# Crossed products and C*-dynamics

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# Preface

Since the work of Connes in the classification of von Neumann algebras and their automorphisms, group actions have received a great deal of attention. Amenable group actions on the hyperfinite $\mathrm{II}_1$-factor were completely classified by Ocneanu, extending earlier results of Connes and Jones. In their work, showing that outer actions have the so-called Rokhlin property was fundamental, as this property allows one to prove classification. For $C^*$-algebras, the picture is more complicated. For once, it is no longer true that (strong) outerness implies the Rokhlin property, and there is little hope to classify general group actions unless they have the Rokhlin property. On the other hand, the Rokhlin property is very restrictive, and there are many C*-algebras that do not admit any action with this property. Several weakenings of the Rokhlin property have been introduced to address this problem. Among them, the weak tracial Rokhlin property and Rokhlin dimension (for which Rokhlin dimension zero is equivalent to the Rokhlin property) have been successfully used to prove structure results for crossed products. Furthermore, actions with these properties seem to be very common.

In this course, we will focus on actions of finite groups, and will only occasionally comment on actions of more general groups. We will introduce the Rokhlin property, provide many examples, and show that Rokhlin actions can be classified. We will also see that there are natural obstructions to the Rokhlin property, and will present some weaker variants of it: the (weak) tracial Rokhlin property and Rokhlin dimension (with and without commuting towers). These properties are flexible enough to cover many relevant examples, and are strong enough to yield interesting structural properties for their crossed products. Finally, we will prove a recent analog of Ocneanu's theorem for amenable group actions on $C^*$-algebras, namely, that for actions on classifiable algebras (which are, in particular, Jiang-Su stable), strong outerness is equivalent to the weak tracial Rokhlin property, and also equivalent to finite Rokhlin dimension (in fact, dimension at most one).

# Introduction

# Part I

# Group actions and crossed products

# Chapter 1

# Introduction

# Chapter 2

# Some preliminaries

## 2.1 Universal $C^*$-algebras

Many of the most relevant $C^*$-algebras can be expressed as universal $C^*$-algebras on relatively simple sets of generators and relations. Unlike for groups, universal $C^*$-algebras do not always exist and there a few subtelties in the theory. Here, we review those aspects that will be needed later, and refer the reader to Blackadar's seminal work [4].

**Definition 2.1.1.** Let $\mathcal{G}$ be a set, which we call the set of *generators*. We do not assume that $\mathcal{G}$ be finite, or even countable. We define a *relation* on $\mathcal{G}$ to be an expression of the form

$$\|p(x_1, x_1^*, \ldots, x_n, x_n^*)\| \leq r,$$

where $p$ is a polynomial on $2n$ noncommuting variables, $n \in \mathbb{N}$, $r \in [0, \infty)$, and $x_1, \ldots, x_n \in \mathcal{G}$.

Let $\mathcal{G}$ be a set of generators and let $\mathcal{R}$ be a set of relations. A *representation* of $(\mathcal{G}, \mathcal{R})$ consists of a Hilbert space $\mathcal{H}$ and a set $\{a_x \colon x \in \mathcal{G}\} \subseteq \mathcal{B}(\mathcal{H})$ such that

$$\|p(a_{x_1}, a_{x_1}^*, \ldots, a_{x_n}, a_{x_n}^*)\| \leq r,$$

whenever $\|p(x_1, x_1^*, \ldots, x_n, x_n^*)\| \leq r$ is a relation in $\mathcal{R}$.

**Definition 2.1.2.** We say that the family $\mathcal{R}$ of relations on a set $\mathcal{G}$ is *admissible* if there exists a non-zero representation of $(\mathcal{G}, \mathcal{R})$, and if there exist constants $r_x \in [0, \infty)$, for every $x \in \mathcal{G}$, such that whenever $\{a_x \in \mathcal{B}(\mathcal{H}) \colon x \in \mathcal{G}\}$ is a representation of $(\mathcal{G}, \mathcal{R})$, then $\|a_x\| \leq r_x$ for all $x \in \mathcal{G}$.

Universal $C^*$-algebras defined by admissible relations exist, as we show next.

**Theorem 2.1.3.** Let $\mathcal{G}$ be a set of generators, and let $\mathcal{R}$ be an admissible set of relations on $\mathcal{G}$. Then there exists a unique $C^*$-algebra $C^*(\mathcal{G} : \mathcal{R})$ containing a generating set $\{a_x \in C^*(\mathcal{G} : \mathcal{R}) \colon x \in \mathcal{G}\}$ satisfying the relations from $\mathcal{R}$, such that whenever $B$ is another $C^*$-algebra containing elements $\{b_x \in B \colon x \in \mathcal{G}\}$ satisfying the relations from $\mathcal{R}$, then there exists a unique homomorphism $\varphi \colon C^*(\mathcal{G} : \mathcal{R}) \to B$ satisfying $\varphi(a_x) = b_x$ for all $x \in \mathcal{G}$.

*Proof.* Denote by $\mathcal{A}$ the free $*$-algebra generated by $\mathcal{G}$. Each representation $\{\pi_x \in \mathcal{B}(\mathcal{H}) \colon x \in \mathcal{G}\}$ of $(\mathcal{G}, \mathcal{R})$ induces a $*$-representation $\pi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$, given by $\pi(x) = \pi_x$ for all $x \in \mathcal{G}$. For $a \in \mathcal{A}$, we define

$$\|a\| = \sup\{\|\pi(a)\| \colon \pi \text{ is a representation of } (\mathcal{G}, \mathcal{R})\}.$$

Observe that $\|a\| \leq \infty$ for all $a \in \mathcal{A}$, because $\mathcal{R}$ is admissible. We let $C^*(\mathcal{G} : \mathcal{R})$ denote the Hausdorff completion of $\mathcal{A}$ with respect to this norm, and note that $C^*(\mathcal{G} : \mathcal{R})$ is a $C^*$-algebra. If $a_x \in C^*(\mathcal{G} : \mathcal{R})$ denotes the image of the canonical generator $x \in \mathcal{A}$, then it is clear that $\{a_x \in C^*(\mathcal{G} : \mathcal{R}) \colon x \in \mathcal{G}\}$ is a generating set that satisfies the relations from $\mathcal{R}$.

Now let $B$ be another $C^*$-algebra containing elements $\{b_x \in B \colon x \in \mathcal{G}\}$ satisfying the relations from $\mathcal{R}$. Then there is a unique $*$-homomorphism $\varphi_0 \colon \mathcal{A} \to B$ given by $\varphi_0(x) = b_x$ for all $x \in \mathcal{G}$. Let $\sigma \colon B \to \mathcal{B}(\mathcal{H})$ be a faithful representation of $B$ on some Hilbert space $\mathcal{H}$. Then $\sigma \circ \varphi_0$ is a representation of $(\mathcal{G}, \mathcal{R})$, and thus

$$\|a\|_{C^*(\mathcal{G}:\mathcal{R})} \leq \|\sigma(\varphi_0(a))\|_{\mathcal{B}(\mathcal{H})} = \|\varphi_0(a)\|_B.$$

It follows that $\varphi_0$ extends uniquely to a homomorphism $\varphi \colon C^*(\mathcal{G} : \mathcal{R}) \to B$ satisfying $\varphi(a_x) = b_x$ for all $x \in \mathcal{G}$.

Finally, uniqueness of $C^*(\mathcal{G} : \mathcal{R})$ follows immediately from its universal property. $\square$

In the definition of admissible representation, the condition that all generators are uniformly bounded in norm is necessary for a universal $C^*$-algebra to exist.

**Example 2.1.4.** There is no "universal $C^*$-algebra generated by a single element". This would correspond to $\mathcal{G} = \{x\}$ and $\mathcal{R} = \emptyset$. The reason is that if such a $C^*$-algebra existed, and $a_x$ were the canonical generator in it, the norm of $a_x$ would have to be larger than the norm of *every* element in *every* $C^*$-algebra. This is of course not possible, so this algebra does not exist.

It is often the case that the relations are described rather informally, particularly when the precise description is clear. We present some examples.

**Examples 2.1.5.** 1. The universal unital $C^*$-algebra generated by a self-adjoint contraction is the universal $C^*$-algebra with $\mathcal{G} = \{1, a\}$ and $\mathcal{R}$ given by

$$\|1a - a\| \leq 0, \|a1 - a\| \leq 0, \|a^* - a\| \leq 0, \quad \text{and} \quad \|a\| \leq 1.$$

This $C^*$-algebra is isomorphic to $C([-1, 1])$ with the canonical generator being the inclusion of $[-1, 1]$ into $\mathbb{C}$.

2. Similarly, the universal $C^*$-algebra generated by a self-adjoint contraction is isomorphic to $C_0([-1,1] \setminus \{0\})$.

3. The universal $C^*$-algebra generated by a unitary has $\mathcal{G} = \{1, u\}$ and $\mathcal{R}$ given by
$$\|1u - u\| \leq 0, \|u1 - u\| \leq 0, \|u^*u - 1\| \leq 0 \text{ and } \|uu^* - 1\| \leq 0.$$
This $C^*$-algebra is isomorphic to $C(S^1)$.

4. If $G$ is a discrete group, then there is a universal $C^*$-algebra generated by a unitary representation of $G$. This algebras has $\mathcal{G} = \{u_g \colon g \in G\}$ and $\mathcal{R}$ given by the relations implying that $u_1$ is the unit, $u_g$ is a unitary with inverse $u_{g^{-1}}$, and $u_g u_h = u_{gh}$ for all $g, h \in G$. This $C^*$-algebra is the *full group $C^*$-algebra* $C^*(G)$ of $G$.

5. Fix $n \in \mathbb{N}$, and set $\mathcal{G} = \{e_{j,k} \colon 1 \leq j, k \leq n\}$ and
$$\mathcal{R} = \{e_{j,k}^* = e_{k,j}, e_{j,k}e_{l,m} = \delta_{k,j}e_{j,m} \colon 1 \leq j, k, l, m \leq n\}.$$
The relations in $\mathcal{R}$ implies that each $e_{j,k}$ is a partial isometry so $\|e_{j,k}\| \leq 1$, and hence the universal $C^*$-algebra exists by Theorem 2.2.1. It is easy to check that this $C^*$-algebra is isomorphic to $M_n$, with $e_{j,k}$ corresponding to the matrix that has a 1 in the $(j,k)$-entry and zeroes elsewhere.

6. Every $C^*$-algebra is a universal $C^*$-algebra. Indeed, for a $C^*$-algebra $A$ we may take $\mathcal{G} = \{x_a \colon a \in A\}$ with relations $\mathcal{R}$ given by
$$\|x_a\| = \|a\|, \|x_a^* - x_{a^*}\| = 0, \text{ and } \|x_a x_b - x_{ab}\| = 0$$
for all $a, b \in A$. Then $C^*(\mathcal{G} : \mathcal{R})$ is naturally isomorphic to $A$. This description of $A$ as a universal $C^*$-algebra is however not very useful in practice.

A number of very familiar constructions in $C^*$-algebras can be described through universal $C^*$-algebras, such as direct sums, tensor products, free products, etc. Crossed products, particularly full crossed products, can also be described as universal $C^*$-algebras; see Theorem 4.1.9.

We close this section by introducing a particularly rich class of universal $C^*$-algebras, namely *graph algebras*.

**Definition 2.1.6.** A *directed graph* is a tuple $\mathcal{E} = (V, E, r, s)$, where $V$ and $E$ are countable sets (usually referred to as the sets of *vertices* and *edges*, respectively), and $r, s \colon E \to V$ are functions (usually referred to as the *range* and *source* functions of an edge).

For a directed graph $\mathcal{E} = (V, E, r, s)$, we define its associated *graph $C^*$-algebra* $C^*(\mathcal{E})$ to be the universal $C^*$-algebra generated by the set
$$\mathcal{G} = \{p_v \colon v \in V\} \cup \{s_e \colon e \in E\}$$
and subject to the following relations

1. $p_v$ is a projection for all $v \in V$;

2. $p_v p_w = 0$ whenever $v, w \in V$ are distinct;

3. $s_e^* s_e = p_{r(e)}$ for all $e \in E$;

4. $s_e^* s_f = 0$ whenever $e, f \in E$ are distinct;

5. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E$;

6. for $v \in V$, if the set $\{e \in E \colon s(e) = v\}$ is not empty, then

$$p_v = \sum_{e \in s^{-1}(v)} s_e s_s e^*.$$

Arguably the first historical examples of a graph algebra (before these were even considered) are the Cuntz algebras $\mathcal{O}_n$ [15], for $n \geq 2$, which are the algebras associated to the graph with one vertex and $n$ loops (see Example 2.1.7). Graph algebras also include many well-studied $C^*$-algebras, such as the compact operators, the Toeplitz algebra, AF-algebras, and all UCT Kirchberg algebras with torsion-free $K_1$. The study of graph algebras is a very active one; see, for example [81]. Indeed, they constitute a particularly tractable and accessible class whose basic structure is well-understood.

**Example 2.1.7.** We collect some elementary examples of graph algebras.

1. The graph algebra associated to the graph with $V = \{*\}$ and $E = \emptyset$ is $\mathbb{C}$.

2. For $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the graph with one vertex and $n$ loops around it. Then $C^*(\mathcal{E}_n)$ is the universal $C^*$-algebra generated by isometries $s_1, \ldots, s_n$ satisfying $\sum_{j=1}^n s_j s_j^* = 1$. Whem $n = 1$, the isometry $s_1$ is a unitary and hence $C^*(\mathcal{E}_1) \cong C(S^1)$. For other values of $n$, the resulting $C^*$-algebra is known as the *Cuntz algebra* and denoted $\mathcal{O}_n$.

3. Let $n \in \mathbb{N}$, and consider the graph $\mathcal{M}_n$ given as follows:

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \cdots \longrightarrow \bullet \xrightarrow{e_{n-1}} \bullet$$

   Then $C^*(\mathcal{M}_n) \cong M_n$.

4. Consider the graph $\mathcal{E}$ given as follows:

$$\cdots \longrightarrow \bullet \xrightarrow{e_{-1}} \bullet \xrightarrow{e_0} \bullet \xrightarrow{e_1} \cdots$$

   Then $C^*(\mathcal{E}) \cong \mathcal{K}$.

## 2.2 Multiplier algebras

Unital $C^*$-algebras are, for many purposes, significantly easier to work with than non-unital ones. When a given $C^*$-algebra $A$ is not unital, one may wish to consider a unital $C^*$-algebra that contains $A$ as an ideal. To avoid working with $C^*$-algebras that are "too big relative to $A$", it is convenient to look at unital algebras that contain $A$ as an *essential* ideal[1]. For an algebra of the form $C_0(X)$, this corresponds to embedding $X$ into a compact space $Y$ as an open *dense* subspace. In the topological setting, there are both a minimal and a maximal way to do this: these are, respectively, the one-point compactification $X^+$ and the Stone-Čech compactification $\beta X$ of $X$. Minimal and a maximal unitizations also exist for an arbitrary $C^*$-algebra $A$: these are the one-dimensional unitization $A^+ \cong A \oplus \mathbb{C}$ and the multiplier algebra $M(A)$.

**Theorem 2.2.1.** Let $A$ be a $C^*$-algebra. Then there exists a unique unital $C^*$-algebra $M(A)$ containing $A$ as an essential ideal such that whenever $A$ is an ideal in some $C^*$-algebra $B$, then there is a unique homomorphism $\varphi\colon B \to M(A)$ extending the identity on $A$ and satisfying $\ker(\varphi) = \{b \in B : bA = \{0\}\}$. In other words, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, hook] & M(A) \\ & B. \arrow[u, dashed, "\exists!\varphi"'] \end{tikzcd}$$

Once the existence of such a $C^*$-algebra is established, its uniqueness follows from the universal property; see Exercise 2.2.6. We will sketch the proof that an algebra satisfying the properties as above exists, using double centralizers. This requires some preparation. For a $C^*$-algebra $A$ and an operator $T \in \mathcal{B}(A)$, we define $T^* \in \mathcal{B}(A)$ by $T^*(a) = T(a^*)^*$ for all $a \in A$.

**Definition 2.2.2.** Let $A$ be a $C^*$-algebra. A *double centralizer* on $A$ is a pair $(L, R)$ consisting of maps $L, R \in \mathcal{B}(A)$ satisfying

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad \text{and} \quad aL(b) = R(a)b$$

for all $a, b \in A$.

We let $M(A)$ denote the set of all double centralizers, endowed with coordinate-wise addition and scalar multiplication, and operations

$$(L_1, R_1)(L_2, R_2) = (L_2 \circ L_1, R_1 \circ R_2) \ \text{ and } \ (L, R)^* = (R^*, L^*),$$

for all $(L, R), (L_1, R_1), (L_2, R_2) \in M(A)$. Finally, set $\|(L, R)\| = \|L\|$ for $(L, R) \in M(A)$.

[1] An ideal $I$ in $B$ is said to be *essential* if $J \cap I \neq \{0\}$ whenever $J$ is a non-trivial ideal in $B$. Equivalently, if $b \in B$ satisfies $bI = \{0\}$, then $b = 0$.

**Proposition 2.2.3.** Let $A$ be a $C^*$-algebra. Then $M(A)$, as defined above, is a unital $C^*$-algebra.

The proof is easy, and is left as an exercise; see Exercise 2.2.6. Perhaps the only subtle part is showing that the adjoint operation is norm-preserving.

**Example 2.2.4.** Let $A$ be a $C^*$-algebra, and let $a \in A$. Define $L_a, R_a \in \mathcal{B}(A)$ by $L_a(b) = ab$ and $R_a(b) = ba$ for all $b \in A$. One easily checks that $(L_a, R_a)$ is a double-centralizer, and that $\|(L_a, R_a)\| = \|a\|$. Moreover, it is easily verified that the map $\iota_A \colon A \to M(A)$ given by $\iota_A(a) = (L_a, R_a)$ for all $a \in A$ is an isometric homomorphism.

**Remark 2.2.5.** If $A$ is an ideal in some $C^*$-algebra $B$, then the identity map on $A$ extends canonically to a homomorphism $\varphi \colon B \to M(A)$ given by $\varphi(b) = (L_b, R_b)$, where $L_b, R_b \in \mathcal{B}(A)$ are given by left and right multiplication by $b \in B$, respectively. It is easy to check that $\ker(\varphi) = \{b \in B \colon bA = 0\}$. Uniqueness of $\varphi$ is also easily verified.

The proof of Theorem 2.2.1 follows by combining Proposition 2.2.3, Example 2.2.4, and Remark 2.2.5.

**Exercise 2.2.6.** Write down a complete proof of Theorem 2.2.1, including uniqueness of $M(A)$, the proof of Proposition 2.2.3, and filling in the details in Example 2.2.4 and Remark 2.2.5.

**Examples 2.2.7.** We list some examples of multiplier algebras, without proof.

1. If $X$ is a locally compact space, then $M(C_0(X)) \cong C_b(X) \cong C(\beta X)$.

2. If $\mathcal{H}$ is a Hilbert space, then $M(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$.

3. If $A$ is a unital $C^*$-algebra, then $A = M(A)$.

Another convenient identification of the elements in $M(A)$ as *single* operators is given as follows.

**Exercise 2.2.8.** Let $\pi \colon A \to \mathcal{B}(\mathcal{H})$ be a non-degenerate, injective representation of a $C^*$-algebra $A$ on a Hilbert space $\mathcal{H}$. Show that $M(A)$ can be canonically identified with

$$\{T \in \mathcal{B}(\mathcal{H}) \colon T\pi(A) \subseteq \pi(A), \pi(A)T \subseteq \pi(A)\}.$$

Yet another presentation of $M(A)$ as operators on $A$, once $A$ is regarded as a Hilbert module over itself, will be given in Section 2.4.

## 2.3 $K$-theory

The area of *noncommutative topology* has largely benefited from taking a notion from topology and extending it to the category of noncommutative $C^*$-algebras, via a suitable rephrasing of the original notion for commutative $C^*$-algebras using the contravariant equivalence of the latter category with that of locally compact Hausdorff spaces. This process has seen great sucess, since often in point-set topology, the natural object to study is a "singular" space, that cannot be described in purely topological terms. A prime example of this situation is the orbit space of a non-proper action. In some of these situations, there is a suitable noncommutative $C^*$-algebra that can play the role of this singular space; in the case of orbit spaces, this is usually the reduced crossed product.

There has been a great amount in noncommutative algebraic topology – that is, extending functors from topological spaces to groups, to general $C^*$-algebras. While attempts to do this for different types of homology or fundamental groups have not been successful, this works out particularly nicely with $K$-theory. In this section we give a brief introduction to $K$-theory for $C^*$-algebras, without assuming any knowledge of $K$-theory for topological spaces. The interested reader is referred to [6] for a much more thorough treatment of this indispensable tool.

