{X/S} := \chi_\ell(X_{\overline{\eta}}) - \chi_\ell(X_{\overline{s}}) + \sum_{q=0}^{2d}(-1)^q \operatorname{Sw} H^q(X_{\overline{\eta}}, \mathbb{Q}_\ell)$$

where $\overline{s}$ and $\overline{\eta}$ are used to denote algebraic closures of the fields, χ_ℓ denotes ℓ-adic Euler-characteristic for $\ell \neq p$ and Sw denotes the Swan conductor of the natural Galois representation acting on the various cohomology groups (cf. introduction of [KS04], [Ser70], 2.1 or [Ser80], chapter VI, for a general discussion on conductors). The major breakthrough in this field was the article [KS04] which proved this relation in full generality (assuming resolution of singularities).

The aim of the current article is to find computational formulas for the order of the discriminant. A partial aim is to connect this to Bloch's conjecture recalled above, which is probably well-known to specialists but which I have not been able to locate in the literature. This amounts to finding relations between the "different", i.e. the localized Chern class mentioned above, and the order of vanishing of the discriminant in the sense of projective duality. This is essentially Porteous' formula.

We recall the setting. Let k be a field and X a smooth geometrically integral variety of dimension $n+1$ over k with a fixed k-embedding $X \subseteq \mathbb{P}^M$. Suppose furthermore that the image of X is non-degenerate. Then we define the discriminant variety (or dual variety) as the subvariety $\Delta_X \subseteq \check{\mathbb{P}}^M$ defined by all the hyperplanes $H \in \check{\mathbb{P}}^M$ such that $X \cap H$ is singular. The variety $\{(x, H) \in X \times \check{\mathbb{P}}^M, x \in (X \cap H)_{\text{sing}}\}$ of singular hyperplanes is the projective bundle $\mathbb{P}(N)$ over X where N is the normal bundle of X in $\mathbb{P}^M$. The map $\mathbb{P}(N) \to \check{\mathbb{P}}^M$ sending (x, H) to H has schematic image Δ_X and the map $\varphi : \mathbb{P}(N) \to \Delta_X$ is called the Gauss morphism. The tautological hyperplane $\mathcal{H}$ in $X \times \check{\mathbb{P}}^M$

is naturally a family over $\check{\mathbb{P}}^M$. Their relations are summed up in the commutative diagram

$$\begin{tikzcd} & \mathbb{P}(N) \arrow[r] \arrow[dl] \arrow[d, "\varphi"] & \mathcal{H} \arrow[r] \arrow[d, "f"] & X \times \check{\mathbb{P}}^M . \arrow[dl, "p"] \\ X & \Delta_X \arrow[r] & \check{\mathbb{P}}^M & \end{tikzcd} \tag{1}$$

In this article we will be mainly concerned with the case when the variety Δ_X is a hypersurface so that the Gauss morphism is proper and generically finite. Then Δ_X is defined by a homogenous polynomial defined up to an element in k^*. When $X = \mathbb{P}^n$ and the projective embedding is the d-th Veronese embedding we can also make a similar construction over the integers. More precisely, for any homogenous polynomial F of degree d in $n+1$ variables with coefficients in a ring R, there is an element $\Delta_F \in R$, given by a universal polynomial in the coefficients of F, which is invertible in R if and only if F defines a smooth hypersurface in $\mathbb{P}^n_R$ (attributed to Demazure in [Sai12], section 2. See the same for a summary of the theory including further references on the topic of discriminants of polynomials, and their precise relations to dual varieties as described above).
The main results are the following. Over a field k, we calculate the localized Chern class of the tautological family of hypersurfaces over $\check{\mathbb{P}}^M$. As an application, we prove the following multiplicity formula for a degenerating family of hypersurface sections:

Proposition 1.1. *[Discriminant-Different formula] Let X be as above and suppose Δ_X is a hypersurface. Also suppose we are given a discrete valuation ring R with spectrum S, and a morphism $\pi : S \to \check{\mathbb{P}}^M$ such that the image is not contained in the discriminant. Denote by H the pullback of the tautological hyperplane section $\mathcal{H} \to \check{\mathbb{P}}^M$, by H_s the special fiber and by $\pi^*\Delta_X$ the pullback of Δ_X to S. Then, for the discrete valuation v on R,*

$$v(\pi^*\Delta_X) \deg \varphi = \deg c^{H_s}_{n+1,H}(\Omega_{H/S}).$$

We also provide the same type of formula for the Deligne discriminant introduced in a letter [Del85] and studied in [Sai88], and in particular remark that this discriminant is not always a discriminant but sometimes the power of the discriminant (Proposition 3.1). The proof is by a global computation followed by a standard specialization argument. We include details since it involves intersection theory over discrete valuation rings.
Part of the purpose of this article is to ask how to compute the localized Chern class in the case of non-regular total spaces. We give a

precise answer using Deligne's discriminant (Theorem 1.4 below) for the case of families of curves. This is one of the main ideas in [Sai88] and the results can be seen as a refinement of those results.
For homogenous polynomial equations and related discriminants we can give a result valid over general discrete valuation rings, possibly of mixed characteristics. Given a homogenous polynomial in $n+1$ variables with coefficients in R, this defines a morphism $\pi : S \to \check{\mathbb{P}}^M_{\mathbb{Z}}$ as above, where we think about the latter space as parameterizing such polynomials. We denote as before H the pullback of $\mathcal{H}$ along π. Writing $\Delta_F = \pi^*\Delta_{d,n}$, we have:

Theorem 1.2. *Suppose that F is a homogenous polynomial of degree d in $n+1$ variables with coefficients in a discrete valuation ring R. For the classical discriminant Δ_F, we have*

$$v(\Delta_F) = \deg c^{H_s}_{n+1,H}(\Omega_{H/S})$$

The interest of these formulas is in the connection with the Bloch's conjecture recalled above, since this computes the right hand side in case H is regular. Precisely, assuming resolution of singularities we find, using [KS04], in any of the above situations:

Corollary 1.3. *Suppose that H is regular and that R has perfect residue field. Then*

$$v(\pi^*\Delta)\deg\varphi = (-1)^n \operatorname{Art}_{H/S}.$$

Remark 1.3.1. In particular, if $\mathbb{P}^1$ is a general line through $P \in \Delta_X$, the above multiplicity is then the definition of the intrinsic multiplicity of the point P in Δ_X which was studied over the complex numbers in [Dim86], [Par91], [Nem88] where formulas in terms of (generalized) Milnor numbers was given, and general formulas in terms of Segre classes was given in [AC93]. In characteristic p, if we assume that the singularities are isolated, a general line through P will define a smooth total space over a pencil, and the formula of Deligne [SGA7-2], Exposé XVI, Proposition 2.1 can be used together with general geometry of pencils to prove that the multiplicity of the discriminant is the total Milnor number in the sense of *idem*. The more general result when the total space is regular around the singular fibers is covered by Bloch's conjecture.

The following theorem computes Deligne's discriminant (cf. section 3 for the definition) for a family of curves.

Theorem 1.4. *Suppose $X \to S$ is a flat projective local complete intersection morphism, with geometrically connected fibers of dimension* *one and* $S = \mathrm{Spec}\ R$ *the spectrum of a discrete valuation ring with perfect residue field. Let* $\Delta_{Del,X/S}$ *be the Deligne discriminant. Then*

$$v(\Delta_{Del,X/S}) = \chi_\ell(X_{\overline{s}}) - \chi_\ell(X_{\overline{\eta}}) + \mathrm{Sw}\, H^1(X_{\overline{\eta}}, \mathbb{Q}_\ell) + \sum_{x \in \pi_0(X^{non-reg})} \mu_{X,x}.$$

where π_0 *denotes the connected components and* $\mu_{X,x}$ *is an invariant of the surface singularity (see definition 1 in section 4).*

In the case (X, x) is also an isolated surface singularity defined by the zero of a holomorphic function f in $\mathbb{C}^3$, $\mu_{X,x}$ is equal to the Milnor number of (X, x) by a formula of Laufer [Lau77] (a similar statement is proved more generally in [Wah81] and [Ste83]). In the special case that X is defined by a ternary form with coefficients in R, the above gives the multiplicity of the associated discriminant.
In the complex geometric setting, when the special fiber has one component with isolated singularities, this resembles the formula in Proposition 1.2, Chapitre II of [Tei73]. In the pure characteristic p situation, with X regular with reduced special fiber, a version of this result can also be found in [Zin77] for discriminants of versal deformations.
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2 Discriminants and localized Chern classes

For convenience of the reader we recall the following construction of [Blo87]. Suppose that Y is an integral scheme of finite type over a regular scheme S. Let $\mathcal{E} = [E \to E']$ be a two-term complex of vector bundles whose map is injective and whose cokernel is locally free of rank n outside of a closed subscheme Z. Let $\mathrm{CH}_i(Y)$ denote the i-th Chow group of Y over S of algebraic cycles of dimension i modulo rational equivalence as in [Ful98], chapter 20.1. Then there is a bivariant class, called the localized Chern class, for any $m > n$, which among other things induces a homomorphism $c^Z_{m,Y}(\mathcal{E}) \cap : \mathrm{CH}_i(Y) \to \mathrm{CH}_{i-m}(Z)$.

This homomorphism only depends on the quasi-isomorphism class of $\mathcal{E}$. Now, let Y be a pure-dimensional integral scheme and $f : Y \to S$ be a flat projective local complete intersection morphism of constant relative dimension n which is generically smooth. Picking any factorization of this morphism $Y \hookrightarrow \mathbb{P}^M_S \to S$ the cotangent bundle $\Omega_{Y/S}$ has a 2-term resolution as above and outside of the f-singular locus $i : Z \subseteq Y$ it is locally free of rank n. We then in particular have an element $c^Z_{n+1,Y}(\Omega_{Y/S}) := c^Z_{n+1,Y}(\Omega_{Y/S}) \cap [Y] \in \mathrm{CH}_{\dim Y - n-1}(Z)$. It has the property that $i_* c^Z_{n+1,Y}(\Omega_{Y/S}) = c_{n+1}(\mathcal{E}) \in \mathrm{CH}_{\dim Y-n-1}(Y)$. This moreover coincides with the "localized top Chern class" in [Ful98].
We consider next a smooth geometrically integral projective variety $X \subseteq \mathbb{P}^M$, with non-degenerate image, of dimension $n+1$ such that Δ_X is a hypersurface in $\check{\mathbb{P}}^M$.