### The Grothendieck group.

The Grothendieck construction allows one to obtain an abelian group from an abelian semigroup in such a way that any group containing a homomorphic image of the semigroup, will also contain a homomorphic image of the Grothendieck envelope. The following is its precise definition:

**Definition 2.3.1.** For be an abelian semigroup $V$, we define $G(V)$ to be the quotient of $V \times V$ under the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ whenever there exists $z \in V$ such that

$$x_1 + y_2 + z = x_2 + y_1 + z.$$

The set $G(V)$ can be thought of as the set of equivalence classes of expressions of the form $x - y$. Addition on $G(V)$ is induced by addition on $V \times V$. Then $G(V)$ is an abelian group, with $-(x - y) = y - x$ for all $x, y \in V$. This group is called the *Grothendieck group of* $V$.

**Example 2.3.2.** It is immediate to check that $G(\mathbb{N}) = \mathbb{Z}$

There is always a canonical semigroup map $\iota_V \colon V \to G(V)$, given by $\iota_V(x) = x$ for all $x \in V$. This map is, however, injective if and only if $V$ has cancellation, meaning that $x + z = y + z$ implies $x = y$, for all $x, y, z \in V$. In the next example, the map $\iota_V$ is not injective.

**Example 2.3.3.** Define on $V = \mathbb{N} \cup \{\infty\}$ a sum extending the operation on $\mathbb{N}$ by setting $x + \infty = \infty$. Then the equivalence relation from Definition 2.3.1 identifies all pairs in $V$, as one can see by taking $z = \infty$. It follows that $G(\mathbb{N} \cup \{\infty\}) = \{0\}$.

Grothendieck groups enjoy an improtant universal property.

**Theorem 2.3.4.** Let $V$ be an abelian semigroup. If $G$ is an abelian group and $\varphi\colon V \to G$ is a semigroup homomorphism, then there exists a unique group homomorphism $\psi\colon G(V) \to G$ satisfying $\psi \circ \iota_V = \varphi$. In other words, the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\iota_V"'] & G \\ G(V). \arrow[ur, dashed, "\exists!\psi"'] & \end{tikzcd}$$

Morever, if $\mathcal{G}(V)$ is another abelian group and $j_V\colon V \to \mathcal{G}(V)$ is a semigroup homomorphism satisfying the same property as above, then there exists an isomorphism $\theta\colon G(V) \to \mathcal{G}(V)$ satisfying $\theta \circ \iota_V = j_V$.

**Exercise 2.3.5.** Give a proof of Theorem 2.3.4.

### The Murray-von Neumann semigroup.

For a $C^*$-algebra $A$ and positive integers $n, m$ with $m \geq n$, we usually identify $M_n(A)$ with a the subalgebra of $M_m(A)$ of those upper-left $n \times n$ matrices with values in $A$. The union of these matrix algebras with these embeddings (but not completion) is usually denoted by $M_\infty(A)$. Note that the completion of $M_\infty(A)$ is isomorphic to $A \otimes \mathcal{K}$.

**Definition 2.3.6.** Let $A$ be a $C^*$-algebra. Given projections $p, q \in A$, we say that $p$ and $q$ are *Murray-von Neumann equivalent*, and write $p \sim_{\mathrm{M-vN}} q$, if there exists a partial isometry $s \in A$ with $s^*s = p$ and $ss^* = q$.

Let $n, m \in \mathbb{N}$ with $n \leq m$. We say that two projections $p \in M_n(A)$ and $q \in M_m(A)$ are $K_0$-*equivalent* if there exists $k \geq 0$ such that $p \oplus 0_{m-n} \oplus 0_k \sim_{\mathrm{M-vN}} q \oplus 0_k$. We denote the $K_0$-equivalence class of $p$ by $[p]_0$.

We define the *Murray-von Neumann semigroup* $V(A)$ of $A$ as

$$V(A) = \{[p]_0 \colon p \in M_\infty(A) \text{ projection}\}.$$

Addition on $V(A)$ is given by $[p]_0 + [q]_0 = [\mathrm{diag}(p, q)]_0$ for projections $p, q \in M_\infty(A)$.

**Remark 2.3.7.** One could define $V(A)$ equivalently using the relations of unitary equivalence (with unitaries taken in the minimal unitization) or homotopy equivalence on the projections of $M_\infty(A)$. Of these relations for projections, homotopy is the strongest, while Murray-von Neumann equivalence is the weakest.

**Lemma 2.3.8.** Let $A$ be a $C^*$-algebra. Then $V(A)$ is an abelian semigroup.

*Proof.* It suffices to show that $\mathrm{diag}(p,q)$ and $\mathrm{diag}(q,p)$ are unitarily equivalent, and this is easily seen by considering the unitary $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. □

**Examples 2.3.9.** Let $\mathcal{H}$ be a Hilbert space.

1. If $\dim(\mathcal{H}) < \infty$, then $V(\mathcal{B}(\mathcal{H})) = \mathbb{N}$, regardless of the dimension of $\mathcal{H}$.
2. If $\dim(\mathcal{H}) = \infty$, then $V(\mathcal{B}(\mathcal{H})) = \mathbb{N} \cup \{\infty\}$.

### The $K_0$-group.

The definition of the $K_0$-group of a $C^*$-algebra is easier when the $C^*$-algebra is unital, so we begin with this case.

**Definition 2.3.10.** Let $A$ be a unital $C^*$-algebra. We define its $K_0$-*group* $K_0(A)$ as the Grothendieck group (Definition 2.3.1) of its Murray-von Neumann semigroup $V(A)$ (Definition 2.3.6).

**Example 2.3.11.** Combining the examples we saw in the previous subsections, we deduce that

$$K_0(M_n) = \mathbb{Z} \quad \text{and} \quad K_0(\mathcal{B}(\ell^2)) = \{0\}.$$

It is easily seen that $K_0$ is a functor from the category of unital $C^*$-algebras (with unital homomorphisms) to the category of abelian groups. The *standard picture* of $K_0(A)$, for a unital $C^*$-algebra $A$, is

$$K_0(A) = \{[p]_0 - [q]_0 \colon p, q \in M_\infty(A) \text{ projections}\}.$$

**Definition 2.3.12.** Let $A$ be a nonunital $C^*$-algebra, and let $\widetilde{A}$ be its minimal unitzation. Then there is a canonical unital homomorphism $\mu\colon \widetilde{A} \to \mathbb{C}$, which induces a homomorphism $K_0(\mu)\colon K_0(\widetilde{A}) \to K_0(\mathbb{C}) = \mathbb{Z}$. We define $K_0(A) = \ker(K_0(\mu))$.

With the above definition, $K_0$ is a functor from the category of all $C^*$-algebras (with arbitrary homomorphisms) to the category of abelian groups.

### The $K_1$-group.

Let $A$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $u \in M_n(A)$ be a unitary. For $m \geq n$, we regard $u$ as a unitary in $M_m(A)$ by identifying $u$ with $u \oplus 1_{m-n}$.

**Definition 2.3.13.** Let $A$ be a unital $C^*$-algebra. Given unitaries $u, v \in A$, we say that $u$ and $v$ are *homotopic*, and write $u \sim_h v$, if there exists a continuous unitary path $w\colon [0,1] \to A$ such that $w(0) = u$ and $w(1) = v$.

Given $n, m \in \mathbb{N}$ with $n \leq m$ and unitaries $u \in M_n(A)$ and $v \in M_m(A)$, we say that $u$ and $v$ are $K_1$-*equivalent*, and write $u \sim_1 v$, if there exists $k \geq 0$ such tha $u \oplus 1_{m-n} \oplus 1_k \sim_h v \oplus 1_k$. The $K_1$-equivalence class of $u$ is denoted $[u]_1$.

The $K_1$-group of a $C^*$-algebra is defined as follows:

**Definition 2.3.14.** We define the $K_1$*-group* $K_1(A)$ of $A$ as

$$K_1(A) = \{[u]_1 \colon u \in \bigcup_{n=1}^{\infty} M_n(\widetilde{A}) \text{ unitary}\}.$$

Addition on $K_1(A)$ is given by $[u]_1 + [v]_1 = [\mathrm{diag}(u, v)]_1$ for unitaries $u, v \in M_\infty(A)$. Then $K_1(A)$ is a group, with $-[u]_1 = [u^*]_1$ and unit given by $[1_{M_n(\widetilde{A})}]_1$ for any $n \in \mathbb{N}$.

With the above definition, $K_1$ is a functor from the category of all $C^*$-algebras (with arbitrary homomorphisms) to the category of abelian groups. Both functors $K_0$ and $K_1$ share a number of properties, some of which we summarize in the next theorem.

**Theorem 2.3.15.** The functors $K_0$ and $K_1$ satisfy the following properties.

1. They commute with direct sums:

   $$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B) \quad \text{and} \quad K_1(A \oplus B) \cong K_1(A) \oplus K_1(B)$$

   for all $C^*$-algebras $A$ and $B$.

2. They commute with direct limits: If $A = \varinjlim((A_j)_{j\in J}, (\varphi_{j,k})_{j,k\in J})$, then

   $$\begin{aligned} K_0(A) &\cong \varinjlim((K_0(A_j))_{j\in J}, (K_0(\varphi_{j,k}))_{j,k\in J}) \\ K_1(A) &\cong \varinjlim((K_1(A_j))_{j\in J}, (K_1(\varphi_{j,k}))_{j,k\in J}) \end{aligned}$$

3. They are stable: for $n \in \mathbb{N}$, let $\iota_n \colon A \to M_n(A)$ be the upper-left corner embedding. Then $K_0(\iota_n)$ and $K_1(\iota_n)$ are isomorphisms.

4. They are homotopy invariant: if $\varphi, \psi \colon A \to B$ are homotopy equivalent homomorphisms, then $K_0(\varphi) = K_0(\psi)$ and $K_1(\varphi) = K_1(\psi)$. In particular, $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$ whenever $A \sim_h B$.

Observe that parts (2) and (3) in the above theorem imply that the natural map $A \to A \otimes \mathcal{K}$, given by $a \mapsto a \otimes e_{1,1}$, induces isomorphisms of the $K$-groups of $A$ and $A \otimes \mathcal{K}$.

Using part (2) in Theorem 2.3.15, it is possible to effortlessly compute the $K$-theory of UHF-algebras.

**Exercise 2.3.16.** The goal of this exercise is to compute the $K$-theory of a UHF-algebra in terms of its supernatural number.

1. Let $n, m \in \mathbb{N}$ with $n|m$, and let $\varphi \colon M_n \to M_m$ be any unital homomorphism. Compute the induced map $K_0(\varphi) \colon K_0(M_n) \to K_0(M_m)$.

2. Let $M_{2^\infty}$ denote the CAR algebra, which is the direct limit of the matrix algebras $M_{2^n}$, for $n \in \mathbb{N}$, we the canonical unital maps. Compute $K_0(M_{2^\infty})$ and $K_1(M_{2^\infty})$.

3. Let $S$ be a supernatural number. Compute the $K$-groups of the UHF-algebra associated to $S$.

**Exercise 2.3.17.** Let $A$ be a $C^*$-algebra. We say that an automorphism $\varphi$ of $A$ is *inner* if there exists a unitary $u \in M(A)$ such that $\varphi = \mathrm{Ad}(u)$. We denote by $\mathrm{Inn}(A)$ the set of all inner automorphisms of $A$.

1. Show that the canonical map $U(M(A)) \to \mathrm{Inn}(A)$ is a group homomorphism.

2. Show that $\mathrm{Inn}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$.

3. The closure $\overline{\mathrm{Inn}}(A)$ of $\mathrm{Inn}(A)$ in $\mathrm{Aut}(A)$ is the group of *approximately inner* automorphisms. Show that $\overline{\mathrm{Inn}}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$.

4. Show that $K_0(\varphi) = \mathrm{id}_{K_0(A)}$ and $K_1(\varphi) = \mathrm{id}_{K_1(A)}$ is $\varphi$ is approximately inner.

5. Is the converse to the previous item true?

For a $C^*$-algebra $A$, denote by $SA$ its suspension, which is isomorphic to $C_0(\mathbb{R}, A)$. Then $K_1(A)$ can be alternatively defined as $K_1(A) = K_0(SA)$. One then defines the higher $K$-groups inductively, by setting $K_n(A) = K_{n-1}(SA)$ for $n \geq 1$.

### Bott periodicity

Perhaps the most fundamental result in $K$-theory for $C^*$-algebras (or more generally, complex Banach algebras) is the fact, known as Bott periodicity, that $K_0(A)$ is naturally isomorphic to $K_1(SA)$. This implies that $K_{n+2}(A)$ is naturally isomorphic to $K_n(A)$ for all $n \in \mathbb{N}$; in other words, complex $K$-theory is periodic with period 2. (For the sake of comparison, we mention here that real $K$-theory is periodic with period 8.)

**Theorem 2.3.18.** (Bott periodicity). Let $A$ be a $C^*$-algebra. Then there is a natural isomorphism $\beta_A \colon K_0(A) \to K_1(SA)$.

For the proof, we refer the reader to Section 9.2 in Blackadar's book [6]. We nevertheless discuss here the easiest case, namely $A = \mathbb{C}$. This involves the Bott projection on the 2-sphere.

**Example 2.3.19.** When $A = \mathbb{C}$, the Bott map is an isomorphism between $K_0(\mathbb{C})$ and $K_0(C_0(\mathcal{R}^2))$. Since $K_0(\mathbb{C})$ is generated by the class of the unit, it suffices to determine what $\beta_{\mathbb{C}}([1]_0)$ is. We identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $p, q \in M_2(C_0(\mathbb{C})^+)$ be the projections given by

$$p(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q(z) = \frac{1}{1+|z|^2} \begin{pmatrix} |z|^2 & z \\ \overline{z} & 1 \end{pmatrix}.$$

Then the map $\beta_{\mathbb{C}}$ is determined by $\beta_{\mathbb{C}}([1]_0) = [q]_0 - [p]_0$.

The projection $q$ is called the *Bott projection*, and it can be identified with the projection coming from a bundle as follows[2]. Identify the one-point compactification of $\mathbb{C}$ with the space $\mathbb{C}P^1$ of subspaces of $\mathbb{C}^2$ of complex dimension one, and let $V$ be the *Bott bundle*, which is given by

$$V = \{(M, v) \in \mathbb{C}P^1 \times \mathbb{C}, v \in M\}.$$

This is a sub-bundle of the trivial 2-dimensional bundle over $\mathbb{C}P^1 \cong S^2$, and hence it is a direct summand in it. Under the natural identifications mentioned here, $q$ is the projection onto this sub-bundle.

### The 6-term exact sequence in $K$-theory

Let

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence of $C^*$-algebras. By functoriality of the $K$-groups, it follows that $K_0(\pi) \circ K_0(\iota) = 0$ and $K_1(\pi) \circ K_1(\iota) = 0$. It can even be shown that $\ker(K_0(\pi)) = \mathrm{Im}(K_0(\iota))$ and $\ker(K_1(\pi)) = \mathrm{Im}(K_1(\iota))$, so that the following sequence is exact, for $j = 0, 1$:

$$K_j(I) \xrightarrow{K_j(\iota)} K_j(A) \xrightarrow{K_j(\pi)} K_j(B).$$

However, $K_0(\iota)$ and $K_1(\iota)$ may fail to be injective, while $K_0(\pi)$ and $K_1(\pi)$ may fail to be surjective, so that the $K$-functors do not preserve short exact sequences. There is, however, a 6-term exact sequence of $K$-groups associated to any short exact sequence of $C^*$-algebras, which resembles the long exact sequences in cohomology.

**Theorem 2.3.20.** Let

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence of $C^*$-algebras. Then there exist group homomorphisms $\delta_0 \colon K_0(B) \to K_1(I)$ (called the *exponential map*) $\delta_1 \colon K_1(B) \to K_0(I)$

[2]Recall that for the algebra of continuous functions on a compact Hausdorff space, matrix-valued projections are in one-to-one correspondence with complex vector bundles over the space.

(called the *index map*), making the following an exact sequence:

$$\begin{tikzcd} K_0(I) \arrow[r, "K_0(\iota)"] & K_0(A) \arrow[r, "K_0(\pi)"] & K_0(B) \arrow[d, "\delta_0"] \\ K_1(B) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_1(\pi)"] & K_1(I) \arrow[l, "K_1(\iota)"] \end{tikzcd}$$

The exponential map $\delta_0 \colon K_0(B) \to K_1(I)$ measures the obstruction to lifting projections in (a matrix algebra over) $B$ to a projection in (a matrix algebra) over $A$, and its name reflects the fact that the way that one obtains a unitary in $\widetilde{I}$ is by taking exponentials.

The index map $\delta_1 \colon K_1(B) \to K_0(I)$ measures the obstruction to lifting unitaries in (a matrix algebra over) $B$ to a unitary in a matrix algebra over $A$. It takes its name from the fact that it generalizes the classical Fredholm index of Fredholm operators on a Hilbert space.

**Exercise 2.3.21.** 1. Let

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence of $C^*$-algebras, and suppose that there is a homomorphism $s \colon B \to A$ satisfying $\pi \circ s = \mathrm{id}_B$. Show that $K_j(A) \cong K_j(I) \oplus K_j(B)$.

2. Let $A$ be a $C^*$-algebra. Use the previous part to compute the $K$-theory of $C(S^1) \otimes A$.

3. Find examples of short exact sequences as in (1) where:

   a) $K_0(\iota)$ is not injective.
   b) $K_1(\iota)$ is not injective.
   c) $K_0(\pi)$ is not surjective.
   d) $K_1(\pi)$ is not surjective.

## 2.4 Hilbert modules and operators

Hilbert modules are a simultaneous generalization of Hilbert spaces and $C^*$-algebras, and are a very useful tool that allow one to give elegant and unified proofs of important parts of the theory. Hilbert modules have had applications in three main areas:

- Rieffel's theory of induced representations and Morita equivalence of $C^*$-algebras [82];
- Kasparov's $KK$-theory [50];
- Woronowicz's theory of quantum groups [107].

In these notes, we will be mostly concerned with the first of these applications, specifically in Chapter 6. Cite Lance and Blackadar.

For a $C^*$-algebra $A$, a Hilbert $A$-module is an $A$-module with an $A$-valued inner product, satisfying axioms analogous to those satisfied by Hilbert spaces (which are Hilbert $\mathbb{C}$-modules[3]). This is the formal definition:

**Definition 2.4.1.** Let $A$ be a $C^*$-algebra and let $\mathcal{E}$ be a right $A$-module. An *$A$-valued inner product* on $\mathcal{E}$ is a map $\langle\cdot,\cdot\rangle_A \colon \mathcal{E}\times\mathcal{E}\to A$ which is linear on the second coordinate and satisfies the following properties for all $\xi,\eta,\zeta\in\mathcal{E}$:

1. $\langle\xi,\eta\cdot a\rangle_A = \langle\xi,\eta\rangle_A a$ for all $\xi,\eta\in\mathcal{E}$ and all $a\in A$;
2. $\langle\xi,\eta\rangle_A^* = \langle\eta,\xi\rangle_A$ for all $\xi,\eta\in\mathcal{E}$;
3. $\langle\xi,\xi\rangle_A\geq 0$ for all $\xi\in\mathcal{E}$ and $\langle\xi,\xi\rangle_A = 0$ if and only if $\xi=0$.

If $\mathcal{E}$ is complete in the norm induced by $\langle\cdot,\cdot,\rangle_A$, then we say that $(\mathcal{E},\langle\cdot,\cdot\rangle_A)$ is a *Hilbert $A$-module*.

When the coefficient algebra is clear from the context, we will write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_A$.

Hilbert modules behave very similarly to Hilbert spaces, and many constructions and arguments can be adapted to this context with only minor changes. There is just one important exception, namely that orthogonality is not nearly as well behaved as in the Hilbert space case. For once, Hilbert submodules are rarely complemented, and it is often the case that for a *proper* submodule $\mathcal{F}\subseteq\mathcal{E}$, one has $\mathcal{F}^\perp=\{0\}$, and thus $\mathcal{F}^{\perp\perp}$ is much larger than $\mathcal{F}$.

**Examples 2.4.2.** Let $A$ be a $C^*$-algebra.

1. If $I$ is an ideal in $A$, we may regard $I$ as a right Hilbert $A$-module, with the $A$-action given by (right) multiplication, and the inner product given by $\langle x,y\rangle_A = x^*y$ for all $x,y\in I$.
2. Let $p\in M(A)$ be a projection, and set $\mathcal{E}=pA$. Then $\mathcal{E}$ is a Hilbert $pAp-A$-bimodule, with left and right actions given by multiplication, and inner products given by
$$_{pAp}\langle pa,pb\rangle = pa(pb)^*, \quad\text{and}\quad \langle pa,pb\rangle_A = (pa)^*pb$$
for all $a,b\in A$.
3. If $(\mathcal{E}_j)_{j\in J}$ is a family of Hilbert $A$-modules, then the algebraic direct sum $\bigoplus\limits_{j\in J}^{\mathrm{alg}}\mathcal{E}_j$ admits a pre-inner product given by
$$\langle(\xi_j)_{j\in J},(\eta_j)_{j\in J}\rangle = \sum_{j\in J}\langle\xi_j,\eta_j\rangle$$

[3]To be precise, Hilbert spaces are the *left* Hilbert $\mathbb{C}$-modules

for $(\xi_j)_{j\in J}, (\eta_j)_{j\in J} \in \bigoplus\limits_{j\in J}^{\mathrm{alg}} \mathcal{E}_j$. The completion of the algebraic direct sum with respect to the induced norm is the Hilbert $A$-module $\bigoplus\limits_{j\in J} \mathcal{E}_j$. When $J$ is finite, no completion is needed.

4. As a special case of the previous example, when $J = \mathbb{N}$ and $\mathcal{E}_j = A$ for all $j \in \mathbb{N}$, we write $\mathcal{H}_A$ for $\bigoplus\limits_{n\in\mathbb{N}} A$. An alternative description of $\mathcal{H}_A$ is

$$\mathcal{H}_A = \left\{ (a_n)_{n\in\mathbb{N}} \colon a_n \in A, \sum_{n\in\mathbb{N}} a_n^* a_n \text{ converges in } A \right\},$$

with the inner product given by $\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \rangle = \sum\limits_{n\in\mathbb{N}} a_n^* b_n$ for all $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathcal{H}_A$.

5. If $\mathcal{E}$ is a Hilbert $A$-module, we define its *dual module* $\mathcal{E}^*$ to be $\mathcal{E}^* = \{\xi^* \colon \xi \in \mathcal{E}\}$ with $\xi^* + \eta^* = (\xi + \eta)^*$ and $\lambda\xi^* = (\overline{\lambda}\xi)^*$ for all $\xi, \eta \in \mathcal{E}$ and all $\lambda \in \mathbb{C}$. We endow $\mathcal{E}^*$ with a left Hilbert $A$-module structure by setting
$$a \cdot \xi^* = (\xi \cdot a)^*, \quad \text{and} \quad {}_A\langle \xi^*, \eta^* \rangle = \langle \eta, \xi \rangle_A.$$

We turn to operators between Hilbert modules.

**Definition 2.4.3.** Let $A$ be a $C^*$-algebra, and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $A$-modules. We say that a function $T \colon \mathcal{E} \to \mathcal{F}$ is *adjointable* if there exists a function $T^* \colon \mathcal{F} \to \mathcal{E}$ satisfying

$$\langle T(\xi), \eta \rangle = \langle \xi, T^*(\eta) \rangle$$

for all $\xi, \eta \in \mathcal{E}$. The operator $T^*$ is called the *adjoint* of $T$, and is uniquely determined by $T$.

We denote by $\mathcal{L}_A(\mathcal{E}, \mathcal{F})$ the set of all adjointable operators from $\mathcal{E}$ to $\mathcal{F}$, and abbreviate $\mathcal{L}_A(\mathcal{E}, \mathcal{E})$ to $\mathcal{L}_A(\mathcal{E})$. We also omit the subscript $A$ whenever confusion is unlikely to arise.

The critical reader will notice that adjointable operators are not assumed to be linear or continuous. Indeed, this is automatic:

**Proposition 2.4.4.** Let $A$ be a $C^*$-algebra, and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $A$-modules, and let $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$. Then $T$ is a continuous, linear, $A$-module map.

*Proof.* We begin with linearity. Given $\xi_1, \xi_2, \eta \in \mathcal{E}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\langle T(\xi_1 + \lambda\xi_2), \eta \rangle &= \langle \xi_1 + \lambda\xi_2, T^*(\eta) \rangle \\
&= \langle \xi_1, T^*(\eta) \rangle + \lambda \langle \xi_2, T^*(\eta) \rangle \\
&= \langle T(\xi_1) + \lambda T(\xi_2), \eta \rangle.
\end{aligned}$$

We deduce that $T$ is linear. A similar computation, using that the inner product respects the $A$-action, shows that $T$ is an $A$-module map. Finally, to check continuity of $T$, we use the Closed Graph Theorem. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ converging to 0, and suppose that there exists $\eta \in \mathcal{F}$ such that $(T(\xi_n))_{n\in\mathbb{N}}$ converges to $\eta$. Then

$$0 = \lim_{n\to\infty} \langle \xi_n, T^*(\eta)\rangle = \lim_{n\to\infty} \langle T(\xi_n), \eta\rangle = \langle \eta, \eta\rangle,$$

and hence $\eta = 0$, as desired. □

Unlike for Hilbert spaces, it is not true that every bounded $A$-module map is automatically adjointable. A counterexample is the canonical inclusion $C_0((0,1]) \hookrightarrow C([0,1])$, regarded as a map between Hilbert $C([0,1])$-modules as in item (1) of Examples 2.4.2. On the other hand, the following is shown just as in the Hilbert space case.

**Proposition 2.4.5.** Let $A$ be a $C^*$-algebra and let $\mathcal{E}$ be a Hilbert $A$-module. Endow $\mathcal{L}(\mathcal{E})$ with the adjoint operation and with the operator norm. Then $\mathcal{L}(\mathcal{E})$ is a $C^*$-algebra.

Again, as in the case of Hilbert spaces, operators of finite rank and compact operators play an important role in the theory, so we proceed to introduce these.

**Definition 2.4.6.** Let $A$ be a $C^*$-algebra, and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $A$-modules. Given $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$, we let $\theta_{\eta,\xi} \in \mathcal{L}_A(\mathcal{E},\mathcal{F})$ be the rank-one operator given by $\theta_{\eta,\xi}(\zeta) = \eta \cdot \langle \xi, \zeta\rangle$ for all $\zeta \in \mathcal{E}$. One checks easily that $\theta_{\eta,\xi}^* = \theta_{\xi,\eta}$, so $\theta_{\eta,\xi}$ is indeed an adjointable operator. We denote by $\mathcal{K}_A(\mathcal{E},\mathcal{F})$ the closed linear span of $\{\theta_{\eta,\xi} \colon \xi \in \mathcal{E}, \eta \in \mathcal{F}\}$ in $\mathcal{L}_A(\mathcal{E},\mathcal{F})$, and call the elements in it *$A$-compact operators*. We abbreviate $\mathcal{K}_A(\mathcal{E},\mathcal{E})$ to $\mathcal{K}_A(\mathcal{E})$. We also omit the subscript $A$ whenever confusion is unlikely to arise.

It is an easy exercise to verify that $\mathcal{K}(\mathcal{E})$ is an ideal in $\mathcal{L}(\mathcal{E})$, and hence a $C^*$-algebra in its own right.

**Remark 2.4.7.** A word of warning is in order. If $T \colon \mathcal{E} \to \mathcal{F}$ is an $A$-compact operator, then $T$ will in general *not* be a compact operator in the usual sense, when $\mathcal{E}$ and $\mathcal{F}$ are regarded as Banach spaces; see Examples 2.4.8.

Next, we compute the algebra $\mathcal{K}(\mathcal{E})$ in some cases of interest.

**Examples 2.4.8.** Let $A$ be a $C^*$-algebra.

1. If $A$ is regarded as a Hilbert $A$-module as in part (1) of Examples 2.4.2, and $a, b \in A$, then one readily verifies that $\theta_{a,b}$ is left multiplication by $ab^*$ (which is an arbitrary element in $A^2 = A$). Moreover, $\|\theta_{a,b}\| = \|ab^*\|$. We deduce that $\mathcal{K}(A)$ is naturally isomorphic to $A$. When $A$ is unital, the operator $\theta_{1,1}$ is the unit of $\mathcal{L}(A)$, and thus $\mathcal{K}(A) = \mathcal{L}(A) = A$. In this case, the operator $\mathrm{id}_A \colon A \to A$ is $A$-compact, although it is not compact unless $A$ is finite-dimensional.

2. Let $\mathcal{E}$ be a Hilbert $A$-module, and denote by $\mathcal{E}^\infty$ the infinite direct sum of countably many copies of $\mathcal{E}$. Then $\mathcal{K}(\mathcal{E}^\infty)$ can be identified with the closure of $\bigcup\limits_{n\in\mathbb{N}} M_n(\mathcal{K}(\mathcal{E}))$ in $\mathcal{L}(\mathcal{E}^\infty)$. In particular, $\mathcal{K}(\mathcal{E}^\infty)$ is isomorphic to $\mathcal{K}(\mathcal{E})\otimes\mathcal{K}$. For $\mathcal{E}=\mathcal{H}_A$, and in combination with the preivous example, this gives the useful identity $\mathcal{K}(\mathcal{H}_A)\cong A\otimes\mathcal{K}$.

A number of results referring to representations of $C^*$-algebras on Hilbert spaces can be generalized in a straightforward manner (mostly even without changes) to representations of $C^*$-algebras on Hilbert modules. For example, universal $C^*$-algebras on generators and relations could have been defined using representations of the relations on Hilbert modules, rather than Hilbert spaces, without changing the outcome. Another example refers to multiplier algebras:

**Proposition 2.4.9.** Let $A$ and $B$ be $C^*$-algebras, let $\mathcal{E}$ be a Hilbert $B$-module, and let $\pi\colon A\to\mathcal{L}_B(\mathcal{E})$ be a non-degenerate, injective representation. Then $M(A)$ can be canonically identified with

$$\{T\in\mathcal{L}_B(\mathcal{E})\colon T\pi(A)\subseteq\pi(A), \pi(A)T\subseteq\pi(A)\}.$$

The proof is identical to that of Exercise 2.2.8, so we omit it. This presentation of $M(A)$ does have an interesting consequence, in the case $A=\mathcal{K}_B(\mathcal{E})$:

**Corollary 2.4.10.** Let $A$ be a $C^*$-algebra, and let $\mathcal{E}$ be a Hilbert $A$-module. Then $M(\mathcal{K}(\mathcal{E}))=\mathcal{L}(\mathcal{E})$. In particular, $M(A)=\mathcal{L}(A)$, when $A$ is regarded as a Hilbert $A$-module.

As for Hilbert spaces, we say that two Hilbert $A$-modules $\mathcal{E}$ and $\mathcal{F}$ are *isomorphic* if there exists a unitary between them, that is, if there exists a bijective map $U\in\mathcal{L}(\mathcal{E},\mathcal{F})$ satisfying $\langle U(\xi),U(\eta)\rangle=\langle\xi,\eta\rangle$ for all $\xi,\eta\in\mathcal{E}$.

Perhaps the most significant result in the theory of Hilbert modules is Kasparov's Stabilization/Absorption Theorem. We say that a Hilbert $A$-module $\mathcal{E}$ is *countably generated* if there exists a countable set $X\subseteq\mathcal{E}$ such that $XA$ is dense in $\mathcal{E}$.

**Theorem 2.4.11.** Let $A$ be a $C^*$-algebra, and let $\mathcal{E}$ be a countably generated Hilbert $A$-module. Then

$$\mathcal{E}\oplus\mathcal{H}_A\cong\mathcal{H}_A.$$

Note that $\mathcal{H}_A$ is countably generated if and only if $A$ is $\sigma$-unital. For this reason, in applications one usually restricts the attention to $\sigma$-unital $C^*$-algebras.

In order to prove Theorem 2.4.11, we will need the following lemma, which is proved in the same way as for Hilbert space. Recall that an element $b$ in some $C^*$-algebra $B$ is said to be *strictly positive (in $B$)* if $\overline{bB}=B$.

**Lemma 2.4.12.** Let $A$ be a $C^*$-algebra, let $\mathcal{E}$ be a countably generated Hilbert $A$-module, and let $T\in\mathcal{K}(\mathcal{E})$ be a positive element. Then $T$ is stricly positive if and only if it has dense range.

*Proof.* If $T$ is strictly positive, then $\overline{T\mathcal{K}(\mathcal{E})} = \mathcal{K}(\mathcal{E})$. Now, since $\mathcal{K}(\mathcal{E})\mathcal{E}$ is dense in $\mathcal{E}$, it follows that $T(\mathcal{E}$ is dense in $\mathcal{E}$, as desired. Conversely, assume $T$ has dense range, and let $\xi, \eta \in \mathcal{E}$. We will show that $\theta_{\xi,\eta}$ belongs to the closure of $T\mathcal{K}(\mathcal{E})$. Let $(\xi_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{E}$ with $\lim\limits_{n\to\infty} T(\xi_n) \to \xi$. Then

$$\theta_{\xi,\eta} = \lim_{n\to\infty} T \circ \theta_{\xi_n,\eta} \in \overline{T\mathcal{K}(\mathcal{E})},$$

as desired. □

**Exercise 2.4.13.** Prove Theorem 2.4.11, as follows.

1. If $A^+$ denotes the one-dimensional unitization of $A$, denote by $\mathcal{E}^+$ the Hilbert $A^+$-module which as a vector space is identical to $\mathcal{E}$, with the obvious extended action and the same $A$-valued inner product as $\mathcal{E}$. Show that if $\mathcal{E}^+ \oplus \mathcal{H}_{A^+} \cong \mathcal{H}_{A^+}$, then $\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A$. Deduce that it is enough to prove the theorem when $A$ is unital (which we will assume from now on).

2. Let $(\xi_n)_{n\in\mathbb{N}}$ be an enumeration of a countable generating set for $\mathcal{E}$, with each element repeated an infinite number of times, and let $(\delta_n)_{n\in\mathbb{N}}$ be the canonical orthonormal basis of $\mathcal{H}_A$. Show that there is a well-defined operator $T \in \mathcal{K}_A(\mathcal{H}_A, \mathcal{E} \oplus \mathcal{H}_A)$ that satisfies $T(\delta_n) = (\xi_n/2^n, \delta_n/4^n)$ for all $n \in \mathbb{N}$.

3. Show that $T$ is injective and has dense range.

4. Show that $T^*T$ has dense range, and is hence strictly positive.

5. Show that there is a unique well-defined operator $U \in \mathcal{L}(\mathcal{H}_A, \mathcal{E} \oplus \mathcal{H}_A)$ satisfying $U((T^*T)^{1/2}\xi) = T(\xi)$ for all $\xi \in \mathcal{E}$.

6. Show that $U$ is a unitary, concluding the proof of the theorem.

## 2.5 Morita equivalence

For a Hilbert $A$-module $\mathcal{E}$, the set $\{\langle \xi, \eta\rangle_A : \xi, \eta \in \mathcal{E}\}$ is not in general closed under sums, and its closed linear span $A_{\mathcal{E}}$ is an ideal in $A$ which does not agree with $A$ in general. If $A_{\mathcal{E}} = A$, then we say that $\mathcal{E}$ is a *full* Hilbert $A$-module. Clearly $\mathcal{E}$ is always a full Hilbert $A_{\mathcal{E}}$-module.

**Definition 2.5.1.** Let $A$ and $B$ be $C^*$-algebras. We say that $A$ and $B$ are *Morita equivalent*, written $A \sim_M B$, if there exists an $A-B$-bimodule $\mathcal{E}$ which is simultaneously a full left Hilbert $A$-module and a full right Hilbert $B$-module, satisfying

$${}_A\langle \xi, \eta\rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta\rangle_B$$

for all $\xi, \eta, \zeta \in \mathcal{E}$. A bimodule $\mathcal{E}$ as above is called an *imprimitivity bimodule*.

In this section, we will prove two important results concerning Morita equivalence. First, and using a construction known as the *linking algebra*, we will show in Theorem 2.5.3 that two $C^*$-algebras are Morita equivalent if and only if there exists a third $C^*$-algebra into which both embed as full corners. And second, we will prove in Theorem 2.5.11 that for $\sigma$-unital $C^*$-algebras, Morita equivalence is the same as stable isomorphism.

The following example will be particularly important for us.

**Example 2.5.2.** Let $A$ be a $C^*$-algebra, and let $p \in M(A)$ be a projection. We have seen in Examples 2.4.2 that $\mathcal{E} = pA$ is a $pAp - A$ bimodule, and it can be easily checked that it satisfies the identity in Definition 2.5.1. As a left Hilbert $pAp$-module, $\mathcal{E}$ is full, while $A_{\mathcal{E}}$ is the ideal $\overline{ApA}$ generated by $p$. In other words, $\mathcal{E}$ induces a Morita equivalence between $pAp$ and $\overline{ApA}$.

In some sense, every Morita equivalence is as in the previous example.

**Theorem 2.5.3.** Let $A$ and $B$ be $C^*$-algebras. Then $A \sim_M B$ if and only if there exist a $C^*$-algebra $C$ and a projection $p \in M(C)$ with $\overline{CpC} = \overline{C(1-p)C} = C$ such that $pCp \cong A$ and $(1-p)C(1-p) \cong B$.

**Exercise 2.5.4.** Prove Theorem 2.5.3. For the "only if" implication, let $\mathcal{E}$ be an imprimitivity bimodule and consider

$$C = \begin{bmatrix} A & \mathcal{E} \\ \mathcal{E}^* & B \end{bmatrix} = \left\{ \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} : a \in A, b \in B, \xi, \eta \in \mathcal{E} \right\}.$$

Define a canonical matrix-type product and involution on $C$. Let $C$ act on $\mathcal{E} \oplus B$ by

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \begin{pmatrix} \zeta \\ c \end{pmatrix} = \begin{pmatrix} a \cdot \zeta + \xi \cdot c \\ \langle \eta, \zeta \rangle_B + bc \end{pmatrix},$$

1. Prove that $C$ is a $C^*$-algebra with the induced operator norm. This $C^*$-algebra is called the *linking algebra* associated to $\mathcal{E}$.

2. Show that $C$ and $p = \begin{pmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{pmatrix}$ satisfy the conclusion of the theorem.

3. Where is fullness of $\mathcal{E}$ used?

For an arbitrary Hilbert $B$-module $\mathcal{E}$, one can define a left Hilbert $\mathcal{K}_B(\mathcal{E})$-module structure on $\mathcal{E}$ by

$$\theta_{\xi,\eta} \cdot \zeta = \Theta_{\xi,\eta}(\zeta) \quad \text{and} \quad {}_{\mathcal{K}_B(\mathcal{E})}\langle \xi, \eta \rangle = \Theta_{\eta,\xi}$$

for all $\xi, \eta, \zeta \in \mathcal{E}$. Then $\mathcal{E}$ is a $\mathcal{K}(\mathcal{E}) - B$-imprimitivity bimodule. The converse to this remark is also true:

**Lemma 2.5.5.** Let $A$ and $B$ be $C^*$-algebras, and let $\mathcal{E}$ be an $A-B$-imprimitivity bimodule. Then there is a natural isomorphism $\mathcal{K}_B(\mathcal{E}) \cong A$.

*Proof.* Given $\xi, \eta \in \mathcal{E}$, define $\varphi(\Theta_{\xi,\eta}) = {}_A\langle \eta, \xi \rangle$. By linearity and continuity, one extends this assignment to a map $\varphi\colon \mathcal{K}_B(\mathcal{E}) \to A$, which is easily seen to be a homomorphism. The map is surjective, because the set of all $A$-inner products spans a dense subspace of $A$, and also injective, since it is injective on the set of finite-rank operators. We omit the details. $\square$

We need to introduce the definition of the (internal) tensor product of Hilbert modules.

**Definition 2.5.6.** Let $A$ and $B$ be $C^*$-algebras, let $\mathcal{E}$ be a Hilbert $A$-module, let $\mathcal{F}$ be a Hilbert $B$-module, and let $\phi\colon A \to \mathcal{L}(\mathcal{F})$ be a homomorphism. We regard $\mathcal{F}$ as a left $A$-module with the action given by $a \cdot \eta = \phi(a)(\eta)$ for all $a \in A$ and all $\eta \in \mathcal{F}$. The algebraic tensor product $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ therefore has the structure of a right $B$-module. Denote by $\mathcal{E} \otimes_\phi \mathcal{F}$ the quotient of $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ by the subspace generated by elements of the form

$$\xi a \otimes \eta - x \otimes \phi(a)(\eta),$$

for $\xi \in \mathcal{E}$, $\eta \in \mathcal{F}$ and $a \in A$. This is a Hilbert $B$-module with action given by $(\xi \otimes \eta)b = \xi \otimes (\eta b)$, and inner product determined by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \phi(\langle \xi_1, \xi_2 \rangle)(\eta_2) \rangle$$

for all $\xi_1, \xi_2 \in \mathcal{E}$ and all $\eta_1, \eta_2 \in \mathcal{F}$. We call $\mathcal{E} \otimes_\phi \mathcal{F}$ the *internal tensor product* of $\mathcal{E}$ and $\mathcal{F}$ with respect to $\phi$.

Implicit in the definition above is the fact that the sesquilinear form defined is indeed an inner product. We omit this verification. The notation $\mathcal{E} \otimes_\phi \mathcal{F}$ is meant to stress the fact that the tensor product depends on the choice of $\phi$. In cases where there is a canonical (or unique) choice, such as in the following example, we may drop $\phi$ from the notation and simply write $\mathcal{E} \otimes_A \mathcal{F}$.

**Example 2.5.7.** Let $\mathcal{H} = \ell^2(\mathbb{N})$, and $B$ be a $C^*$-algebra. We regard $\mathcal{H}$ as a Hilbert $\mathbb{C}$-module and $B$ as a $B$-module. Let $\phi\colon \mathbb{C} \to M(B) = \mathcal{L}(B)$ be the unique unital homomorphism. Then $\mathcal{H} \otimes_{\mathbb{C}} B$ is canonically isomorphic to $\mathcal{H}_B$.