Proposition 2.1. *Let $\varphi : \mathbb{P}(N) \to \Delta_X$ be the Gauss morphism. Then*

$$\varphi_* c^{\mathbb{P}(N)}_{n+1,\mathcal{H}}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M}) = \deg \varphi \cdot [\Delta_X]$$

in $\mathrm{CH}_{M-1}(\Delta_X) = \mathbb{Z} \cdot [\Delta_X]$.

The following proposition is equivalent, and is a consequence of Porteous' formula. The below proof which we have chosen is by computing degrees of the discriminant variety which seems more direct to the author.

Proposition 2.2. *[[Kle77], III.8] Let $X \subseteq \mathbb{P}^M$ be such that Δ_X is a hypersurface. Consider the tautological hyperplane section $\mathcal{H}$ over $\check{\mathbb{P}}^M$. Then*

$$c^{\mathbb{P}(N)}_{n+1,\mathcal{H}}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M}) = [\mathbb{P}(N)] \in \mathrm{CH}_{M-1}(\mathbb{P}(N)) = \mathbb{Z} \cdot [\mathbb{P}(N)].$$

Proof. (of Proposition 2.1) We suppose that Δ_X is a hypersurface. Δ_X is integral so it is obvious that

$$\varphi_* c^{\mathbb{P}(N)}_{n+1,\mathcal{H}}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M}) = c[\Delta_X]$$

for some integer c. To determine c, it suffices to calculate the class of $\varphi_* c^{\mathbb{P}(N)}_{n+1,\mathcal{H}}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M})$ in $\mathrm{CH}_{M-1}(\check{\mathbb{P}}^M) = \mathrm{Pic}(\check{\mathbb{P}}^M) = \mathbb{Z}$, where the map is the natural one sending Δ_X to $\deg \Delta_X$. Denote by L and L' the natural tautological line bundles $\mathcal{O}(1)$ on X and $\check{\mathbb{P}}^M$. Then $\mathcal{H}$ is cut out by the section of $L \otimes L'$ on $X \times \check{\mathbb{P}}^M$ determined by the dual of $(L \otimes L')^{-1} \to \mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}$ where $\mathcal{E} = \mathcal{O}^M$. To compute c, we simply compute the class $p_* i_* c^{\mathbb{P}(N)}_{n+1,\mathcal{H}}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M}) = p_* c_{n+1}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M})$ in $\mathrm{Pic}(\check{\mathbb{P}}^M)$. We use the resolution

$$0 \to L \otimes L'|^{-1}_{\mathcal{H}} \to \Omega_X|_{\mathcal{H}} \to \Omega_{\mathcal{H}/\check{\mathbb{P}}^M} \to 0$$

and obtain by the Whitney sum formula:

$$c_{n+1}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M})) = \sum c_{n+1-i}(\Omega_X)c_1(L\otimes L')^i \cap [\mathcal{H}].$$

Since $c_1(L\otimes L')\cap[X\times\check{\mathbb{P}}^M] = [\mathcal{H}]$, we have

$$\sum c_{n+1-i}(\Omega_X)c_1(L\otimes L')^i\cap[\mathcal{H}] = \sum c_{n+1-i}(\Omega_X)c_1(L\otimes L')^{i+1}\cap[X\times\check{\mathbb{P}}^M].$$

Binomial expanding $c_1(L\otimes L')^{i+1} = (c_1(L)+c_1(L'))^{i+1}$ we see that the only terms that will give a contribution after applying pushforward are the terms of the form $c_{n+1-i}(\Omega_X)(i+1)c_1(L)^i c_1(L')$. After rewriting we obtain that the class is given by a generic hyperplane times the number

$$(-1)^{n+1}\int_X \frac{c(T_X)}{(1+c_1(L))^2}$$

where $c(E) = 1+c_1(E)+c_2(E)+\dots$ is the total Chern class. This was calculated by Katz in [SGA7-2], XVII, Corollaire 5.6,, and is equal to $\deg\varphi\deg\Delta_X$ (correcting the typo in *idem*, the term $(-1)^r$ should be $(-1)^{\dim X}$)). Also see [GKZ94], Chapter 2, Theorem 3.4, for a more suggestive formulation.

□

Notice that even for a surface X the Gauss morphism might not be birational. Indeed, by [SGA7-2], Exposé XVII, Proposition 3.5 and Corollaire 3.5.0 the Gauss morphism associated to a smooth surface is birational if and only if it is generically unramified if and only if there is a non-degenerate quadratic singularity in some singular hyperplane section. An explicit example of a smooth hypersurface whose dual is a hypersurface but the Gauss morphism completely ramified is given by a special Fermat hypersurface (cf. *idem* 3.4.2).