**Example 2.5.8.** More generally, let $B$ be a $C^*$-algebra and let $\mathcal{E}$ be a Hilbert $B$-module. Let $\phi\colon \mathbb{C} \to \mathcal{L}(\mathcal{E})$ be the unique unital homomorphism. Then $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}$ is canonically isomorphic to the $B$-module $\mathcal{E}^\infty$ from item (2) in Examples 2.4.8.

**Exercise 2.5.9.** Let $A$ and $B$ be $C^*$-algebras, let $\mathcal{F}$ be a Hilbert $B$-module, and let $\phi\colon A \to \mathcal{K}(\mathcal{F})$ be a homomorphism[4]. Show that there is a canonical isomorphism

$$u\colon \mathcal{H}_A \otimes_\phi \mathcal{F} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{F}$$

determined on elementary tensors by $u((\xi \otimes a) \otimes \eta) = \xi \otimes \phi(a)(\eta)$ for all $\xi \in \mathcal{H}$, all $a \in A$ and all $\eta \in \mathcal{F}$. In particular, when $\mathcal{F} = B$ and $\phi\colon A \to B$ is a homomorphism, we get $\mathcal{H}_A \otimes_\phi B \cong \mathcal{H}_B$.

[4]This is equivalent to $\phi\colon A \to \mathcal{L}(\mathcal{F})$ being *nondegenerate* in the sense that $\phi(A)\mathcal{F}$ is dense in $\mathcal{F}$.

In the previous exercise, if the argument is done using $\mathbb{C}$ instead of $\mathcal{H}$, one obtains the isomorphisms $A \otimes_\phi \mathcal{F} \cong \mathcal{F}$ and $A \otimes_\phi B \cong B$.

**Proposition 2.5.10.** Let $A$ and $B$ be $C^*$-algebras, and let $\mathcal{E}$ be an imprimitivity bimodule. Then there is a canonical isomorphism $\varphi\colon \mathcal{E}^* \otimes_A \mathcal{E} \to B$ of Hilbert $B$-modules given by $\varphi(\xi^*, \eta) = \langle \xi, \eta \rangle_B$ for all $\xi, \eta \in \mathcal{E}$.

Since $A = A \otimes e_{1,1} \sim_M A \otimes \mathcal{K}$, it is clear that stable isomorphism implies Morita equivalence. The converse is clearly not true, since for a non-separable Hilbert space $\mathcal{H}$ one has $\mathbb{C} \sim_M \mathcal{K}(\mathcal{H})$ although $\mathbb{C}$ and $\mathcal{K}(\mathcal{H})$ are not stably isomorphic. A fundamental result in the theory of Morita equivalence, due to Brown-Green-Rieffel, asserts that the converse does hold for $\sigma$-unital $C^*$-algebras:

**Theorem 2.5.11.** Let $A$ and $B$ be $\sigma$-unital $C^*$-algebras. Then $A \sim_M B$ if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.

*Proof.* Let $\mathcal{E}$ be an imprimitivity bimodule.

**Claim:** *there is an isomorphism $\mathcal{H}_A \otimes_A \mathcal{E} \cong \mathcal{H}_B$ of Hilbert $B$-modules.* Since $B$ is $\sigma$-unital, the module $\mathcal{E}$, and hence also $\mathcal{E}^*$, is countably generated. In particular, we have $\mathcal{E}^* \oplus \mathcal{H}_A \cong \mathcal{H}_A$ by Theorem 2.4.11. Hence, using Proposition 2.5.10 at the fourth step, we get

$$\begin{aligned}
\mathcal{H}_A \otimes_A \mathcal{E} &\cong \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}_A \otimes_A \mathcal{E} \\
&\cong \mathcal{H} \otimes_{\mathbb{C}} (\mathcal{H}_A \oplus \mathcal{E}^*) \otimes_A \mathcal{E} \\
&\cong (\mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}_A \otimes_A \mathcal{E}) \oplus (\mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}^* \otimes_A \mathcal{E}) \\
&\cong (\mathcal{H}_A \otimes_A \mathcal{E}) \oplus (\mathcal{H} \otimes_{\mathbb{C}} B) \\
&\cong \mathcal{H}_B,
\end{aligned}$$

thus proving the claim.

Recall that $\mathcal{H}_A \otimes_A \mathcal{E} \cong \mathcal{E}^\infty$ as Hilbert $B$-modules (see Example 2.5.8). Combined with the claim above, we deduce that $\mathcal{E}^\infty \cong \mathcal{H}_B$ as Hilbert $B$-modules.

In the following computation, we use Lemma 2.5.5 at the first step; item (2) in Examples 2.4.8 at the second step; the isomorphism $\mathcal{E}^\infty \cong \mathcal{H}_B$ at the third step; and the isomorphism $\mathcal{K}_B(\mathcal{H}_B) \cong B \otimes \mathcal{K}$ at the fourth step (see last claim in item (2) of Examples 2.4.8), to get

$$A \otimes \mathcal{K} \cong \mathcal{K}_B(\mathcal{E}) \otimes \mathcal{K} \cong \mathcal{K}_B(\mathcal{E}^\infty) \cong \mathcal{K}_B(\mathcal{H}_B) \cong B \otimes \mathcal{K}.$$

$\square$

Together with Example 2.5.2, we deduce the following.

**Corollary 2.5.12.** Let $A$ be a $\sigma$-unital $C^*$-algebra, and let $p \in M(A)$ be a projection. Then $pAp \otimes \mathcal{K} \cong \overline{ApA} \otimes \mathcal{K}$. In particular, if $p$ is full in $A$, then $pAp \otimes \mathcal{K} \cong A \otimes \mathcal{K}$.

# Chapter 3

# Group actions

## 3.1 $C^*$-dynamical systems

In this section, we introduce the notion of group action on a $C^*$-algebra, and present a number of examples of them. A large source of examples comes from topological dynamics, while inner actions on noncommutative $C^*$-algebras also play an important role in the theory.

For a $C^*$-algebra $A$, we write $\mathrm{Aut}(A)$ for its automorphism group. If $X$ is any set and $f\colon X \to \mathrm{Aut}(A)$ is a function, and to avoid cumbersome notation, we usually write $f_x$ in place of $f(x)$.

**Definition 3.1.1.** Let $G$ be a topological group, and let $A$ be a $C^*$-algebra. An *(strongly continuous) action* of $G$ on $A$ is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$ such that for every $a \in A$, the map $\alpha^a\colon G \to A$ given by $\alpha^a(g) = \alpha_g(a)$, for all $g \in G$, is continuous. In this case, we say that the triple $(G, A, \alpha)$ is a *$C^*$-dynamical system*, and that the pair $(A, \alpha)$ is a $G$-$C^*$-algebra.

In these notes, actions will always be strongly continuous, and we will not mention it explicitly. Notice that continuity is automatic if the acting group is discrete.

The study of group actions is a generalization of the study of automorphisms of $C^*$-algebras, in view of the following easy observation.

**Remark 3.1.2.** Let $A$ be a $C^*$-algebra. Then there is a one-to-one correspondence between $\mathrm{Aut}(A)$ and $\mathbb{Z}$-actions on $A$. Indeed, given $\varphi \in \mathrm{Aut}(A)$, the associated action $\alpha^\varphi\colon \mathbb{Z} \to \mathrm{Aut}(A)$ is given by $\alpha^\varphi_n(a) = \varphi^n(a)$ for all $n \in \mathbb{Z}$, with the convention that $\varphi^0 = \mathrm{id}_A$.

Similarly, there are a one-to-one correspondences:

- between pairs of commuting automorphisms of $A$ and $\mathbb{Z}^2$-actions on $A$;
- between pairs of automorphisms of $A$ and $\mathbb{F}_2$-actions on $A$;
- between automorphisms of $A$ of order $n$ and $\mathbb{Z}_n$-actions on $A$.

## 3.2 Topological actions

It is well-known that the categories of commutative $C^*$-algebras and that of locally compact Hausdorff spaces are equivalent, via the Gelfand transform. Not surprisingly, for a fixed locally compact group $G$, the categories of commutative $G$-$C^*$-algebras and that of $G$-topological spaces are equivalent. We recall first the definition of a topological action. For a locally compact Hausdorff space, we denote by $\mathrm{Homeo}(X)$ the group of homeomorphisms of $X$.

**Definition 3.2.1.** Let $G$ be a locally compact group and let $X$ be a locally compact Hausdorff space. An *action* of $G$ on $X$ is a group homomorphism $\sigma\colon G \to \mathrm{Homeo}(X)$ such that the map $\widetilde{\sigma}\colon G\times X \to X$ given by $\widetilde{\sigma}(g,x) = \sigma_g(x)$ for all $(g,x) \in G \times X$, is continuous. When no confusion can arise, we often omit the symbol $\sigma$, and just write $G \curvearrowright X$ to mean that $G$ acts on $X$.

**Theorem 3.2.2.** Let $G$ be a locally compact group and let $X$ be a locally compact Hausdorff space. If $\sigma\colon G \to \mathrm{Homeo}(X)$ is a topological action, then the formula $\sigma_g^*(f) = f \circ \sigma_g^{-1}$, for $g \in G$ and $f \in C_0(X)$, defines an action of $G$ on $C_0(X)$.

Moreover, the assignment $\sigma \mapsto \sigma^*$ defines a one-to-one correspondence between $G$-actions on $X$ and $G$-actions on $C_0(X)$.

We now give some relevant examples of topological actions.

**Examples 3.2.3.** Let $G$ be a locally compact Hausdorff space.

1. There is unique action of $G$ on the one-point space $\{*\}$. More generally, every locally compact Hausdorff space $X$ carries an action of $G$, namely the trivial on $\mathrm{id}_X\colon G \to \mathrm{Homeo}(X)$.
2. The action $\mathtt{Lt}\colon G \to \mathrm{Homeo}(G)$ given by $\mathtt{Lt}_g(h) = gh$ for all $g,h \in G$, is called the *left translation action.* With a slight abuse of notation, we also write $\mathtt{Lt}\colon G \to \mathrm{Aut}(C_0(G))$ for the induced action.
3. More generally, if $H$ is a subgroup of $G$, then $H$ acts on $G$ via left translation.
4. The action $\mathrm{Ad}\colon G \to \mathrm{Homeo}(G)$ given by $\mathrm{Ad}_g(h) = ghg^{-1}$ for all $g,h \in G$, is called the *conjugation action.* This action is trivial if and only if $G$ is abelian.
5. Let $\theta \in \mathbb{R}$ be a number. Then the homeomorphism $r_\theta \in \mathrm{Homeo}(S^1)$ given by $r_\theta(\zeta) = e^{2\pi i\theta}\zeta$, for all $\zeta \in S^1$, defines a $\mathbb{Z}$-action on $S^1$. We call this action the *rotation action (by angle $\theta$).* When $\theta$ is irrational, we call it an *irrational rotation.*
6. Define the *boundary* $\partial\mathbb{F}_2$ of $\mathbb{F}_2 = \langle a,b\rangle$ as the set of right-infinite reduced words on $\{a,a^{-1},b,b^{-1}\}$, and endow it with the topology in which two infinite words are close if they agree on some long initial segment. Then $\partial\mathbb{F}_2$ is a Cantor set, and $\mathbb{F}_2$ acts on it by left concatenation (followed, potentially, by reduction). This is called the *boundary action* of $\mathbb{F}_2$.

## 3.3 Actions on noncommutative $C^*$-algebras

We now turn to actions on noncommutative $C^*$-algebras.

**Notation 3.3.1.** If $A$ is a unital $C^*$-algebra, we denote by $\mathcal{U}(A)$ its unitary group. When $A$ is not necessarily unital, we write $M(A)$ for its multiplier algebra. There is a canonical group homomorphism $\mathrm{Ad}\colon \mathcal{U}(M(A)) \to \mathrm{Aut}(A)$ given by $\mathrm{Ad}_u(a) = uau^*$ for all $u \in \mathcal{U}(M(A))$ and all $a \in A$. The image of Ad is denoted by $\mathrm{Inn}(A)$ and it is called the *group of inner automorphisms* of $A$. It is routine to verify that $\mathrm{Inn}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$.

**Definition 3.3.2.** Let $G$ be a locally compact Hausdorff group, let $A$ be a $C^*$-algebra, and let $z\colon G \to \mathcal{U}(M(A))$ be a homomorphism such that for every $a \in A$, the map $z^a\colon G \to A$ given by $z^a(g) = z_g a$ for all $g \in G$, is continuous. Then the map $\mathrm{Ad}(z)\colon G \to \mathrm{Aut}(A)$ given by $\mathrm{Ad}(z)_g(a) = z_g a z_g^*$ for all $g \in G$ and all $a \in A$, is called the *inner action associated to $z$*.

If $\alpha$ is an inner action, in the sense of the definition above, then clearly $\alpha_g \in \mathrm{Inn}(A)$ for all $g \in G$. However, the converse does not hold in general. The following two exercises give counterexamples:

**Exercise 3.3.3.** Set

$$u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

1. Prove that there is a well-defined action $\alpha\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(M_2)$ determined by $\alpha_{(1,0)} = \mathrm{Ad}(u)$ and $\alpha_{(0,1)} = \mathrm{Ad}(v)$.

2. Prove that this is not an inner action, although $\alpha_g \in \mathrm{Inn}(M_2)$ for all $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Exercise 3.3.4.** Define a continuous function $u\colon S^1 \to M_2$ by

$$u_\zeta = \frac{1}{2}\begin{pmatrix} \zeta+1 & i(\zeta-1) \\ i(\zeta-1) & -\zeta-1 \end{pmatrix}$$

for all $\zeta \in S^1$.

1. Prove that $u_\zeta$ is a unitary for all $\zeta \in S^1$.

2. Show that there is a well-defined action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(C(S^1, M_2))$ whose nontrivial automorphism is given by conjugation by $u$.

3. Prove that this is not an inner action, although $\mathrm{Ad}(u)$ is an inner automorphism.

In some special situations, however, every action by inner automorphisms is indeed an inner action. One such situation is given by the following lemma.

**Lemma 3.3.5.** Let $A$ be a unital $C^*$-algebra with trivial center, let $G$ be a finite cyclic group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action satisfying $\alpha_g \in \mathrm{Inn}(A)$ for all $g \in G$. Then $\alpha$ is an inner action.

As Exercise 3.3.3 shows, the assumption that $G$ be a finite cyclic group is necessary in the previous lemma, while Exercise 3.3.4 shows that the assumption that $A$ have trivial center is also necessary.

We finish this section by giving a number of methods for constructing new actions from old ones.

**Theorem 3.3.6.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

1. Let $B$ be another $C^*$-algebra, and let $\beta\colon G \to \mathrm{Aut}(B)$ be another action. Let $\otimes_\gamma$ be a $C^*$-tensor product such that for all $g \in G$, the automorphism $\alpha_g \odot \beta_g$ of the algebraic tensor product $A \odot B$ extends to an isomorphism $\alpha_g \otimes_\gamma \beta_g$ of $A \otimes_\gamma B$. Then there is a well-defined action $\alpha \otimes_\gamma \beta\colon G \to \mathrm{Aut}(A \otimes_\gamma B)$ given by $(\alpha \otimes_\gamma \beta)_g = \alpha_g \otimes_\gamma \beta_g$ for all $g \in G$.

2. Let $I$ be a $G$-invariant ideal of $A$, that is, an ideal satisfying $\alpha_g(I) = I$ for all $g \in G$, and denote by $\pi\colon A \to A/I$ the canonical quotient map. Then $\alpha$ induces an action $\overline{\alpha}\colon G \to \mathrm{Aut}(A/I)$ such that for all $g \in G$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha_g"] \arrow[d, "\pi"'] & A \arrow[d, "\pi"] \\ A/I \arrow[r, "\overline{\alpha}_g"'] & A/I. \end{tikzcd}$$

Furthermore,

3. Let $(\Lambda, \leq)$ be a directed set, and let $((A_\lambda)_{\lambda\in\Lambda}, (\iota_{\mu,\lambda})_{\mu\leq\lambda})$ be a direct system of $C^*$-algebras, and denote by $(A, (\iota_{\lambda,\infty})_{\lambda\in\Lambda})$ its direct limit. For $\lambda \in \Lambda$, let $\alpha^{(\lambda)}\colon G \to \mathrm{Aut}(A_\lambda)$ be the action and assume that $\iota_{\lambda,\mu} \circ \alpha_g^{(\lambda)} = \alpha_g^{(\mu)} \circ \iota_{\lambda,\mu}$ for all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, and all $g \in G$. Then there is a canonical action $\alpha\colon G \to \mathrm{Aut}(A)$ satisfying $\alpha_g \circ \iota_{\lambda,\infty} = \iota_{\lambda,\infty} \circ \alpha_g^{(\lambda)}$ for all $g \in G$ and all $\lambda \in \Lambda$.

The next exercise shows that the assumption on $\otimes_\gamma$ in part (2) of Theorem 3.3.6 is not automatic.

**Exercise 3.3.7.** Let $A_0$ be a $C^*$-algebra such that $A_0 \otimes_{\max} A_0$ is not isomorphic to $A_0 \otimes_{\min} A_0$. (One could take, for example, $A_0$ to be the reduced group $C^*$-algebra of $\mathbb{F}_2$.) Set $A = A_0 \oplus A_0$, and let $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ be the flip action. Denote by $\otimes_\gamma$ the $C^*$-norm on the algebraic tensor product $A \odot A$ satisfying

$$A \otimes_\gamma A = (A_0 \otimes_{\max} A_0) \oplus (A_0 \otimes_{\min} A_0).$$

Show that $\alpha_1 \odot \alpha_1$ does not extend to an isomorphism of $A \otimes_\gamma A$.

Using Theorem 3.3.6, we can introduce a special family of actions on UHF-algebras.

**Example 3.3.8.** For $n \in \mathbb{N}$, let $d_n \in \{2, 3, \ldots\}$ be a natural number, and denote by $M_d$ the UHF-algebra which is the direct limit of $M_{d_1} \otimes \cdots \otimes M_{d_n}$, for $n \in \mathbb{N}$. Let $G$ be a locally compact group, and for every $n \in \mathbb{N}$, let $u_n \colon G \to \mathcal{U}(M_{d_n})$ be a unitary representation. Let $\alpha^{(n)} \colon G \to \mathrm{Aut}(M_{d_n})$ be the inner action associated to $u_n$, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{d_1} \otimes \cdots \otimes M_{d_n})$ be the tensor product of $\alpha^{(1)}, \ldots, \alpha^{(n)}$ (see part (1) of Theorem 3.3.6). Part (3) of Theorem 3.3.6 implies that is a well-defined direct limit action $\beta \colon G \to \mathrm{Aut}(M_d)$.

Actions on UHF-algebras of this form are called *product-type actions*. For compact groups, they have been completely classified in terms of their equivariant $K$-theory by Handelman and Rossmann; see [39].

The class considered in the example above is a particular case of a more general construction of group actions on AF-algebras obtained as certain direct limits of actions on matrix algebras:

**Definition 3.3.9.** Let $A$ be an AF-algebra, let $G$ be a locally compact group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. We say that $\alpha$ is an *AF-action* if there exist an AF-approximation $A = \varinjlim(A_n, \varphi_n)$ of $A$, and actions $\alpha^{(n)} \colon G \to \mathrm{Aut}(A_n)$, for $n \in \mathbb{N}$, making the connecting maps $\varphi_n$ equivariant, and such that $\alpha$ is (conjugate to) the direct limit of $(\alpha_n, \varphi_n)_{n \in \mathbb{N}}$.

**Exercise 3.3.10.** Let $A$ be an AF-algebra, let $G$ be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Show that $A \rtimes_\alpha G$ is an AF-algebra.

For many decades, it was an open problem to decide whether every finite group action on an AF-algebra is necessarily an AF-action. In [5], Blackadar solved this problem negatively by constructing an action of $\mathbb{Z}_2$ on the UHF-algebra of type $2^\infty$ whose crossed product is not AF – such an action cannot be an AF-action by Exercise 3.3.10. The main idea is to write said UHF-algebra in an unusual way as a direct limit of $C^*$-algebras that are not finite-dimensional and do not even have trivial $K_1$. Blackadar's construction revealed that the structure theory of AF-algebras is really much richer than that of their algebraic analogs, and among other things it allowed him to prove that the $2^\infty$ UHF-algebra contains Cartan subalgebras that are themselves not AF.

The following exercise has the purpose of making the reader become familiar with Blackadar's celebrated example. Recall that $K_0(M_n(C(S^1))) \cong K_1(M_n(C(S^1))) \cong \mathbb{Z}$ for all $n \in \mathbb{N}$.

**Exercise 3.3.11.** For $k \in \mathbb{N}$, we identify $M_k(C(S^1))$ as

$$M_k(C(S^1)) = \{f \colon [0,1] \to M_k \text{ continuous} \colon f(0) = f(1), \}$$

and define the *positive twice-around embedding* $\phi_k^+ \colon M_k(C(S^1)) \to M_{2k}(C(S^1))$ by

$$\phi_k^+(f)(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \begin{pmatrix} f(t/2) & 0 \\ 0 & f((t+1)/2) \end{pmatrix} \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}^*$$

for all $f \in M_k(C(S^1))$ and all $t \in [0,1]$. Similarly, the *negative twice-around embedding* $\phi_k^- : M_k(C(S^1)) \to M_{2k}(C(S^1))$ is given by $\phi_k^-(f)(t) = \phi_k^+(f)(1-t)$ for all $f \in M_k(C(S^1))$ and all $t \in [0,1]$.

1. Let $k \in \mathbb{N}$. Show that

$$K_0(\phi_k^\pm)\colon K_0(M_k(C(S^1)) \to K_0(M_{2k}(C(S^1)))$$

is multiplication by 2, while $K_1(\phi_k^\pm)$ is multiplication by the sign of $\phi_k^\pm$.

2. For $n \in \mathbb{N}$, set $A_n = M_{4^n}(C(S^1))$ and let $\psi_n\colon A_n \to A_{n+1}$ be

$$\psi_n(f) = \begin{pmatrix} \phi_{4^n}^+(f) & 0 \\ 0 & \phi_{4^n}^-(f) \end{pmatrix} \otimes 1_{M_2}$$

for all $f \in A_n$. Let $A = \varinjlim(A_n, \psi_n)$. Show that $A$ has the same $K$-theory as the UHF-algebra of type $2^\infty$.

3. Prove that $A$ is an AF-algebra, and conclude that $A$ is isomorphic to the UHF-algebra of type $2^\infty$. [1]

4. For $n \in \mathbb{N}$, set

$$u_n = \mathrm{diag}\,(1, -1, -1, 1) \otimes 1_{4^{n-1}} \in A_n$$

and let $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A_n)$ be the inner action determined by $u_n$. Show that $\psi_n\colon A_n \to A_{n+1}$ is equivariant, and conclude that there is a well-defined limit action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$.

5. With $B_n = M_{2^{2n+1}}(C(S^1))$, show that $A_n^{\alpha_n}$ is isomorphic to $B_n \oplus B_n$.

6. Show that under the above identification, the restriction of $\psi_n$ to $A_n^\alpha$ corresponds to the map $\theta_n\colon B_n \oplus B_n \to B_{n+1} \oplus B_{n+1}$ given by

$$\theta_n(f, g) = \mathrm{diag}\left(\phi_{2^{2n+1}}^+(f), \phi_{2^{2n+1}}^-(g), \phi_{2^{2n+1}}^+(g), \phi_{2^{2n+1}}^-(f)\right)$$

for all $f, g \in B_n$.

7. Show that $K_0(A^\alpha) \cong K_1(A^\alpha) \cong K_0(A)$. Conclude that $A^\alpha$ is not AF, and hence that $\alpha$ is not an AF-action.

Bernoulli shifts are extremely important in topological dynamics and ergodic theory. Since they also play a fundamental role within noncommutative $C^*$-algebras, we formally define them next.

[1] This item is significantly harder than the others, and it becomes easier if one uses the classification of $C^*$-algebras of tracial rank zero Theorem A.4.1.

**Example 3.3.12.** Let $D$ be a unital $C^*$-algebra, and let $G$ be a discrete group. We denote by $D^{\otimes G}$ the infinite tensor product of copies of $D$ indexed by the elements of $G$[2]. We define an action $\beta_{G,D}\colon G \to \mathrm{Aut}(D^G)$ on simple tensors by

$$\beta_{G,D}(g)(d_{h_1} \otimes \cdots \otimes d_{h_n} \otimes 1 \otimes \cdots) = d_{gh_1} \otimes \cdots \otimes d_{gh_n} \otimes 1 \otimes \cdots$$

for all $g, h_1, \ldots, h_n \in G$. We call $\beta_{G,D}$ the *Bernoulli shift of $G$ with base $D$*.

Another relevant family of examples of actions of noncommutative $C^*$-algebras is that of the so-called *gauge actions*. There is no formal definition of what a gauge action is, but they typically are actions of $S^1$ (or $(S^1)^n$ for some $n \in \mathbb{N}$) on a $C^*$-algebra that is defined by generators and relations, where the action multiplies some of the generators by a scalar of modulus one, in such a way that the relations are preserved. Crossed products of gauge actions are typically "less complicated" than the original algebras where they act, and this is usually a very helpful feature of these actions.

In the next example, given $n \in \mathbb{N}$ we write the elements of $(S^1)^n$ as tuples $\overline{\zeta} = (\zeta_1, \ldots, \zeta_n)$.

**Examples 3.3.13.** The following are examples of gauge-type actions.

1. Identify $C(S^1)$ with the universal unital $C^*$-algebra generated by a unitary $u$, and define an action $\gamma\colon S^1 \to C(S^1)$ by $\gamma_\zeta(u) = \zeta u$ for all $\zeta \in S^1$. This action is just $\mathtt{Lt}\colon S^1 \to C(S^1)$.

2. For $\theta \in \mathbb{R}$, let $A_\theta$ denote the associated rotation algebra, which is the universal unital $C^*$-algebra generated by unitaries $u, v$ satisfying $uv = e^{2\pi i\theta} vu$. Then there is an action $\gamma\colon S^1 \to \mathrm{Aut}(A_\theta)$ determined by $\gamma_\zeta(u) = \zeta u$ and $\gamma_\zeta(v) = v$ for all $\zeta \in S^1$.

3. Identify $C^*(\mathbb{F}_n)$ with the universal unital $C^*$-algebra generated by unitaries $u_1, \ldots, u_n$ without any further relations. Then there is an action $\gamma\colon (S^1)^n \to \mathrm{Aut}(C^*(\mathbb{F}_n))$ determined by $\gamma_{\overline{\zeta}}(u_j) = \zeta_j u_j$ for all $\overline{\zeta} \in (S^1)^n$ and all $j = 1, \ldots, n$.

4. Let $n \in \mathbb{N}$, and consider the Cuntz algebra $\mathcal{O}_n$, which is the universal unital $C^*$-algebra generated by isometries $s_1, \ldots, s_n$ satisfying $\sum\limits_{j=1}^{n} s_j s_j^* = 1$. Then there is an action $\gamma\colon (S^1)^n \to \mathrm{Aut}(\mathcal{O}_n)$ determined by $\gamma_{\overline{\zeta}}(s_j) = \zeta_j s_j$ for all $\overline{\zeta} \in (S^1)^n$ and all $j = 1, \ldots, n$.

Examples (1) and (4) above are particular cases of the gauge action on a graph algebra.

[2]Formally, this infinite tensor product is defined as the direct limit of algebras of the form $\otimes_{g\in F} D$, for $F \subseteq G$ finite, where for finite sets $F \subseteq F' \subseteq G$ the connecting map $\otimes_{g\in F} D \to \otimes_{g\in F'} D$ is $x \mapsto x \otimes 1_{F'\setminus F}$.

**Example 3.3.14.** Let $\mathcal{E} = (V, E, r, s)$ be a directed graph (see Definition 2.1.6). We denote a generic element of the group $(S^1)^E = \{f \colon E \to S^1\}$ by $z$, and for $e \in E$ we write $z_e$ for the evaluation of $z$ at $e$. The gauge action on $C^*(\mathcal{E})$ is the action $\gamma \colon (S^1)^E \to \mathrm{Aut}(C^*(\mathcal{E}))$ determined on generators by

$$\gamma_z(p_v) = p_v \quad \text{and} \quad \gamma_z(s_e) = z_e s_e$$

for all $v \in V$ and all $e \in E$.

The restriction of $\gamma$ to the (constant) copy of $S^1$ is sometimes also referred to as the gauge action of $S^1$ on $C^*(\mathcal{E})$.

# Chapter 4

# Full and reduced crossed products

## 4.1 Covariant representations and crossed products

One of the main goals of these lecture notes is to study the structure of crossed products by certain classes of actions. This section is devoted to the construction of full and reduced crossed product, and we also prove some elementary properties.

**Definition 4.1.1.** Let $G$ be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action on a $C^*$-algebra $A$. A *covariant representation* of $(G, A, \alpha)$ is a triple $(\mathcal{H}, u, \varphi)$, where $\mathcal{H}$ is a Hilbert space, $u\colon G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous unitary representation[1], and $\varphi\colon A \to \mathcal{B}(\mathcal{H})$ is a representation satisfying

$$\varphi(\alpha_g(a)) = u_g \varphi(a) u_g^*$$

for all $g \in G$ and all $a \in A$.

A covariant representation $(\mathcal{H}, u, \varphi)$ is called *regular* if there exist a Hilbert space $\mathcal{H}_0$, a representation $\varphi_0\colon A \to \mathcal{B}(\mathcal{H}_0)$ and an identification $\mathcal{H} \cong L^2(G, \mathcal{H}_0)$ under which $u$ and $\varphi$ are given by

$$u_g(\xi)(h) = \xi(g^{-1}h) \quad \text{and} \quad \varphi(a)(\xi)(g) = \varphi_0(\alpha_{g^{-1}}(a(g)))(\xi(g))$$

for all $g, h \in G$, for all $\xi \in \mathcal{H}$ and for all $a \in C_c(G, A, \alpha)$.

A covariant representation $(\mathcal{H}, u, \varphi)$ is said to be *non-degenerate* if $\varphi$ is non-degenerate.

**Remark 4.1.2.** The previous definition also has a recipe for constructing regular covariant representations, and in particular shows that covariant representations exist. Namely, starting from any representation $\varphi\colon A \to \mathcal{B}(\mathcal{H})$, we

[1] This means that for every $\xi \in \mathcal{H}$, the map $u^\xi\colon G \to \mathcal{H}$ given by $u^\xi(g) = u_g(\xi)$ for all $g \in G$, is continuous.

take $\mathcal{H}^G = L^2(G, \mathcal{H}_0)$ and let $\lambda^{\mathcal{H}} \colon G \to \mathcal{U}(\mathcal{H}^G)$ and $\varphi^G \colon A \to \mathcal{B}(\mathcal{H}^G)$ be given by

$$\lambda^{\mathcal{H}}_g(\xi)(h) = \xi(g^{-1}h) \quad \text{and} \quad \varphi^G(a)(\xi)(g) = \varphi(\alpha_{g^{-1}}(a(g)))(\xi(g))$$

for all $g, h \in G$, for all $\xi \in \mathcal{H}$ and for all $a \in C_c(G, A, \alpha)$. Then the tripe $(\mathcal{H}^G, \lambda^{\mathcal{H}}, \varphi^G)$ is a (regular) covariant representation.

The full crossed product of a $C^*$-dynamical system $(G, A, \alpha)$ is defined to be the universal object with respect to covariant representations of the system, while the reduced crossed product is defined to be the universal object with respect to regular covariant representations; see Definition 4.1.8 below. The rigourous definition in the case that $G$ is locally compact requires that one develops some theory of integration on Banach spaces. Since the focus of these notes will be on discrete groups, we omit much of this discussion here.

**Definition 4.1.3.** Let $G$ be a locally compact group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action on a $C^*$-algebra $A$. Let $\mu$ denote a left Haar measure on $G$, and let $\Delta \colon G \to \mathbb{R}$ denote its associated modular function. Endow the vector space $C_c(G, A)$ of continuous compactly supported functions from $G$ to $A$ with the product given by the following twisted convolution

$$(a * b)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, d\mu(h)$$

for all $a, b \in C_c(G, A)$ and all $g \in G$, and with the following twisted convolution

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*)$$

for all $a \in C_c(G, A)$ and all $g \in G$. Finally, we define the norm of $a \in C_c(G, A)$ by $\|a\|_1 = \int_G \|a(g)\|\, d\mu(g)$. The resulting object is denoted by $C_c(G, A, \alpha)$.

**Exercise 4.1.4.** Prove that $C_c(G, A, \alpha)$ is a normed $*$-algebra, and that it is unital if and only if $G$ is discrete and $A$ is unital.

**Remark 4.1.5.** When $G$ is discrete, $C_c(G, A, \alpha)$ is sometimes denoted by $A[G]$, and it admits a more concise description as follows:

$$A[G] = \left\{ \sum_{g \in F} a_g \delta_g \colon F \subseteq G \text{ finite, and } a_g \in A \text{ for all } g \in F \right\},$$

with multiplication given by $au_g bu_h = a\alpha_g(b)u_{gh}$ and involution given by $(au_g)^* = u_{g^{-1}}a^* = \alpha_{g^{-1}}(a^*)\delta_{g^{-1}}$ for all $a, b \in A$ and all $g, h \in G$.

It turns out that $*$-representations of $C_c(G, A, \alpha)$ on Hilbert space are in one-to-one correspondence with covariant representations of $(G, A, \alpha)$, via the *integrated form*.

**Theorem 4.1.6.** Let $G$ be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action on a $C^*$-algebra $A$. Then there is a canonical one-to-one correspondence between (non-degenerate) covariant representations of $(G, A, \alpha)$ and (non-degenerate) $*$-representations of $C_c(G, A, \alpha)$. If $(\mathcal{H}, u, \varphi)$ is a covariant representations of $(G, A, \alpha)$, then the associated $*$-representation $\varphi \rtimes u\colon C_c(G, A, \alpha) \to \mathcal{B}(\mathcal{H})$ is called the *integrated form of* $(u, \varphi)$, and is given by

$$(\varphi \rtimes u)(a)\xi = \int_G \varphi(a(g))(u_g(\xi))\ d\mu(g)$$

for all $a \in C_c(G, A, \alpha)$ and all $\xi \in \mathcal{H}$.

*Proof.* Let $(\mathcal{H}, u, \varphi)$ be a covariant representations of $(G, A, \alpha)$. We need to show that the formula in the statement determines a $*$-representation of $C_c(G, A, \alpha)$. Given $a, b \in C_c(G, A, \alpha)$ and $\xi \in \mathcal{H}$, we have

$$\begin{aligned}
(\varphi \rtimes u)(ab)\xi &= \int_G \varphi(ab(g))u_g(\xi)\ d\mu(g)\\
&= \int_G\int_G \varphi(a(h))\varphi(\alpha_h(b(h^{-1}g)))u_g(\xi)\ d\mu(h)d\mu(g)\\
&= \int_G\int_G \varphi(a(h))u_h\varphi(b(h^{-1}g))u_{h^{-1}g}(\xi)\ d\mu(h)d\mu(g)\\
&= \left(\int_G \varphi(a(h))u_h\left(\int_G \varphi(b(k))u_k(\xi)\ d\mu(k)\right) d\mu(h)\right)\\
&= \left((\varphi \rtimes u)(a) \circ (\varphi \rtimes u)(a)\right)\xi,
\end{aligned}$$

so $\varphi \rtimes u$ is multiplicative. One checks analogously that $\varphi \rtimes u$ is $*$-preserving.

We proceed to show that any $*$-representation of $C_c(G, A, \alpha)$ is the integrated form of a covariant representation of $(G, A, \alpha)$. We only prove the result when $G$ is discrete and $A$ is unital, and refer the reader to [99] for a proof in the general case.

Assume then that $G$ is discrete and that $A$ is unital. We denote by $\delta_g \in C_c(G, A, \alpha)$ the corresponding Kronecker delta, and by $\iota\colon A \to C_c(G, A, \alpha)$ the $*$-homomorphism given by $\iota(a) = a\delta_1$ for $a \in A$.In the case of discrete groups, integrated forms take a particularly nice form. Indeed, a generic element of $C_c(G, A, \alpha)$ has the form $\sum\limits_{g\in G} a_g\delta_g$, where $a_g \in A$ for all $g \in G$ and $a_g \neq 0$ for at most finitely many $g \in G$. If $(\mathcal{H}, u, \varphi)$ is a covariant representations of $(G, A, \alpha)$, one easily checks that

$$(\varphi \rtimes u)(\sum_{g\in G} a_g\delta_g) = \sum_{g\in G} \varphi(a_g)u_g.$$

Let $\psi\colon C_c(G, A, \alpha) \to \mathcal{B}(\mathcal{H})$ be an $*$-representation. Set $\varphi = \psi \circ \iota$ and let $u\colon G \to \mathcal{B}(\mathcal{H})$ be given by $u(g) = \psi(\delta_g)$ for all $g \in G$. Then $\varphi$ is a representation of $A$, and $u$ is a unitary representation of $G$.

We claim that $\psi = \varphi \rtimes u$. To check this, it suffices to show that they agree on elements of the form $a\delta_g$, for $a \in A$ and $g \in G$. This is readily checked:

$$(\varphi \rtimes u)(a\delta_g) = \varphi(a)u_g = \psi(a\delta_1)\psi(\delta_g) = \psi(a\delta_g),$$

thus concluding the proof. □

The reader is encouraged to prove the above result in the case that $G$ is discrete but $A$ is not necessarily unital, using limits along an approximate identity to define the unitary representation.

**Exercise 4.1.7.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Show that any $*$-representation of $C_c(G, A, \alpha)$ is the integrated form of a covariant representation of $(G, A, \alpha)$.

We are now ready to introduce full and reduced crossed products.

**Definition 4.1.8.** Let $G$ be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action on a $C^*$-algebra $A$.

We define the *full crossed product* $A \rtimes_\alpha G$ of $(G, A, \alpha)$ to be the completion of $C_c(G, A, \alpha)$ with respect to the norm

$$\|a\|_{A\rtimes_\alpha G} = \sup\{\|(\varphi \rtimes u)(a)\colon (\varphi, u) \text{ is a covariant representation of } (G, A, \alpha)\}.$$

We define the *reduced crossed product* $A \rtimes_{\lambda,\alpha} G$ of $(G, A, \alpha)$ to be the completion of $C_c(G, A, \alpha)$ with respect to the norm

$$\|a\|_{A\rtimes_{\lambda,\alpha} G} = \sup\{\|(\varphi \rtimes u)(a)\colon (\varphi, u) \text{ is a regular covariant representation of } (G, A, \alpha)\}.$$

By definition, there is a canonical quotient map $\kappa\colon A \rtimes_\alpha G \to A \rtimes_{\lambda,\alpha} G$.

It follows directly from the definitions that representations of $A \rtimes_\alpha G$ are in one-to-one correspondence with covariant representations of $(G, A, \alpha)$, and that representations of $A \rtimes_{\lambda,\alpha} G$ are in one-to-one correspondence with regular covariant representations of $(G, A, \alpha)$. These are generalizations of the corresponding facts for the full and reduced group $C^*$-algebras $C^*(G)$ and $C^*_\lambda(G)$ of $G$, which are, respectively, the full and reduced crossed products of the trivial action of $G$ on $\mathbb{C}$. Section 4.2 contains the explicit computations of a number of crossed products. In particular, Example 4.2.1 is a generalization of the observation that $\mathbb{C} \rtimes G = C^*(G)$ and $\mathbb{C} \rtimes_\lambda G = C^*_\lambda(G)$.

Full crossed products by discrete groups admit a very natural description as universal $C^*$-algebras with generators and relations, as we show next.

**Theorem 4.1.9.** Let $G$ be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action on a $C^*$-algebra $A$. Then $A \rtimes_\alpha G$ is canonically isomorphic to the universal $C^*$-algebra generated by the set $\{au_g\colon a \in A, g \in G\}$, where $u_g$ is a unitary for all $g \in G$, subject to the relations $au_gbu_h = a\alpha_g(b)u_{gh}$ for all $a, b \in A$ and all $g, h \in G$.

*Proof.* Denote by $B$ the universal $C^*$-algebra described in the statement; we will show that there is a canonical isomorphism $B \cong A \rtimes_\alpha G$. By universality of $B$, and since $A \rtimes_\alpha G$ is generated by the same generators and relations as $B$, there is a surjective map $\pi\colon B \to A \rtimes_\alpha G$, which maps an element $au_g$ to $a\delta_g \in C_c(G, A, \alpha)$, for all $a \in A$ and all $g \in G$.

Observe that there is a canonical inclusion $\iota\colon C_c(G, A, \alpha) \to B$ determined by sending $\iota(a\delta_g) = au_g$ for all $a \in A$ and all $g \in G$. We obtain a $*$-representation of $C_c(G, A, \alpha)$ on $B$, which by Theorem 4.1.6 must be the integrated form of some covariant representation $(\varphi, u)$, where $\varphi\colon A \to B$ is a homomorphism and $u\colon G \to M(B)$ is a unitary representation. By the universal property of the crossed product, the integrated form of $(\varphi, u)$ extends to a homomorphism $\theta\colon A \rtimes_\alpha G \to B$ satisfying $\theta(a\delta_g) = (\varphi \rtimes u)(a\delta_g) = au_g$. It follows that $\pi$ and $\theta$ are mutual inverses, and thus $B$ is canonically isomorphic to $A \rtimes_\alpha G$. $\square$

Observe that in the context of the theorem above, the elements $u_g$, for $g \in G$, belong to the multiplier algebra of $A \rtimes_\alpha G$, and $u_1$ is its unit. In particular, $A$ is contained in $A \rtimes_\alpha G$ as $Au_1$.