Consider the discriminant $\Delta_{d,n}$ in $\check{\mathbb{P}}^M_{\mathbb{Z}}$, $M = \binom{n+d}{d}-1$ parameterizing singular hypersurfaces of degree d in $\mathbb{P}^n_{\mathbb{Z}}$. This discriminant can be defined as the resultant of the partial derivatives times a normalizing factor, $d^{((-1)^{n+1}-(d-1)^{n+1})/d}$ (cf. [GKZ94], Chapter 13, Proposition 1.7). We then have the following proposition, which basically follows from the definition given (cf. [Sai12], Section 2, in particular Lemma 2.5 and Proposition 2.8. It is called "divided discriminant" in the latter terminology):

Proposition 2.3. $[\Delta_{d,n}] = \varphi_* c_{n,\mathcal{H}}^{\mathcal{H}_{sing}}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M_{\mathbb{Z}}})$.

Thus the discriminant $\Delta_{d,n}$ is an integral polynomial defined up to sign and is controlled by the localized Chern class.

Proof. (of Theorem 1.1 and 1.2) The proof is more or less a standard specialization argument but we give the details since intersection theory over discrete valuation rings requires some extra care. Let T be the spectrum of one of the following: The spectrum of the base field in the general case, and $\mathbb{Q}, \mathbb{F}_p$ or the integers localized at p, $\mathbb{Z}_{(p)}$ in the case of polynomial hypersurfaces. Denote by Δ the corresponding discriminant variety in $\check{\mathbb{P}}^M_T$ which we suppose is a hypersurface. We suppose Proposition 2.3 for the case of polynomial hypersurfaces. Let now R be a discrete valuation ring with spectrum S and suppose we are given a morphism $\pi : S \to \check{\mathbb{P}}^M_T$ such that $\pi(\eta) \notin \Delta$, but $\pi(s) \in \Delta$. Without loss of generality we can suppose that $S \to T$ is faithfully flat, and by base change we can also suppose that $T = S$, and $\pi : S \to \check{\mathbb{P}}^M_S$ is a section of the natural projection. Consider the Cartesian diagram (see the diagram (1) for the notation)

$$\begin{tikzcd} H'_s \arrow[r, "i"] \arrow[d] & f^{-1}(\Delta) \arrow[d] \\ H \arrow[r, "j"] & \mathcal{H} \end{tikzcd}.$$

Then, as $j^*\mathcal{H} = H$, by defining properties of bivariant classes

$$c_{n+1}^{H'_s}(\Omega_{H/S}) \cap [H] = c_{n+1}^{f^{-1}(\Delta)}(\Omega_{\mathcal{H}/\mathbb{P}^n_S}) \cap [j^*\mathcal{H}] = i^* c_{n+1}^{f^{-1}(\Delta)}(\Omega_{\mathcal{H}/\check{\mathbb{P}}^M_S}) \cap [\mathcal{H}]. \tag{2}$$

Also consider the Cartesian diagram

$$\begin{tikzcd} H'_s \arrow[r, "i"] \arrow[d, "f'"] & f^{-1}(\Delta) \arrow[d, "f"] \\ S \times_{\check{\mathbb{P}}^M_S} \Delta \arrow[r, "i'"] & \Delta \end{tikzcd}.$$

By base change we have

$$\begin{aligned} {f'}_* i^* c_{n+1}^{f^{-1}(\Delta)}(\Omega_{\mathcal{H}/\mathbb{P}^n_S}) \cap [H] &= i'^* f_* c_{n+1}^{f^{-1}(\Delta)}(\Omega_{\mathcal{H}/\mathbb{P}^n_S}) \cap [\mathcal{H}] \\ &= i'^* \deg \varphi[\Delta], \end{aligned} \tag{3}$$

and $i'^* \deg \varphi[\Delta] = \deg \varphi[S \times_{\check{\mathbb{P}}^M_S} \Delta] = \deg \varphi v(\Delta)[s] \in \mathrm{CH}_0(S \times_{\check{\mathbb{P}}^M_S} \Delta) \simeq \mathrm{CH}_0(s)$. On the other hand, under the identification $\mathrm{CH}_0(S \times_{\mathbb{P}^n_S} \Delta) \simeq \mathrm{CH}_0(s) \simeq \mathbb{Z}$, we have

$$\deg c_{d+1}^{H'_s}(\Omega_{H/S}) \cap [H] = \deg c_{d+1}^{H_s}(\Omega_{H/S}) \cap [H].$$

Combining the last equality with (2) and (3) proves the proposition.

□

3 Families of curves and Deligne's discriminant

Let $X \subseteq \mathbb{P}^M$ be an embedding with non-degenerate image, with X a smooth projective geometrically integral surface over a field k and let L be $\mathcal{O}_X(1)$. Consider the set $\mathbb{P}(N) = \{(x, H), x \in (X \cap H)_{\text{sing}}\}$. The projection $f : \mathcal{H} \to \check{\mathbb{P}}^M$ is now a relative curve whose singularities form the space $\mathbb{P}(N)$ for N the normal bundle of X in $\mathbb{P}^M$ and the image of $\mathbb{P}(N)$ in $\check{\mathbb{P}}^M$ is the discriminant of the embedding determined by L. If k is of characteristic 0, a theorem of Ein (cf. [Ein86], partially attributed to Landman and Zak) states that the dual variety of a smooth surface is a hypersurface. The running hypothesis here is that Δ_X is a hypersurface.
Consider, for a line bundle M on $\mathcal{H}$, the Deligne-Riemann-Roch-isomorphism (cf. [Del87], p. 170)