**Remark 4.1.10.** In particular, when $A$ is unital, a crossed product of the form $A \rtimes_\alpha \mathbb{Z}$ is the universal unital $C^*$-algebra generated by a unital copy of $A$ together with a unitary $u$ satisfying $uau^* = \alpha_1(a)$.

In previous examples, equality between full and reduced crossed products was deduced from simplicity of the full crossed product. A very general instance in which full and reduced crossed products agree is when the acting group is amenable, as we show in the next theorem, whose proof is very similar to the proof that the universal map $C^*(G) \to C^*_\lambda(G)$ is an isomorphism when $G$ is amenable.

**Theorem 4.1.11.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. If $G$ is amenable, then the natural map $\kappa\colon A \rtimes_\alpha G \to A \rtimes_{\lambda,\alpha} G$ is an isomorphism.

*Proof.* Let $x \in C_c(G, A, \alpha)$, let $(\mathcal{H}, \varphi, u)$ be a covariant representation, and let $\varepsilon > 0$. Use $\varphi$ to construct the regular covariant representation $(\mathcal{H}^G, \lambda^{\mathcal{H}}, \varphi^G)$ as in Remark 4.1.2. Using the notation from Theorem 4.1.6, we will show that

$$\|(\varphi \rtimes u)(x)\| - \varepsilon \le \|(\varphi^G \rtimes \lambda^{\mathcal{H}})(x)\|.$$

Recall that $\mathcal{H}^G$ is just $L^2(G, \mathcal{H})$. Let $v \in \mathcal{U}(\mathcal{H}^G)$ be the unitary given by

$$v(\xi)(g) = u_g^{-1}(\xi(g))$$

for all $\xi \in \mathcal{H}^G$ and all $g \in G$.

**Claim:** *For all $g \in G$ and all $a \in A$, we have*

$$v(\lambda_g \otimes u_g)v^* = \lambda_g \otimes 1_{\mathcal{H}}, \quad \textit{and} \quad v(1_{L^2(G)} \otimes \varphi(a))v^* = \varphi^G(a).$$

The proof of the claim is a routine verification, which we omit.

It follows that $(\mathcal{H}^G, \lambda \otimes u, 1_{L^2(G)} \otimes \varphi)$ is a regular covariant representation that induces the same norm as $(\mathcal{H}^G, \lambda^{\mathcal{H}}, \varphi^G)$ on $C_c(G, A, \alpha)$. It therefore suffices to show that

$$\|(\varphi \rtimes u)(x)\| - \varepsilon \leq \|((1 \otimes \varphi \rtimes (\lambda \otimes u))(x)\|.$$

Without loss of generality, we may assume that $\|(\varphi \rtimes u)(x)\| = 1$. Choose $\xi_0 \in \mathcal{H}$ with $\|\xi_0\| = 1$ and

$$\|(\varphi \rtimes u)(x)\xi_0\| > 1 - \varepsilon/2.$$

Set $F = \operatorname{supp}(x) \cup \{1\}$ and

$$\delta = \left(\frac{1 - \varepsilon/2}{1 - \varepsilon}\right)^2 - 1 > 0,$$

and use amenability of $G$ to find a compact subset $K \subseteq G$ satisfying $\mu(F^{-1}K) < (1+\delta)\mu(K)$. Let $\xi \in \mathcal{H}^G = L^2(G, \mathcal{H})$ be given by

$$\xi(g) = \begin{cases} \xi_0, & \text{for } g \in F^{-1}K \\ 0, & \text{else.} \end{cases}$$

Then $\|\xi\| = \mu(F^{-1}K)^{1/2}\|\xi_0\| < (1+\delta)^{1/2}\mu(K)^{1/2}$. One shows that

$$((1 \otimes \varphi) \rtimes \lambda \otimes u)(x)\xi(g) = (u \rtimes \varphi)(x)(\xi_0)$$

for all $g \in K$, from which it follows that

$$\|((1 \otimes \varphi) \rtimes \lambda \otimes u)(x)\xi\| \geq \mu(K)^{1/2}\|(u \rtimes \varphi)(x)(\xi_0)\| \geq \mu(K)^{1/2}\left(1 - \frac{\varepsilon}{2}\right).$$

We conclude that

$$\begin{aligned} \|((1 \otimes \varphi \rtimes (\lambda \otimes u))(x)\| &> \frac{\mu(K)^{1/2}\left(1 - \frac{\varepsilon}{2}\right)}{(1+\delta)^{1/2}\mu(K)^{1/2}} \\ &= (1+\delta)^{-1/2}\left(1 - \frac{\varepsilon}{2}\right) = (1 - \varepsilon), \end{aligned}$$

as desired. $\square$

When $A = \mathbb{C}$, the converse to Theorem 4.1.11 is true: if $C^*(G) = C^*_\lambda(G)$, then $G$ is amenable. For actions on arbitrary $C^*$-algebras, however, this need not be the case. For example, $C_0(G) \rtimes G = C_0(G) \rtimes_\lambda G$ regardless of $G$, by Proposition 4.2.3.

Full and reduced crossed products exhibit nice functoriality properties with respect to a number of constructions in $C^*$-algebras. We record a few of them, and leave some of the proofs to the reader.

**Proposition 4.1.12.** Let $G$ be a locally compact group, let $A$ and $B$ be $C^*$-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then there are natural isomorphisms

$$(A \otimes_{\max} B) \rtimes_{\alpha \otimes_{\max} \mathrm{id}_B} G \cong (A \rtimes_\alpha G) \otimes_{\max} B$$

and

$$(A \otimes_{\min} B) \rtimes_{\lambda, \alpha \otimes_{\min} \mathrm{id}_B} G \cong (A \rtimes_{\lambda,\alpha} G) \otimes_{\min} B.$$

**Exercise 4.1.13.** Let $G$ be a locally compact group, let $A$ and $B$ be $C^*$-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

1. Prove Proposition 4.1.12 by comparing the representations of the objects involved.

2. Can Proposition 4.1.12 be generalized to the case when $G$ acts non-trivially on $B$?

**Proposition 4.1.14.** Let $G$ be a locally compact group, let $A$, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that $A$ can be written as a direct limit $(A, (\iota_{n,\infty})_{n\in\mathbb{N}}) = \varinjlim(A_n, \iota_n)$, and that there exist actions $\alpha^{(n)}\colon G \to \mathrm{Aut}(A_n)$, for $n \in \mathbb{N}$, which make the connecting maps $\iota_n$ equivariant, and satisfy $\alpha_g(\iota_{n,\infty}(a)) = \iota_{n,\infty}(\alpha_g^{(n)}(a))$ for all $g \in G$, for all $n \in \mathbb{N}$ and for all $a \in A_n$. Then

$$A \rtimes_\alpha G = \varinjlim A_n \rtimes_{\alpha^{(n)}} G \quad \text{and} \quad A \rtimes_{\lambda,\alpha} G = \varinjlim A_n \rtimes_{\lambda,\alpha^{(n)}} G.$$

**Exercise 4.1.15.** Prove Proposition 4.1.14 in the case that all the maps $\iota_n$ are inclusions.

Discussion about extensions: goes well with full crossed products, not so with reduced. What is needed later is that the crossed product of an invariant ideal is an ideal in the crossed product.

For future use, we give here explicit descriptions of approximate identities in full crossed products (and hence also on reduced crossed products). This description has two very useful consequences: first, when $A$ is discrete then any approximate identity of $A$ is an approximate identity of $A \rtimes_\alpha G$. And second, if $G$ is first countable[2] and $A$ is $\sigma$-unital, then $A \rtimes_\alpha G$ is also sigma unital.

**Proposition 4.1.16.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Let $(a_j)_{j\in J}$ be an approximate unit for $A$, and let $(f_k)_{k\in K}$ be an approximate unit for $C^*(G)$. Then $(a_j f_k)_{j\in J, k\in K}$ is an approximate unit for $A \rtimes_\alpha G$.

*Proof.* In order to check that $(a_j f_k)_{j\in J,k\in K}$ is an approximate unit for $A \rtimes_\alpha G$, it suffices to show that it is an approximate unit for elements of the form $fa$, for $f \in C_c(G)$ and $a \in A$.

[2] Recall that for a locally compact group $G$, its full group algebra $C^*(G)$ is $\sigma$-unital if and only if $G$ is first countable.

Accordingly, let $S \subseteq C_c(G)$ and $F \subseteq A$ be finite subsets, and let $\varepsilon > 0$. Without loss of generality, we assume that $\|a\| = 1$ for all $a \in F$. Find $k \in K$ such that

$$\|f_k * f - f\| < \varepsilon/2 \quad \text{and} \quad \max_{g \in \operatorname{supp}(f_k)} \|\alpha_g(a) - a\| < \varepsilon/2$$

for all $f \in S$ and all $a \in F$. Find $j \in J$ such that

$$\left\| a_j \left( \int_G \alpha_g(a) f_k(g) \, d\mu(g) \right) - \int_G \alpha_g(a) f_k(g) \, d\mu(g) \right\| < \varepsilon/2$$

for all $a \in F$. Given $a \in F$ and $f \in S$, we have

$$\begin{aligned}
\|(a_j f_k)(af) - af\| &= \left\| \left( a_j \int_G \alpha_g(a) f_k(g) \, d\mu(g) \right) (f_k * f) - af \right\| \\
&\leq \left\| a_j \int_G \alpha_g(a) f_k(g) \, d\mu(g) - a \right\| + \|f_k * f - f\| \\
&< \left\| \int_G \alpha_g(a) f_k(g) \, d\mu(g) - a \right\| + \varepsilon/2 \\
&= \left\| \int_G (\alpha_g(a) - a) f_k(g) \, d\mu(g) \right\| + \varepsilon/2 \\
&\leq \max_{g \in \operatorname{supp}(f_k)} \|\alpha_g(a) - a\| \int_G f_k(g) \, d\mu(g) + \varepsilon/2 \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon,
\end{aligned}$$

as desired. □

**Corollary 4.1.17.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action.

1. If $G$ is discrete, then any approximate identity of $A$ is an approximate identity of $A \rtimes_\alpha G$.

2. If $G$ is first countable and $A$ is $\sigma$-unital, then $A \rtimes_\alpha G$ is also sigma unital.

## 4.2 Examples and computations

This section is devoted to the computation of a number of examples of full and reduced crossed products. We begin with the trivial action on a $C^*$-algebra.

**Example 4.2.1.** Let $A$ be a $C^*$-algebra, let $G$ be a locally compact group, and let $\mathrm{id}_A\colon G \to \operatorname{Aut}(A)$ be the trivial action. Then there are canonical isomorphisms

$$A \rtimes_{\mathrm{id}_A} G \cong A \otimes_{\max} C^*(G) \quad \text{and} \quad A \rtimes_{\lambda, \mathrm{id}_A} G \cong A \otimes_{\min} C^*_\lambda(G).$$

*Proof.* Covariant representations for $(G, A, \mathrm{id}_A)$ on a Hilbert space $\mathcal{H}$ are in natural one-to-one correspondence with pairs $(\varphi, \pi)$ of commuting representations $\varphi\colon A \to \mathcal{B}(\mathcal{H})$ and $\pi\colon C^*(G) \to \mathcal{B}(\mathcal{H})$. Thus, the universal object for these representations is the maximal tensor product of $A$ and $C^*(G)$. Similarly, regular covariant representations of $(G, A, \mathrm{id}_A)$ have the form $(\varphi \otimes \mathrm{id}_{L^2(G)}, \mathrm{id}_{\mathcal{H}} \otimes \lambda)$, where $\varphi\colon A \to \mathcal{B}(\mathcal{H})$ is some representation and $\lambda\colon G \to \mathcal{U}(L^2(G))$ is the left regular representation. The universal $C^*$-algebra that these representations generate is thus the minimal tensor product of $A$ and $C^*_\lambda(G)$. □

The notation $\rtimes$ that we use for crossed products is inspired by the following fact.

**Exercise 4.2.2.** Let $G = N \rtimes H$ be a semidirect product of discrete compact groups. Show that there are canonical isomorphisms

$$C^*(G) \cong C^*(N) \rtimes H \quad \text{and} \quad C^*_\lambda(G) \cong C^*_\lambda(N) \rtimes_\lambda H.$$

For the next computation, we recall that $M_n$ is the universal unital $C^*$-algebra generated by $\{e_{j,k}\colon j, k = 1, \ldots, n\}$, subject to the relations

$$e_{j,k}e_{l,m} = \delta_{k,m}e_{j,m}, \quad e^*_{j,k} = e_{k,j}, \quad \sum_{j=1}^{n} e_{j,j} = 1$$

for all $j, k, l, m = 1, \ldots, n$. Elements $e_{j,k}$ as above are called *matrix units* for $M_n$.

**Proposition 4.2.3.** Let $G$ be a locally compact group. Then there are canonical isomorphisms

$$C_0(G) \rtimes_{\mathtt{Lt}} G \cong C_0(G) \rtimes_{\lambda,\mathtt{Lt}} G \cong \mathcal{K}(L^2(G)).$$

*Proof.* We only prove the proposition in the discrete case; the proof for arbitrary $G$ can be found in ???.

For $g \in G$, we denote by $u_g$ the canonical unitary implementing $\mathtt{Lt}_g$ in the full crossed product, and we denote by $\chi_g \in c_0(G)$ the characteristic function of $\{g\}$. For $g, h \in G$, set $e_{g,h} = \chi_g u_{gh^{-1}} \in c_0(G) \rtimes_{\mathtt{Lt}} G$.

Let $F \subseteq G$ be a finite set.

**Claim:** *There is a canonical isomorphism $\varphi_F\colon \mathcal{B}(\ell^2(F)) \to C^*(\{e_{g,h}\colon g, h \in F\})$ satisfying $\varphi_F(1) = \sum_{g \in F} e_{g,g}$.* For this, it is enough to show that the set $\{e_{g,h}\colon g, h \in F\}$ satisfies the relations for matrix units in $B(\ell^2(F))$. For $g, h, k, m \in F$, we have

$$\begin{aligned} e_{g,h}e_{k,m} &= \chi_g u_{gh^{-1}} \chi_k u_{km^{-1}} \\ &= \chi_g \mathtt{Lt}_{gh^{-1}}(\chi_k) u_{gh^{-1}km^{-1}} \\ &= \chi_g \chi_{gh^{-1}k} u_{gh^{-1}km^{-1}} \\ &= \chi_g \delta_{h,k} u_{gm^{-1}}. \end{aligned}$$

On the other hand, it is clear that $e_{g,h}^* = e_{h,g}$ for all $g, h \in F$. The claim thus follows.

Observe that if $F_1 \subseteq F_2$ are finite subsets of $G$, and we regard $\mathcal{B}(\ell^2(F_1))$ as a (non-unital) subalgebra of $\mathcal{B}(\ell^2(F_2))$, then $\varphi_{F_2}|_{\mathcal{B}(\ell^2(F_1))} = \varphi_{F_1}$. We deduce that the family $(\varphi_F)_{F \subseteq G \text{ finite}}$ induces a well-defined contractive homomorphism

$$\varphi\colon \bigcup_{F \subseteq G \text{ finite}} \mathcal{B}(\ell^2(F)) \to C_c(G, c_0(G), \mathtt{Lt}).$$

By continuity, $\varphi$ extends to a homomorphism $\psi\colon \mathcal{K}(\ell^2(G)) \to c_0(G) \rtimes_{\mathtt{Lt}} G$. This homomorphism is surjective, since its range contains the dense set $\operatorname{span}\{e_{g,h}\colon g, h \in G\}$. Since $\mathcal{K}(\ell^2(G))$ is simple, $\psi$ must also be injective and hence an isomorphism.

Finally, since $c_0(G) \rtimes_{\mathtt{Lt}} G$ is simple and $c_0(G) \rtimes_{\lambda,\mathtt{Lt}} G$ is a quotient of it (via $\kappa$), they must be isomorphic. $\square$

The previous proposition can be generalized

**Proposition 4.2.4.** Let $G$ be a locally compact group, let $H$ be a closed subgroup, and let $G$ act on $G/H$ by translation. Prove that there are canonical isomorphisms

$$C_0(G/H) \rtimes_{\mathtt{Lt}} G \cong C^*(H) \otimes \mathcal{K}(L^2(G/H))$$

and

$$C_0(G/H) \rtimes_{\lambda,\mathtt{Lt}} G \cong C^*_\lambda(H) \otimes \mathcal{K}(L^2(G/H)).$$

**Exercise 4.2.5.** Prove Proposition 4.2.4 in the case that $G$ is discrete.

**Example 4.2.6.** Let $\theta \in \mathbb{R}$ be a number, and let $r_\theta\colon \mathbb{Z} \to \operatorname{Aut}(C(S^1))$ denote the rotation action from part (5) of Examples 3.2.3. Then $C(S^1) \rtimes_{r_\theta} \mathbb{Z}$ is isomorphic to $A_\theta$, that is, the universal $C^*$-algebra generated by two unitaries $u, v$ satisfying $uv = e^{2\pi i\theta} vu$, and the same is true for $C(S^1) \rtimes_{\lambda, r_\theta} \mathbb{Z}$

*Proof.* Let $v$ denote the canonical generating unitary of $C(S^1)$. Then $r_\theta(v) = e^{\pi i\theta} v$, and hence the result follows from Remark 4.1.10. The reduced crossed product, being a quotient of the simple $C^*$-algebra $A_\theta$, is therefore isomorphic to $A_\theta$ itself. $\square$

**Example 4.2.7.** Let $\mathbb{Z}_2$ act on $S^1$ via conjugation, and denote by $\alpha\colon \mathbb{Z}_2 \to \operatorname{Aut}(C(S^1))$ the induced action. We will show that

$$C(S^1) \rtimes_\alpha \mathbb{Z}_2 \cong \{f \in C([-1,1], M_2)\colon f(1), f(-1) \text{ are diagonal}\}.$$

Denote by $S^1_+$ the closed upper-half circle, and observe first that the algebra

$$B = \left\{ f \in C(S^1_+, M_2)\colon f(1), f(-1) \text{ are of the form } \begin{bmatrix} x & y \\ y & x \end{bmatrix} \right\}$$

is isomorphic to what we want to get. Define $\varphi\colon C(S^1) \to B$ by

$$\varphi(f) = \begin{bmatrix} f(z) & 0 \\ 0 & f(\overline{z}) \end{bmatrix}$$

for all $f \in C(S^1)$ and all $z \in S^1_+$, and let $v \in \mathcal{U}(B)$ be the unitary given by $v(z) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for all $z \in S^1_+$. One checks that $v^2 = 1$ and that $v\varphi(f)v^* = \varphi(\alpha_1(f))$ for all $f \in C(S^1)$. By the universal property of $C(S^1) \rtimes_\alpha \mathbb{Z}_2$, there is a unique homomorphism $\psi\colon C(S^1) \rtimes_\alpha \mathbb{Z}_2 \to B$ extending $\varphi$ and satisfying $\psi(u) = v$. One checks that this homomorphism is given by

$$\psi(f_0 + f_1 u) = \begin{bmatrix} f_0(z) & f_1(z) \\ f_1(\overline{z}) & f_0(\overline{z}) \end{bmatrix}$$

for all $f_0, f_1 \in C(S^1)$ and all $z \in S^1_+$. The rest of the proof consists in showing that $\psi$ is an isomorphism, which we leave as an exercise.

**Exercise 4.2.8.** Complete the details in Example 4.2.7.

**Exercise 4.2.9.** Let $\mathbb{Z}_2$ act on $[-1, 1]$ via multiplication by -1, and denote by $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(C([-1, 1]))$ the induced action. Compute $C([-1, 1]) \rtimes_\alpha \mathbb{Z}_2$.

The following computation will be very relevant in chapter 11. The action we consider there is a particular case of a product type action, as defined in Example 3.3.8.

**Proposition 4.2.10.** Let $G$ be a finite group, and set $m = |G|$. We denote by $D$ the UHF-algebra of type $m^\infty$, which we identify with the infinite tensor product of $\mathcal{B}(\ell^2(G))$. Let $\lambda\colon G \to \mathcal{U}(\ell^2(G))$ denote the left regular representation, and let $\delta\colon G \to \mathrm{Aut}(D)$ be the action given by

$$\delta_g = \otimes_{n=1}^\infty \mathrm{Ad}(\lambda_g)$$

for all $g \in G$. Then $D \rtimes_\delta G$ is isomorphic to $D$.