$$(\det Rf_* M)^{12} \simeq \langle \omega, \omega \rangle \langle M, M\omega^{-1} \rangle^6$$

over the locus away from the discriminant. Here $\det Rf_*$ denotes the determinant of the cohomology and $\langle M, N \rangle$ denotes the Deligne brackets. The latter can be defined as the line bundle

$$\det Rf_*((M-1) \otimes (N-1)), \tag{4}$$

where $\det Rf_*(A - B) := \det Rf_* A \otimes (\det Rf_* B)^{-1}$. Alternatively we can define it étale locally on the base as the line bundle generated by symbols $\langle \ell, \ell' \rangle$ where ℓ (resp. ℓ') is a rational section of M (resp. N) such that $\operatorname{div}(\ell) \cap \operatorname{div}(\ell') = \emptyset$. This is subject to some relations (cf. [Sai88] for a discussion on discriminants and its relation to the above theorem, and the formalism introduced in [Del87] for the Deligne brackets). The purpose of this section is to provide a natural interpretation of the discriminant as the degeneration of this rational isomorphism over $\check{\mathbb{P}}^M$. In [Del85] Deligne calls this the discriminant section, and verifies that it corresponds to the usual discriminant in the case of degree d-curves in $\mathbb{P}^2$.
To calculate the degeneration we can suppose that k is moreover algebraically closed. Let $\deg \Delta_X$ be the degree of Δ_X in $\check{\mathbb{P}}^M$. Then

$$\deg \varphi \deg \Delta_X = \deg c_2(X) + 4g_H - 4 + \deg X \tag{5}$$

where φ is the Gauss morphism as above, and g_H is the genus of a generic hyperplane section H of X. This follows immediately from the "class formula"

$$\deg \varphi \deg \Delta_X = \chi_\ell(X) - 2\chi_\ell(X \cap H_0) + \chi_\ell(X \cap H_0 \cap H_1)$$

in [SGA7-2], Exposé XVII, Proposition 5.7.2, since for generic smooth hyperplane sections H_0 and H_1, and we have $\chi_\ell(X) = \deg c_2(X)$ (by the Lefschetz trace formula), $\chi_\ell(X \cap H_0) = 2 - 2g_H$ and $\chi_\ell(X \cap H_0 \cap H_1) = \deg X$.
The following appears in [Del85] under the headline "remarque inutile" in the special case of curves of degree d in $\mathbb{P}^2$. The general, possibly surprising, result is that in fact the Deligne discriminant is not a discriminant but the power of a discriminant in the setting of projective duality. However, if one accepts that the localized Chern class is the discriminant times the degree of the Gauss morphism, this is not surprising since the Deligne-isomorphism specializes to the Grothendieck-Riemann-Roch theorem in the Picard group where there is the same c_2-term which was calculated in the previous section. Thus this perhaps makes the following proposition into another remarque inutile, but it was also the motivating example behind this note.

Proposition 3.1 (Remarque inutile)**.** *The Deligne-isomorphism extends to an isomorphism*

$$\det Rf_*(M)^{12} \simeq \langle \omega, \omega \rangle \langle M, M\omega^{-1} \rangle^6 \otimes \mathcal{O}(\deg \varphi \Delta_X)$$

over $\check{\mathbb{P}}^M$*.*

Proof. It is not difficult to prove that the order of degeneration is independent of the choice of line bundle (cf. [Eri11], proof of Proposition 3.3). To calculate the degeneration we can assume that we really consider the isomorphism $\det Rf_*\omega^{12} \simeq \langle \omega, \omega \rangle$ over the smooth locus (the Mumford isomorphism). It is also clear that the degeneration is of the form $\mathcal{O}(c\Delta_X)$, for some integer c, for the same reason as in the previous section. We need to calculate c, which we will do by calculating the degree of the various line bundles appearing in the isomorphism. Write $\mathcal{L} = L \otimes L'$ with notation as in the previous section. In this case we have by adjunction, $K = \omega := \omega_{\mathcal{H}/\check{\mathbb{P}}^M} = \omega_X \otimes \mathcal{L}|_{\mathcal{H}}$ which admits a resolution

$$0 \to \omega_X \to \omega_X \otimes \mathcal{L} \to \omega_X \otimes \mathcal{L}|_{\mathcal{H}} \to 0$$

and so

$$\det Rf_*\omega = \det Rp_*(\omega_X \otimes \mathcal{L}) \otimes \det Rp_*\omega_X^{-1} = L'^{\mathrm{rk}(Rp_*\omega_X \otimes L)} \otimes \text{trivial sheaf}$$

and by Riemann-Roch for surfaces and the Noether formula we have

$$\begin{aligned} \mathrm{rk}(Rp_*(\omega_X \otimes L)) &= \chi(\omega_X \otimes L) = \frac{1}{2}(L+K)L + 1 + p_a \\ &= \frac{1}{2}(L+K)L + \frac{1}{12}\left(\deg c_2(X) + K^2\right). \end{aligned}$$