*Proof.* For $n \in \mathbb{N}$, set $D_n = \bigotimes_{j=1}^{n} \mathcal{B}(\ell^2(G))$ and let $\delta^{(n)}\colon G \to \mathrm{Aut}(D_n)$ denote the restriction of $\delta$ to $D_n$. Denote by $\iota_n\colon D_n \to D_{n+1}$ the map given by $\iota_n(d) = d \otimes 1$ for all $d \in D_n$, and note that $(D, \delta)$ is the equivariant direct limit of the systems $(D_n, \iota_n, \delta^{(n)})$. We denote by

$$\iota_n \rtimes G\colon D_n \rtimes_{\delta^{(n)}} G \to D_{n+1} \rtimes_{\delta^{(n+1)}} G$$

the homomorphism induced by $\iota_n$. By Proposition 4.1.14, there is a natural isomorphism

$$\varinjlim (D_n \rtimes_{\delta^{(n)}} G, \iota_n \rtimes G) \cong D \rtimes_\delta G.$$

We begin by computing $D_n \rtimes_{\delta^{(n)}} G$. For $g \in G$, set $\lambda_g^{(n)} = \bigotimes_{j=1}^{n} \lambda_g$, so that $\delta_g^{(n)} = \mathrm{Ad}(\lambda_g^{(n)})$. For $g \in G$, we write $u_g \in D_n \rtimes_{\delta^{(n)}} G$ and $v_g \in C^*(G)$ for the canonical unitaries. We define a linear map

$$\varphi_n \colon D_n \rtimes_{\delta^{(n)}} G \to D_n \otimes C^*(G)$$

by $\varphi_n(du_g) = d\lambda_g^{(n)} \otimes v_g$ for all $d \in D_n$ and all $g \in G$. We claim that $\varphi_n$ is an isomorphism. To check multiplicativity, we let $d, e \in D_n$ and $g, h \in G$. Then

$$\begin{aligned}
\varphi_n(du_g)\varphi_n(eu_h) &= (d\lambda_g^{(n)} \otimes v_g)(e\lambda_h^{(n)} \otimes v_h) \\
&= (d\lambda_g^{(n)} e\lambda_h^{(n)}) \otimes v_{gh} \\
&= d\delta_g^{(n)}(e)\lambda_{gh}^{(n)} \otimes v_{gh} \\
&= \varphi_n(d\delta_g^{(n)}(e)u_{gh}) \\
&= \varphi_n((du_g)(eu_h)),
\end{aligned}$$

as desired. A similar computation shows that $\varphi_n$ is $*$-preserving. The map $\varphi_n$ is clearly surjective, since its image contains all elementary tensors. Injectivity can be deduced from the fact that both the domain and the codomain of $\varphi_n$ are finite-dimensional $C^*$-algebras with the same linear dimension. One can also see this directly by constructing its inverse, which is the map $\psi_n \colon D_n \otimes C^*(G) \to D_n \rtimes_{\delta^{(n)}} G$ given by $\psi_n(d \otimes v_g) = d\lambda_{g^{-1}}^{(n)} u_g$ for all $d \in D_n$ and all $g \in G$. This proves the claim.

It follows that there is an isomorphism

$$D \rtimes_\delta G = \varinjlim (D_n \otimes C^*(G), \varphi_{n+1} \circ (\iota_n \rtimes G) \circ \psi_n).$$

The connecting map is given by

$$(\varphi_{n+1} \circ (\iota_n \rtimes G) \circ \psi_n)(d \otimes v_g) =$$

□

Similar arguments can be used to compute the crossed products of other product-type actions. The following computation will be relevant in chapter 12.

**Exercise 4.2.11.** Let $G$ be a finite group, and set $m = |G|$. We denote by $E$ the UHF-algebra of type $(m+1)^\infty$, which we identify with the infinite tensor product of $\mathcal{B}(\ell^2(G) \oplus \mathbb{C})$. Let $\lambda \oplus 1 \colon G \to \mathcal{U}(\ell^2(G) \oplus \mathbb{C})$ denote the direct sum of the left regular representation and the trivial representation, and let $\beta \colon G \to \mathrm{Aut}(E)$ be the action given by

$$\beta_g = \otimes_{n=1}^{\infty} \mathrm{Ad}(\lambda_g \oplus 1)$$

for all $g \in G$. Compute $E \rtimes_\beta G$, at least when $G = \mathbb{Z}_2$.

## 4.3 Conditional expectations in crossed products by discrete groups

In this section, we will establish the existence of a canonical conditional expectation from full and reduced crossed products by discrete groups back into the original coefficient algebra. We will also prove a distinguishing feature of the reduced crossed product, namely, faithfulness of its canonical conditional expectation.

**Lemma 4.3.1.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Let $\varphi\colon A \to \mathcal{B}(\mathcal{H})$ be a representation and let $(\mathcal{H}^G, \lambda^{\mathcal{H}}, \varphi^G)$ be its associated regular covariant representation as in Remark 4.1.2. Let $F \subseteq G$ be a finite set, let $a_g \in A$, for $g \in F$, and set

$$a = \sum_{g\in F} a_g u_g \in C_c(G, A, \alpha) \subseteq A \rtimes_{\lambda,\alpha} G.$$

1. For $\xi \in \mathcal{H}^G$ and $g \in G$, we have
$$((\varphi^G \rtimes \lambda^{\mathcal{H}})(a)\xi)(g) = \sum_{h\in G} \varphi(\alpha_{g^{-1}}(a_h)(\xi(h^{-1}g)).$$
2. For $g \in G$, let $s_g \in \mathcal{B}(\mathcal{H}, \mathcal{H}^G)$ be the isometry which sends $\xi$ to $\xi\delta_g$, for all $\xi \in \mathcal{H}$. For all $g, h \in G$, we have
$$s_g^*(\varphi^G \rtimes \lambda^{\mathcal{H}})s_h = \varphi(\alpha_{g^{-1}}(a_{gh^{-1}})).$$

The proof of the previous lemma is routine, so we leave it as an exercise.

**Exercise 4.3.2.** Prove Lemma 4.3.1.

**Lemma 4.3.3.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action, and fix $g \in G$. Then there exists a unique contractive linear map $E_g\colon A \rtimes_{\lambda,\alpha} G \to A$ satisfies $E_g(a) = a_g$ for all $a = \sum_{g\in F} a_g u_g \in C_c(G, A, \alpha)$ as in Lemma 4.3.1. Moreover, for every representation $\varphi\colon A \to \mathcal{B}(\mathcal{H})$, one has
$$s_g^*(\varphi^G \rtimes \lambda^{\mathcal{H}})(b)s_h = \varphi(\alpha_{g^{-1}}(E_{gh^{-1}}(b)).$$
for all $g, h \in G$, and for all $b \in A \rtimes_{\lambda,\alpha} G$.

*Proof.* Define a linear map $E_g^{(0)}\colon C_c(G, A, \alpha) \to A$ by evaluating at $g$. Then $E_g$ is continuous with respect to the $\|\cdot\|_\infty$ norm on $C_c(G, A, \alpha)$. It therefore suffices to show that $\|\cdot\|_\infty \leq \|\cdot\|_\lambda$ norm on $C_c(G, A, \alpha)$. For $a$ as in the statement, and for an isometric representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$, we have

$$\|a_g\| = \|\pi_0(a_g)\| = \|s_1^*(\varphi^G \rtimes \lambda^{\mathcal{H}})(a)s_{g^{-1}}\| \leq \|(\varphi^G \rtimes \lambda^{\mathcal{H}})(a)\| \leq \|a\|_\lambda.$$

The last identity follows immediately from part (2) of Lemma 4.3.1. ∎

It follows that for any $a \in A \rtimes_{\lambda,\alpha} G$, and hence for every $a \in A \rtimes_\alpha G$, there are well-defined coefficients $a_g \in A$, for $g \in G$. (This is obvious for elements in $C_c(G, A, \alpha)$.) However, $a$ cannot in general be recovered as $\sum_{g\in G} a_g u_g$, since this series does not in general converge in the crossed product. However, as we show next, an element in the reduced crossed product is zero if and only if all of its coefficients are zero.

**Theorem 4.3.4.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

1. If $a \in A \rtimes_{\lambda,\alpha} G$ satisfies $E_g(a) = 0$ for all $g \in G$, then $a = 0$.
2. If $a \in A \rtimes_{\lambda,\alpha} G$, then $\|E_g(a^*a)\|^2 \leq \|E_1(a^*a)\|$ for all $g \in G$.
3. If $a \in A \rtimes_{\lambda,\alpha} G$ satisfies $E_1(a^*a) = 0$, then $a = 0$.

*Proof.* (1). Let $\varphi\colon A \to \mathcal{B}(\mathcal{H})$ be a representation. If $a \in A \rtimes_{\lambda,\alpha} G$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_g^*(\varphi^G \rtimes \lambda^{\mathcal{H}})(a)s_h = 0$ for all $g, h \in G$, and hence $(\varphi^G \rtimes \lambda^{\mathcal{H}})(a) = 0$. Since $\varphi$ is arbitrary, it follows that $a = 0$.

(2). For a finite subset $F \subseteq G$ and an element $a = \sum_{g\in F} a_g u_g \in C_c(G, A, \alpha)$, we have

$$a^*a = \sum_{g,h} \alpha_{g^{-1}}(a_g a_h^*) u_{g^{-1}h}$$

and thus

$$E_1(a^*a) = \sum_{g\in G} a_g a_g^* = \sum_{g\in G} E_g(a)E_g(a^*).$$

Thus $E_1(a^*a) \geq \alpha_{g^{-1}}(E_g(a)E_g(a)^*)$ for all $g \in G$ and hence $\|E_1(a^*a)\| \geq \|E_g(a)\|^2$. By continuity, this holds for all elements in $A \rtimes_{\lambda,\alpha} G$.

(3). If $E_1(a^*a) = 0$, then $E_g(a) = 0$ for all $g \in G$ by part (2), and hence $a = 0$ by part (1). $\square$

Among the maps $E_g$, for $g \in G$, the one corresponding to the unit of the group is special. For once, it is a completely positive map, and it is even a conditional expectation in the sense of the following definition.

**Definition 4.3.5.** Let $B$ be a $C^*$-algebra, let $A \subseteq B$ be a $C^*$-subalgebra, and let $E\colon B \to A$ be a positive, linear map. We say that $E$ is a *conditional expectation* if $E(a) = a$ for all $a \in A$, and $E(ab) = aE(b)$ for all $a \in A$ and all $b \in B$.

When $A$ and $B$ are commutative von Neumann algebras and $E$ is unital, it can be shown that a conditional expectation in the sense of Definition 4.3.5 is given by a conditional expectation at the level of the underlying measure spaces, in the sense of probability theory.

**Proposition 4.3.6.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then the map $E_1\colon A \rtimes_{\lambda,\alpha} G \to A$ is a conditional expectation.

*Proof.* Positivity of $E_1$ follows by taking $g = h = 1$ in the displayed identity of Lemma 4.3.3. Since $A$ is identified with $Au_1 \subseteq A \rtimes_{\lambda,\alpha} G$, it is clear that $E(a) = a$ for all $a \in A$. Finally, given $a \in A$ and $b \in C_c(G, A, \alpha)$ of the form $b = \sum\limits_{g \in G} b_g u_g$, it is clear that $E(ab) = ab_1 = aE(b)$. Since $E$ is continuous, it follows that this identity holds for all $b \in A \rtimes_{\lambda,\alpha} G$, as desired. $\square$

**Definition 4.3.7.** Let $G$ be a discrete group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then the maps

$$E = E_1\colon A \rtimes_{\lambda,\alpha} G \to A \quad \text{and} \quad E \circ \kappa\colon A \rtimes_{\alpha} G \to A$$

are called the *canonical conditional expectations* on the reduced and full crossed products, respectively.

# Chapter 5

# Duality theory for abelian group actions

In this chapter, we will prove a generalization of Pontryagin's duality $\widehat{\widehat{G}} \cong G$ in the context of crossed products of $C^*$-algebras, usually known as *Takai duality*. In its most classical form, it is a result about crossed products by abelian groups, although generalization for arbitrary groups, using coactions, exist ???. For the sake of clarity of the exposition, we will restrict ourselves to the abelian case.

The original proof of Takai duality, due to Takai, has been subsequently simplified by other authors. In these notes, we have chosen to follow Raeburn's proof [80], which is based on the universal property of the (full) crossed product. Accordingly, we devote the next section to establishing said universal property and discussing an important example, namely $G$ acting on $C_0(G)$ by translation (see also Proposition 4.2.3).

## 5.1 The universal property of the full crossed products

In this section, we prove that the full crossed product of a $C^*$-dynamical system can be completely described through a universal property intimately related to covariant representations of the system. This presentation is not surprising, but it has the advantage of making certain computations of crossed products much easier. Additionally, a relatively elementary proof of Takai duality can be given using this picture of the crossed product.

**Notation 5.1.1.** Let $G$ be a locally compact group, let $B$ be a unital $C^*$-algebra, and let $u\colon G \to \mathcal{U}(B)$ be a unitary representation. We will usually also denote by $u\colon C_c(G) \to B$ the non-degenerate $*$-representation given by $u(f) = \int_G f(g)u_g \, d\mu(g)$ for all $f \in C_c(G)$.

**Theorem 5.1.2.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then there exist homomorphisms

$$\iota_A\colon A \to M(A \rtimes_\alpha G) \quad \text{and} \quad \iota_G\colon G \to \mathcal{U}(M(A \rtimes_\alpha G))$$

satisfying

1. $\iota_A(\alpha_g(a)) = \iota_G(g)\iota_A(a)\iota_G(g)^{-1}$ for all $a \in A$ and all $g \in G$;
2. $A \rtimes_\alpha G = \overline{\operatorname{span}}\{\iota_A(a)\iota_G(f) : a \in A, f \in C_c(G)\}$;
3. Whenever $(\mathcal{H}, u, \varphi)$ is a covariant representation of $(G, A, \alpha)$, then

$$\varphi = (\varphi \rtimes u) \circ \iota_A \quad \text{and } u = (\varphi \rtimes u) \circ \iota_G.$$

Moreover, suppose that $B$ is another $C^*$-algebra and that $j_A : A \to M(B)$ and $j_G : G \to \mathcal{U}(M(B))$ are maps satisfying conditions (a), (b) and (c) above. Then there exists an isomorphism $\psi : A \rtimes_\alpha G \to B$ such that $\psi \circ \iota_A = j_A$ and $\psi \circ \iota G = j_G$.

*Proof.* For $g \in G$, let $\iota_G(g) : C_c(G, A, \alpha) \to C_c(G, A, \alpha)$ be given by $\iota_G(g)(x)(h) = \alpha_g(x(g^{-1}h))$ for all $x \in C_c(G, A, \alpha)$ and all $h \in G$. It is easily seen that $\iota_G(g)$ determines an invertible multiplier of $A \rtimes_\alpha G$, and that the resulting map $\iota_G : G \to M(A \rtimes_\alpha G)$ is a unitary representation.

For $a \in A$, let $\iota_A(a) : C_c(G, A, \alpha) \to C_c(G, A, \alpha)$ be given by $\iota_A(a)(x) = ax$ for all $x \in C_c(G, A, \alpha)$. It is easily seen that $\iota_A(a)$ determines an invertible multiplier of $A \rtimes_\alpha G$, and that the resulting map $\iota_A : A \to M(A \rtimes_\alpha G)$ is a homomorphism. The maps $\iota_G$ and $\iota_A$ clearly satisfy conditions (1), (2) and (3) above.

Now let $(B, j_A, j_G)$ be a triple as in the statement. Since $(j_A, j_G)$ is a covariant representation and $j_A$ is non-degenerate, there exists a non-degenerate homomorphism $j_A \rtimes j_G : A \rtimes_\alpha G \to B$ satisfying

$$(j_A \rtimes j_G) \circ \iota_A = j_A \quad \text{and} \quad (j_A \rtimes j_G) \circ \iota_G = j_G.$$

Reversing the roles of $(B, j_A, j_G)$ and $(A \rtimes_\alpha G, \iota_A, \iota_G)$ gives a non-degenerate homomorphism $\iota_A \rtimes \iota_G : B \to A \rtimes_\alpha G$ satisfying

$$(\iota_A \rtimes \iota_G) \circ j_A = \iota_A \quad \text{and} \quad (\iota_A \rtimes \iota_G) \circ j_G = \iota_G.$$

It is then immediate that $j_A \rtimes j_G$ and $\iota_A \rtimes \iota_G$ are mutual inverses. $\square$

**Remark 5.1.3.** Instead of developing the theory of crossed products as we did in Section 4.1, we could have defined crossed products through the universal property from the previous theorem. Had we taken that route, we would have had to show that there exists a least one (and hence precisely one) crossed product. Of course our work in Section 4.1 shows that one crossed product can be obtained by completing $C_c(G, A, \alpha)$ with respect to its universal norm, but there is a slightly shorter way of proving its existence. We can find a set $\mathcal{S}$ of covariant representations of $(G, A, \alpha)$ such that every cyclic covariant representation is equivalent to an element of $\mathcal{S}$. Then set

$$\iota_A = \bigoplus_{(\mathcal{H},\varphi,u)\in\mathcal{S}} \varphi \quad \text{and} \quad \iota_G = \bigoplus_{(\mathcal{H},\varphi,u)\in\mathcal{S}} u,$$

which act on the Hilbert space $\bigoplus_{(\mathcal{H},\varphi,u)\in\mathcal{S}} \mathcal{H}$. Then take $A \rtimes_\alpha G$ to be the closed linear span of $\{\iota_A(a)\iota_G(f) \colon a \in A, f \in C_c(G)\}$.

We will now have another look at the crossed product of $\mathtt{Lt} \colon G \to \mathrm{Aut}(C_0(G))$ from the perspective of Theorem 5.1.2. Recall that $M(\mathcal{K}(L^2(G)))$ is just $\mathcal{B}(L^2(G))$.

**Example 5.1.4.** Let $G$ be a locally compact group. Let $\lambda \colon G \to \mathcal{U}(L^2(G))$ be its left regular representation, and let $m \colon C_0(G) \to M(\mathcal{K}(L^2(G))$ be the multiplication function. Then $(\mathcal{K}(L^2(G)), m, \lambda)$ is the crossed product of the translation action $\mathtt{Lt} \colon G \to \mathrm{Aut}(C_0(G))$. Checking that it satisfies the properties in Theorem 5.1.2 is easy, with the exception of (c). Indeed, (a) is an elementary computation that amounts to the identity $\mathrm{Ad}(\lambda_g) \circ m = m \circ \mathtt{Lt}_g$, valid for all $g \in G$. For (b), it suffices to observe that $m \rtimes \lambda$ maps $C_c(G, C_c(G))$ into the space of kernels in $C_c(G \times G)$, which is known to be dense in $\mathcal{K}(L^2(G))$. Finally, (c) can be established using induced representations, and we omit the argument.

## 5.2 Dual actions and Takai duality

We begin by showing the existence of dual actions, using the universal property of crossed products established in Theorem 5.1.2.

**Proposition 5.2.1.** Let $G$ be a locally compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Then there exists an action $\widehat{\alpha} \colon \widehat{G} \to \mathrm{Aut}(A \rtimes_\alpha G)$ which is determined by

$$\widehat{\alpha}_\chi(\iota_A(a)\iota_G(f)) = \iota_A(a)\iota_G(\chi f)$$

for $\chi \in \widehat{G}$, for $a \in A$ and for $f \in C_c(G)$.

*Proof.* Fix $\chi \in \widehat{G}$, and define $\iota_G^\chi \colon G \to \mathcal{U}(M(A \rtimes_\alpha G))$ by $\iota_G^\chi(g) = \chi(g)\iota_G(g)$ for all $g \in G$. We claim that $(A \rtimes_\alpha G, \iota_A, \iota_G^\chi)$ satisfies the universal property for $A \rtimes_\alpha G$ from Theorem 5.1.2. Parts (a) and (b) are clear, while (c) follows since $(\mathcal{H}, u, \varphi)$ is covariant if and only if $(\mathcal{H}, \chi^{-1}u, \varphi)$ is covariant, and one has

$$u = (\varphi \rtimes \chi^{-1}u) \circ \iota_G^\chi.$$

It follows that there is an automorphism $\widehat{\alpha}_\chi \in \mathrm{Aut}(A \rtimes_\alpha G)$ satisfying $\widehat{\alpha}_\chi(\iota_A(a)\iota_G(f)) = \iota_A(a)\iota_G(\chi f)$ for all $a \in A$ and for all $f \in C_c(G)$. It is easy to check that the resulting map $\widehat{\alpha} \colon \widehat{G} \to \mathrm{Aut}(A \rtimes_\alpha G)$ is a homomorphism of groups. Finally, for $\chi \in \widehat{G}$, for $a \in A$ and for $f \in C_c(G)$, we have

$$\|\iota_A(a)\iota_G(\chi f) - \iota_A(a)\iota_G(f)\| \leq \|a\|\|\iota_G(\chi f) - \iota_G(f)\| \leq \|a\|\mu(\mathrm{supp}(f)\|\chi f - f\|_\infty.$$

Now, recall that the topology of $\widehat{G}$ is uniform convergence on compact subsets of $G$. It thus follows that the action of $\widehat{G}$ on $C_c(G)$ my multiplication is continuous with respect to $\|\cdot\|_\infty$. We conclude that $\widehat{\alpha}$ is continuous. $\square$

When $G$ is discrete, there is a very concrete description of the dual action in terms of the canonical generators identified in Theorem 4.1.9.