If we have a relative Cartier divisor D, then $\langle \mathcal{O}(D), M, N\rangle \simeq \langle M|_D, N|_D\rangle$ for the multiple index Deligne brackets (cf. [Elk89], III.2.6), so we also have

$$\langle \omega, \omega\rangle = \langle \mathcal{L}, \omega_X\mathcal{L}, \omega_X\mathcal{L}\rangle_{X\times\check{\mathbb{P}}^M/\check{\mathbb{P}}^M}.$$

Using the formulas

$$\langle f^*N, M_1, \ldots, M_n\rangle = N^{M_1.\ldots.M_n}$$

and

$$\langle f^*N_1, f^*N_2, M_1, \ldots, M_n\rangle \simeq \mathcal{O}$$

where the upper indices indicate the intersection number (restricted to any fiber, cf. loc. cit. IV. 2. 1.a and IV. 2.2 b iii), we deduce that the line bundle $\langle\omega,\omega\rangle$ is L' to the power of $K^2 + 4L(L+K) - L^2$. By the adjunction formula we have $L(K+L) = 2g_H - 2$. This in turn means that the Mumford isomorphism induces an abstract isomorphism

$$\det Rf_*\omega^{12} \otimes \langle\omega,\omega\rangle^{-1} = \mathcal{O}(\deg\varphi\Delta_X)$$

since the various powers of L' are:

$$(6L(K+L) + K^2 + \deg c_2(X)) - (K^2 + 4L(K+L) - L^2)$$

$$= \deg c_2(X) + 4g_H - 4 + \deg X = \deg\varphi\deg\Delta_X,$$

by the class formula (5). □

We finish this section by a general remark on intersection theory and Deligne brackets. If we have two Cartier divisors D and D' on a relative curve X over a discrete valuation ring R, whose supports don't intersect on the generic fiber, there is a well-known intersection product between the two (cf. [Ser75] and [SGA7-2], Exposé X, Définition 1.5 and Proposition 1.6 iii). The latter reference only considers the case when one of the divisors is concentrated on the special fiber, but the argument goes through anyway):

$$D.D' = \sum(-1)^i\chi(\mathrm{Tor}_i(\mathcal{O}_D, \mathcal{O}_{D'})) = \chi(\mathcal{O}_D \otimes \mathcal{O}_{D'}). \qquad (6)$$

If D is concentrated on the special fiber, and L is any line bundle on L, one defines

$$D.L = \deg_D L|_D \qquad (7)$$

(see *idem*, Proposition 1.6 ii). In any of the two above situations, this defines a generic trivialization of $\langle\mathcal{O}(D), \mathcal{O}(D')\rangle$.We record the relation to the Deligne brackets for the next section (this statement is implicit in [Fal84]).

Lemma 3.2. *Let $X \to S$ be a flat proper relative dimension 1 local complete intersection morphism, with geometrically connected fibers, over the spectrum of a discrete valuation ring R. Suppose that D and D' are two Cartier divisors on X such that generically they determine non-vanishing sections of $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$. Then the order of degeneration of this section is given by the intersection theory of divisors described in (6) and (7).*

Proof. By bilinearity we can suppose that both D and D' are effective and integral and not equal. Then we have an isomorphism of virtual objects

$$(\mathcal{O}_X - \mathcal{O}(D)) \otimes (\mathcal{O}_X - L) \simeq \mathcal{O}_D(D) \otimes (L - \mathcal{O}_X) = L(D)|_D - \mathcal{O}(D)|_D.$$

Applying the determinant to this is the definition of the Deligne bundles in (4).
First suppose that D is concentrated on the special fiber. The order of degeneration is given by the Zariski Euler characteristic of $L(D)|_D - \mathcal{O}(D)|_D$ (this is proved by devissage). By the Riemann-Roch theorem (cf. [Ful98], Example 18.3.4) this $\deg_D L|_D$, i.e. the intersection product in (7). The same argument goes through when D and D' are two horizontal divisors which are not equal on the generic fiber, defining a generic trivialization of $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$, and proves that the degeneration is given by (6). □

In either case, we denote the above numbers by $D.D'$ (or $D.L$). The definition for $D.L$ as the degree of L along D also makes sense when D is not a Cartier divisor. As above, one verifies that, for Cartier divisors D and D', $D.\pi^* D' = \pi_* D.D'$, where π_* has to be interpreted in the sense of pushforward of cycles of dimension 1.

4 Multiplicity of Deligne's discriminant

In this section, let $X \to S$ be as in the theorem, i.e. a flat proper local complete intersection morphism with $S = \operatorname{Spec} R$ and R a discrete valuation ring R with perfect residue field, with smooth geometrically connected fibers of dimension 1. In particular X could be the scheme associated to

$$F(Z_1, Z_2, Z_3) = \sum_{i+j+k=d} a_{ijk} Z_1^i Z_2^j Z_3^k = 0, a_{ijk} \in R$$

whenever F is generically smooth. Denote by $\Delta = \Delta_{Del,X/S}$ the associated Deligne discriminant. We are interested in the order $v(\Delta)$, and for the purposes of this section we can suppose R is henselian with algebraically closed residue field, or even its completion so that X is necessarily excellent. It should also be noted that properties such as regularity, normality or even openness of the regular locus in X over the original R can be read off from base change to the completion of the henselization (the first two properties are contained in Lemma 2.1.1 of [CES03], the last point follows from the fact that being an open immersion is a fpqc-local statement).

By [Liu02], Corollary 8.3.51, we have a desingularization $\pi : X' \to X$, i.e. X' is regular and π is a proper birational and an isomorphism over the regular locus. Both X and X' are local complete intersections so their dualizing sheaves are line bundles, and we write $\omega_{X'/S} = \pi^* \omega_{X/S} + \sum b_i E_i = \pi^* \omega_{X/S} + \Gamma$ for the exceptional divisors E_i so that Γ is the discrepancy. Using [SGA7-2], Exposé X (as considered in Lemma 3.2), we have an intersection product Γ^2. We consider also the Zariski cohomology Euler characteristic(s) $\chi(\mathrm{cone}[\mathcal{O}_X \to R\pi_* \mathcal{O}_{X'}]) = \chi(\pi_* \mathcal{O}_{X'}/\mathcal{O}_X) - \chi(R^1 \pi_* \mathcal{O}_{X'}) =: -p_g$. The individual terms do not depend on the choice of regular model. In case X is normal, by Zariski's main theorem, $\pi_* \mathcal{O}_{X'}/\mathcal{O}_X = 0$ and p_g is nothing but the usual genus of the singularities defined as the dimension of the $k(s)$-module $R^1 \pi_* \mathcal{O}_{X'}$. Also consider the dimension one contribution Y given by the Weil divisor $\pi_* \Gamma$. It is necessarily contained in the non-normal locus and independent of the resolution. Then we have:

Lemma 4.1. *Let X be as above. Then*

$$v(\Delta_{Del,X/S}) - v(\Delta_{Del,X'/S}) = 12 p_g + \Gamma^2 + 2 \deg_Y \omega_{X/S}.$$

Proof. Denoting by λ the determinant of the cohomology, the difference is measured by the difference of the two Mumford isomorphisms, extended over S, which we write in the form

$$\lambda(\mathcal{O}_{X'})^{12} \simeq \langle \omega', \omega' \rangle + v(\Delta_{Del,X'/S})$$

and

$$\lambda(\mathcal{O}_X)^{12} \simeq \langle \omega, \omega \rangle + v(\Delta_{Del,X/S}).$$

The difference $\lambda(\mathcal{O}_{X'})^{12} - \lambda(\mathcal{O}_X)^{12}$ is computed by considering the lengths of the cohomology groups of the cone of $\mathcal{O}_X \to R\pi_* \mathcal{O}_{X'}$. This is $-12 p_g$. We now prove that for line bundles L and M on X,

$$\langle \pi^* L, \pi^* M \rangle \simeq \langle L, M \rangle,$$

i.e. that the (identity) isomorphism on the generic point extends to a global isomorphism. Using the cohomological definition of the Deligne brackets, the projection formula shows that the obstruction to obtain a global isomorphism is measured by the Zariski Euler characteristics

$$\chi((L-1)\otimes(M-1)\otimes\pi_*\mathcal{O}_{X'}/\mathcal{O}_X)$$

and

$$\chi((L-1)\otimes(M-1)\otimes R^1\pi_*\mathcal{O}_{X'}).$$

But the sheaves $\pi_*\mathcal{O}_{X'}/\mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_{X'}$ are concentrated on the special fiber, and it follows directly from Riemann-Roch (for singular curves) that both of these numbers are 0.

We are now ready to calculate the difference $\langle\omega_{X'/S},\omega_{X'/S}\rangle - \langle\omega_{X/S},\omega_{X/S}\rangle$. By the above and Lemma 3.2 this is $\langle\omega_{X'/S},\omega_{X'/S}\rangle - \langle\pi^*\omega_{X/S},\pi^*\omega_{X/S}\rangle = 2\Gamma\cdot\pi^*\omega_{X/S}+\Gamma^2$. Since $\Gamma\cdot\pi^*\omega_{X/S} = \pi_*\Gamma\cdot\omega_{X/S} = \deg_Y\omega_{X/S}$ by the projection formula the lemma follows. □

Since now X' is a regular surface over S, using T. Saito's formula in [Sai88], since the generic fiber doesn't change,

$$v(\Delta_{Del,X'/S}) \quad = \quad -\operatorname{Art}_{X'/S} = \chi_\ell(X'_{\overline{s}}) - \chi_\ell(X_{\overline{\eta}}) + \operatorname{Sw} H^1(X_{\overline{\eta}},\mathbb{Q}_\ell).$$

This gives

$$v(\Delta) = 12p_g + \Gamma^2 + 2\deg_Y\omega_{X/S} + \chi_\ell(X'_{\overline{s}}) - \chi_\ell(X_{\overline{\eta}}) + \operatorname{Sw} H^1(X_{\overline{\eta}},\mathbb{Q}_\ell).$$

We recall a formula of Laufer for the Milnor number of a normal surface singularity (cf. [Lau77]), in the complex setting. For $x\in X$ an isolated singular point, X a normal complex hypersurface in $\mathbb{C}^3$, the Milnor number of x in X is shown to be equal to

$$\mu_{X,x} = 12p_g + \Gamma^2 - b_1(E) + b_2(E)$$

where $(X',E)\to(X,x)$ is a desingularization with Betti numbers $b_i(E)$.

Definition 1. We set

$$\mu_X = 12p_g + \Gamma^2 + 2\deg_Y\omega_{X/S} - b_1(E) + b_2(E),$$

where the numbers are determined by some choice of resolution $X'\to X$, and b_i denote ℓ-adic Betti numbers. If $x\in\pi_0(X^{non-reg})$, we set

$$\mu_{X,x} = 12p_{g,x} + \Gamma_x^2 + 2\deg_{Y_x}\omega_{X/S} - b_1(E_x) + b_2(E_x)$$

so that

$$\mu_X = \sum_{x\in\pi_0(X^{non-reg})}\mu_{X,x}.$$

Here the subscript x denotes the various contributions local on $x\in X$.

If x is an isolated singularity, then the above definition reduces to Laufer's formula in the complex setting, which then can be interpreted as some type of Milnor number. These numbers are also easily verified to be invariant under blow-up of closed points on X', and since any regular two models are related by such blow-ups (cf. [Lic68], Theorem 1.15, p. 392), they are independent of the choice of X'.

Proposition 4.2. *With the above notation,*

$$v(\Delta) = -\operatorname{Art}_{X/S} + \mu_X$$

or

$$v(\Delta) = \chi_\ell(X_{\overline{s}}) - \chi_\ell(X_{\overline{\eta}}) + \operatorname{Sw} H^1(X_{\overline{\eta}}, \mathbb{Q}_\ell) + \sum_{x \in \pi_0(X^{non-reg})} \mu_{X,x}.$$

Proof. Standard rewriting gives $\chi_\ell(X'_{\overline{s}}) = \chi_\ell(\tilde{X}_{\overline{s}}) + \sum \chi_\ell(E_{x_i}) - \sum B(x)$ where $B(x)$ is the number of (geometric) branches of x in X_s under the map $X'_s \to X_s$ and $\tilde{X}_s$ is the strict transform of X_s. The formula $\chi_\ell(\tilde{X}_{\overline{s}}) - \chi_\ell(X_{\overline{s}}) = \sum(B(x) - 1)$ implies the proposition. □

We explicit the numbers $\mu_{X,x}$ in one case. For this, write $E = \pi^{-1}(x) = \cup C_i$ as a union of irreducible curves.

Proposition 4.3. *Suppose x is an isolated singularity on X. Then*

$$\mu_{X,x} = 12p_g + \Gamma^2 - 2g - b + r.$$

Here:

- *$g = \sum g(\tilde{C}_{i,red})$ is the sum of the genus of the normalizations of the reduced irreducible components $C_{i,red}$ of E.*
- *b is the number of loops in the dual graph of a normal crossings model (the reduced components are smooth), defined as the graph whose vertices are the components of E and we connect two vertices by an edge for each intersection.*
- *r is the number of components of E.*

Proof. Since ℓ-adic cohomology on depends on the reduced structure, we can suppose E is reduced. Using the argument of [Zin77], Lemma 3.1, the Betti numbers can be computed in the case of a normal crossings model. One finds that $b_1(E) = 2g + 1 - r + \#E_{sing}$, $b_2(E) = r$. The number of loops of the dual graph is its first Betti number, and hence we have

$$1 - b = \#\text{nodes} - \#\text{edges} = r - \#E_{sing},$$

so that $b_1(E) = 2g + b$. We conclude since $\mu_{X,x}$, $12p_g, \Gamma^2 + r, g$ and b are clearly independent of blowups in regular closed points. □

Example 1. *Suppose $x \in X$ is a rational double point singularity, and consider a minimal desingularization $(X', E) \to (X, x)$. Basically by definition $p_g = g = 0$, and by [Lip69], p. 258, the dual graph of this desingularization is a tree, so $b = 0$. Finally, it is well-known that $\Gamma = 0$, so $\mu_{X,x} = r$. This is one of the miracles behind the usual formula of Ogg since the minimal Weierstrass equation is the unique Weierstrass model of a smooth planar degree three curve which only has rational double points as singularities, and the minimal resolution is given by the Néron model. The article [Kol97] approaches the question of finding "minimal models" of equations over discrete valuation rings, from the point of view of geometric invariant theory. These models are minimal, amongst other things, in the sense that the discriminant is minimal, and in the case of Weierstrass models corresponds to minimal Weierstrass models. It would be interesting to understand better the geometry of such minimal models for the purpose of this article.*

Example 2. *When X is given by a family of ternary homogeneous polynomials, denoted F, then*

$$v(\Delta_{Del,X/S}) = v(\Delta_F) = \deg c_{2,X}^{X_s}(\Omega_{X/S}),$$

by Proposition 1.2 and Proposition 3.1. More generally, J. Franke (unpublished work on the functorial Riemann-Roch theorem) proved this relation directly without any polynomial assumption, and it was also revisited in the regular case in [Eri]. Thus Theorem 1.4 can be seen as a computation of the error terms in the naive expectation from Bloch's conjecture whenever the total space is not regular. As the Artin conductor is defined using vanishing cycles, is it possible to compute the localized Chern class in in terms of the Euler characteristic of a similar constructible sheaf even in the non-regular case?

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