**Proposition 5.2.2.** Let $G$ be a discrete abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. For $\chi \in \widehat{G}$, for $a \in A$ and for $g \in G$, we have
$$\widehat{\alpha}_\chi(au_g) = a\chi(g)u_g.$$

*Proof.* Recall that $A \rtimes_\alpha G$ is generated by linear combinations of elements of the form $au_g$, for $a \in A$ and $g \in G$. For $\chi \in \widehat{G}$, the formula for the dual action given in Proposition 5.2.1 entails that $\widehat{\alpha}_\chi(au_g) = a\widehat{\alpha}_\chi(u_g)$, so we may assume that $a = 1$. In terms of the map $\iota_G$ from Theorem 5.1.2, the unitary $u_g$ is just $\iota(\delta_g)$, where $\delta_g \in C_c(G) \subseteq C^*(G)$ is the Kronecker delta. Since $\chi\delta_g = \chi(g)\delta_g$ in $C_c(G)$, we conclude that $\widehat{\alpha}_\chi(u_g) = \chi(g)u_g$, as desired. □

In particular, we see that $\widehat{\alpha}$ leaves $A$ fixed and acts on the unitaries by multiplication by the character.

We now look at a concrete example:

**Example 5.2.3.** Let $\mathbb{Z}$ act on the one-point space. Then $\mathbb{C} \rtimes_\alpha \mathbb{Z} = C^*(\mathbb{Z}) \cong C(S^1)$, where the last identification is given by the Fourier transform. Denote by $u \in C(S^1)$ the canonical generating unitary, which is just the inclusion of $S^1$ into $\mathbb{C}$. Then $\widehat{\alpha}\colon S^1 \to \mathrm{Aut}(C(S^1))$ is determined by $\widehat{\alpha}_z(u)(w) = zu(w) = zw = u(wz)$ for all $w \in S^1$. In other words, $\widehat{\alpha}$ is canonically identified with $\mathtt{Rt}\colon S^1 \to \mathrm{Aut}(C(S^1))$. (And also with $\mathtt{Lt}$, since $G$ is abelian.)

The previous example is a particular case of a much more general fact:

**Proposition 5.2.4.** Let $G$ be a locally compact abelian group, acting trivially on $\mathbb{C}$. Show that the dual action $\widehat{\mathrm{id}}\colon \widehat{G} \to \mathrm{Aut}(C^*(G))$ can be identified with the left translation action of $\widehat{G}$ on itself.

*Proof.* Let $\mathcal{F}_0\colon C_c(G) \to C_0(\widehat{G})$ be given by
$$\mathcal{F}_0(f)(\tau) = \int_G f(g)\overline{\tau(g)}\, d\mu(g)$$
for all $f \in C_c(G)$ and all $\tau \in \widehat{G}$. It is readily verified that $\mathcal{F}$ is a $*$-homomorphism (with respect to convolution), which is contractive for the universal norm of $C^*(G)$ and has dense range. Thus it extends to an isomorphism $\mathcal{F}\colon C^*(G) \to C_0(\widehat{G})$, known as the Fourier transform. It suffices to check that $\mathcal{F}_0$ is equivariant with respect to $\widehat{\mathrm{id}}$ and $\mathtt{Lt}$.

For $\chi, \tau \in \widehat{G}$ and $f \in C_c(G)$, we have
$$\mathcal{F}_0(\widehat{\mathrm{id}}_\chi(f))(\tau) = \mathcal{F}_0(\chi f)(\tau) = \int_G f(g)\chi(g)\overline{\tau(g)}\, d\mu(g) = \mathcal{F}_0(f)(\chi^{-1}\eta),$$
as desired. □

In turn, one could wonder what is the dual of the translation action – this is the *double dual* of the trivial action. One should notice right away that one cannot possibly get the trivial action back, since (double) dual actions are never trivial for $G \neq \{1\}$. However, one gets a very natural action on the compact operators.

**Proposition 5.2.5.** Let $G$ be a locally compact abelian group, and identify $L^2(G)$ and $L^2(\widehat{G})$ via the Fourier transform. Then

$$\widehat{\mathtt{Lt}}\colon \widehat{G} \to \mathrm{Aut}(\mathcal{K}(L^2(G))) \cong \mathrm{Aut}(\mathcal{K}(L^2(\widehat{G})))$$

is the action of conjugation by the left regular representation $\lambda\colon \widehat{G} \to \mathcal{U}(L^2(\widehat{G}))$.

**Exercise 5.2.6.** Prove Proposition 5.2.5.

Putting Proposition 5.2.4 and Proposition 5.2.2, we see that in the case of the trivial action, the passage to the dual generates a copy of $\mathcal{K}(L^2(G))$ with conjugation by the left regular representation. Takai's theorem shows that this phenomenon occurs in full generality:

**Theorem 5.2.7.** (Takai duality) Let $G$ be a locally compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then there exists a canonical equivariant isomorphism:

$$\tau\colon (A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}) \cong (A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \mathrm{Ad}(\lambda)).$$

*Proof.* The proof is technical and long. For the sake of brevity and clarity, we will sketch the core of the argument, given by a series of claims whose verifications we leave to the reader.

We will construct maps $j_{A\rtimes_\alpha G}\colon A \rtimes_\alpha G \to M(A\otimes\mathcal{K}(L^2(G)))$ and $j_{\widehat{G}}\colon \widehat{G} \to \mathcal{U}(M(A \otimes \mathcal{K}(L^2(G))))$ such that the triple $(A \otimes \mathcal{K}(L^2(G)), j_{A\rtimes_\alpha G}, j_{\widehat{G}})$ satisfies the universal property for $A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}$. We denote by $(A \rtimes_\alpha G, \iota_A, \iota_G)$ the universal triple given by Theorem 5.1.2.

Let $\alpha, \alpha^{-1}\colon A \to C_b(G, A) \subseteq M(C_0(G, A))$ be the (coaction) maps given by

$$\alpha(a)(g) = \alpha_g(a) \quad \text{and} \quad \alpha^{-1}(a)(g) = \alpha_g^{-1}(a)$$

for all $a \in A$ and all $g \in G$. We define map $j_A\colon A \to M(A \otimes \mathcal{K}(L^2(G)),$ $j_G\colon G \to \mathcal{U}(M(A \otimes \mathcal{K}(L^2(G))))$ and $j_{\widehat{G}}\colon \widehat{G} \to \mathcal{U}(M(A \otimes \mathcal{K}(L^2(G))))$ by

$$j_A(a) = (\mathrm{id}_A \otimes m) \circ \alpha^{-1}, \quad j_G = 1_A \otimes \lambda, \quad \text{and} \quad j_{\widehat{G}} = 1_A \otimes m,$$

where we regard canonically $\widehat{G}$ as a subset of $C_b(G)$, which acts on $\mathcal{K}(L^2(G))$ by multiplication.

**Claim 1:** *$(j_A, j_G)$ is a covariant pair for $(G, A, \alpha)$.* It therefore makes sense to consider the integrated form $j_A \rtimes j_G\colon A \rtimes_\alpha G \to M(A\otimes\mathcal{K}(L^2(G)))$. This will be our map $j_{A\rtimes_\alpha G}$. We now proceed to show that $(A \otimes \mathcal{K}(L^2(G)), j_A \rtimes j_G, j_{\widehat{G}})$ satisfies the conditions in Theorem 5.1.2. The following two claims show that conditions (a) and (b) are satisfied.

**Claim 2:** $(j_A \rtimes j_G, j_{\widehat{G}})$ *is a covariant pair for* $(\widehat{G}, A \rtimes_\alpha G, \widehat{\alpha})$. The proof of this claim reduces to showing that $j_{\widehat{G}}$ commutes with $j_A$, since the interaction with $j_G$ is clear from Example 4.2.1.

**Claim 3:** $\operatorname{span}\{j_A(a)j_G(f)j_G(\widehat{f})\colon a \in A, f \in C_c(G), \widehat{f} \in C_c(\widehat{G})\}$ *is dense in* $A \otimes \mathcal{K}(L^2(G))$. Example 4.2.1 implies that the span of the elements of the form $j_G(f)j_G(\widehat{f})$ is dense in $\mathcal{K}(L^2(G))$, so it suffices to $j_A(a)j_G(f)$ spans a dense subset of $A \otimes C_0(G)$.

It remains to check condition (c). For this, let $(\mathcal{H}, v, \varphi \rtimes u)$ be a covariant representation of $(\widehat{G}, A \rtimes_\alpha G, \widehat{\alpha})$. We want to construct a homomorphism $\psi\colon A \otimes \mathcal{K}(L^2(G)) \to \mathcal{B}(\mathcal{H})$ such that $\psi \circ (j_A \rtimes j_G) = \varphi \rtimes u$ and $\psi \circ j_{\widehat{G}} = v$.

The representation $v\colon \widehat{G} \to \mathcal{U}(\mathcal{H})$ can be integrated to a non-degenerate homomorphism from $C^*(\widehat{G}) \cong C_0(G)$, which we extend to a unital homomorphism $\theta\colon C_b(G) \to B(\mathcal{H})$ satisfying $\theta(\chi) = v_\chi$ for all $\chi \in \widehat{G}$.

**Claim 4:** $(\mathcal{H}, u, \theta)$ *is a covariant representation of* $(G, C_0(G), \mathtt{Lt})$. This is relatively straightforward from the fact that $(v, \varphi \rtimes u)$ is a covariant pair for $(\widehat{G}, A \rtimes_\alpha G, \widehat{\alpha})$.

Since $v$ commutes with $\varphi$ (because $\widehat{\alpha}$ leaves $A$ fixed), so does $\theta$. Hence we get a homomorphism $\theta \otimes \varphi\colon C_0(G) \otimes A \to B(\mathcal{H})$. We set $\pi = (\theta \otimes \varphi) \circ \alpha$.

**Claim 5:** $\pi$ *commutes with* $\theta$ *and with* $u$. Commutation with $\theta$ is clear since $\theta$ itself commutes with $\theta \otimes \varphi$. Commutation with $u$ is somewhat more delicate.

It follows that $\pi$ commutes with $\theta \rtimes u$, so we get an induced homomorphism

$$\psi = \pi \otimes (\theta \rtimes u)\colon A \otimes \mathcal{K}(L^2(G)) \to \mathcal{B}(\mathcal{H}).$$

**Claim 6:** *We have* $\psi \circ j_A = \varphi$, $\psi \circ j_G = u$ *and* $\psi \circ j_{\widehat{G}} = v$. It thus follows that $\psi \circ (j_A \rtimes j_G) = \varphi \rtimes u$, and hence part (c) in Theorem 5.1.2 is verified.

Denote by $\iota_{A \rtimes_\alpha G}\colon A \rtimes_\alpha G \to M(A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G})$ the canonical inclusion. We deduce that there exists an isomorphism

$$\tau\colon A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}$$

satisfying $\tau(j_A(a)j_G(f)j_{\widehat{G}}(\widehat{f})) = \iota_{A \rtimes_\alpha G}(\iota_A(a)\iota_G(f))\iota_{\widehat{G}}(\widehat{f})$ for all $a \in A$, all $f \in C_c(G)$ and all $\widehat{f} \in C_c(\widehat{G})$. It remains to show that $\tau$ is equivariant.

Fix $g \in G$, $a \in A$, $f \in C_c(G)$ and $\widehat{f} \in C_c(\widehat{G})$. Note that

$$\widehat{\widehat{\alpha}}_g(\iota_{A \rtimes_\alpha G}(\iota_A(a)\iota_G(f))\iota_{\widehat{G}}(\widehat{f})) = \iota_{A \rtimes_\alpha G}(\iota_A(a)\iota_G(f))\iota_{\widehat{G}}(g\widehat{f}).$$

Thus, the result follows from the following claim.

**Claim 7:** *We have*

$$(\alpha_g \otimes \operatorname{Ad}(\lambda_g))(j_A(a)j_G(f)j_{\widehat{G}}(\widehat{f})) = j_A(a)j_G(f)j_{\widehat{G}}(g\widehat{f}).$$

This concludes the sketch of the proof. $\square$

The proof of Theorem 5.2.7 given above is admittedly quite involved. For finite groups, however, it takes a much simpler form, and the reader is encouraged to attempt the following illuminating exercise.

**Exercise 5.2.8.** Give a complete proof of Takai duality for $G = \mathbb{Z}_2$ and $A$ unital. In this case, the isomorphism can be described very explicitly.

We will explore a number of consequences of Takai duality in the next chapters, particularly to computations of the $K$-theory of certain crossed product. Here, we apply it to obtain a satisfactory description of the $G$-invariant ideals in $A$.

**Proposition 5.2.9.** Let $G$ be a locally compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then the assignment $I \mapsto I \rtimes_{\alpha|_I} G$ defines a one-to-one correspondence between the $G$-invariant ideals in $A$ and the $\widehat{G}$-invariant ideals in $A \rtimes_\alpha G$.

*Proof.* Let $I$ be a $G$-invariant ideal in $A$. One easily checks that $I \rtimes_{\alpha|_I} G$ is isomorphic to

$$\overline{\mathrm{span}}(\iota_A(x)\iota_G(f)\colon x \in I, f \in C_c(G)\},$$

and therefore $I \rtimes_{\alpha|_I} G$ is canonically an ideal in $A \rtimes_\alpha G$. The identification above also makes it obvious that $I \rtimes_{\alpha|_I} G$ is $\widehat{G}$-invariant, so the assignment is well-defined.

Suppose now that $J$ is a $\widehat{G}$-invariant ideal in $A \rtimes_\alpha G$. Taking its crossed product and reasoning as above, we deduce that $J \rtimes_{\widehat{\alpha}|_J} \widehat{G}$ is a $G$-invariant ideal in $A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}$. This double crossed product is equivariantly isomorphic to $(A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \mathrm{Ad}(\lambda))$ by Theorem 5.2.7, so there exists a $G$-invariant ideal $I_J$ in $A$ such that

$$J \rtimes_{\widehat{\alpha}|_J} \widehat{G} = I_J \otimes \mathcal{K}(L^2(G)).$$

We claim that $I_{I \rtimes_{\alpha|_I} G} = I$ and that $J = I_J \rtimes_{\alpha|_{I_J}} G$. Let $j_A, j_G$ and $j_{\widehat{G}}$ be the maps constructed in the proof of Theorem 5.2.7, that show that $A \otimes \mathcal{K}(L^2(G))$ satisfies the universal property of the double crossed product. We will use the fact that the Takai isomorphism

$$\tau\colon A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}$$

satisfies $\tau(j_A(a)j_G(f)j_{\widehat{G}}(\widehat{f})) = \iota_{A\rtimes_\alpha G}(\iota_A(a)\iota_G(f))\iota_{\widehat{G}}(\widehat{f})$ for all $a \in A$, all $f \in C_c(G)$ and all $\widehat{f} \in C_c(\widehat{G})$.

Let $I$ be an $\alpha$-invariant ideal in $A$. Using $\alpha$-invariance, one checks that $j_A(I)$ is contained in $M(I \otimes \mathcal{K}(L^2(G)))$. It then follows that the restriction of $\tau$ to $I \otimes \mathcal{K}(L^2(G))$ is an isomorphism between $I \otimes \mathcal{K}(L^2(G))$ and $I \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}$. This shows that $I_{I \rtimes_{\alpha|_I} G} = I$.

The other identity is proved similarly, and is left as an exercise. $\square$

We finish this chapter with two examples.

**Example 5.2.10.** Let $\theta \in \mathbb{R}$, and consider the rotation algebra

$$A_\theta = C^*(\{u, v \text{ unitaries with } uv = e^{2\pi i\theta} vu\});$$

see Example 4.2.6. The gauge action $\gamma\colon S^1 \to \mathrm{Aut}(A_\theta)$ described in part (2) in Examples 3.3.13 can be immediately identified with the dual action of $r_\theta\colon \mathbb{Z} \to \mathrm{Aut}(C(S^1))$, once $C(S^1) \rtimes_{r_\theta} \mathbb{Z}$ is identified canonically with $A_\theta$. It follows that there is an equivariant isomorphism

$$(A_\theta \rtimes S^1, \widehat{\gamma}) \cong (C(S^1) \otimes \mathcal{K}(L^2(S^1)), r_\theta \otimes \mathrm{Ad}(\lambda)).$$

Although we do not have enough tools to prove all the claims in the following example, we choose to present it here due to its historical relevance.

**Example 5.2.11.** Let $n \in \mathbb{N}$ with $n \geq 1$, and consider the Cuntz algebra $\mathcal{O}_n$ with its gauge action $\gamma\colon S^1 \to \mathrm{Aut}(\mathcal{O}_n)$ from part (4) of Examples 3.3.13. Then there is a canonical identification of $\mathcal{O}_n \rtimes_\gamma S^1$ with $M_{n^\infty} \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$, where the dual of $\gamma$ becomes the bilateral shift.

Possibly add: Landstat, and Pedersen's result on equivariant isomorphism of the crossed product.

# Chapter 6

# Compact group actions

In this chapter, we define and study fixed point algebras, with special emphasis on the case of compact group actions. We will define the strong Connes spectrum of an action of a compact abelian group, and will prove that the spectrum is full if and only if the fixed point algebra is Morita equivalent to the crossed product.

## 6.1 Fixed point algebras

**Definition 6.1.1.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We define the *fixed point algebra* of $\alpha$ by

$$A^\alpha = \{a \in A \colon \alpha_g(a) = a \text{ for all } g \in G\}.$$

When $\alpha$ is clear from the context, we sometimes write $A^G$ in place of $A^\alpha$.

Fixed point algebras are the noncommutative analog of orbit spaces, since for a locally compact Hausdorff space $X$ and an action of $G$ on $X$ there is a canonical identification $C_0(X/G) \cong C_0(X)^G$.

In particular, when $G$ is not compact, the fixed point algebra may be very small (and in some cases even empty). The study of fixed point algebras is therefore most meaningful in the case of compact group actions. In this case, there is a canonical conditional expectation from the original algebra to the fixed point algebra, which is in some sense "dual" to the one constructed in Theorem 4.3.4.

**Proposition 6.1.2.** Let $G$ be a compact group with Haar probability measure $\mu$, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then there is a canonical faithful conditional expectation $E\colon A \to A^\alpha$ given by $E(a) = \int\limits_G \alpha_g(a)\, d\mu(g)$ for all $a \in A$.

*Proof.* That $E$ is a conditional expectation is proved analogously to Proposition 4.3.6, and is left as an exercise. We show that $E$ is faithful as follows. Let

$a \in A$ be nonzero. Choose a positive linear functional $\phi\colon A \to \mathbb{C}$ such that $\phi(a^*a) > 0$. By continuity, there exists an open neighborhood $U$ of $e \in G$ such that $\phi(\alpha_g(a^*a)) > 0$ for all $g \in U$. Then

$$\phi(E(a^*a)) = \int_G \alpha_g(a^*a)\ d\mu(g) \geq \int_U \alpha_g(a^*a)\ d\mu(g) > 0.$$

Thus $E(a^*a) \neq 0$ whenever $a \neq 0$, as desired. □

**Definition 6.1.3.** Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a compact group $G$ on a $C^*$-algebra $A$, and let $a \in A^\alpha$. We denote by $c_a\colon G \to A$ the continuous function that is constantly equal to $a$. We denote by $c\colon A^\alpha \to C(G, A) \subseteq A \rtimes_\alpha G$ the resulting map.

Fauthfulness of the conditional expectation is very helpful when checking whether a certain equivariant map is injective. Concretely, it suffices to check injectivity on the fixed point algebra.

**Corollary 6.1.4.** Let $G$ be a compact group, let $A$ and $B$ be $C^*$-algebras, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions, and let $\varphi\colon A \to B$ be an equivariant homomorphism. Then $\varphi$ is injective if and only if $\varphi|_{A^\alpha]}\colon A^\alpha \to B^\beta$ is injective.

*Proof.* It is clear that $\varphi|_{A^\alpha]}$ is injective if $\varphi$ is. Conversely, let $a \in A$ be a positive element satisfying $\varphi(a) = 0$. Since $E$ is natural with respect to equivariant homomorphisms (meaning that $E \circ \varphi = \varphi \circ E$), it follows that $\varphi(E(a)) = 0$. Since $E(a) \in A^\alpha$ and $\varphi|_{A^\alpha]}$ is injective, we deduce that $E(a) = 0$. Using positivity of $a$ and faithfulness of $E$ (Proposition 6.1.2), we conclude that $a = 0$, as desired. □

**Theorem 6.1.5.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then $c\colon A^\alpha \to A \rtimes_\alpha G$ is an injective homomorphism, and there exists a unique projection $p \in M(A \rtimes_\alpha G)$ such that $c(A^\alpha) = p(A \rtimes_\alpha G)p$.

*Proof.* It is clear that $c$ is injective, so it suffices to check that it is multiplicative. Given $a, b \in A^\alpha$ and $g \in G$, we use the operation in $C(G, A, \alpha)$ to get

$$(c_a * c_b)(g) = \int_G c_a(h)\alpha_h(c_b(h^{-1}g))\ d\mu(h) = \int_G ab\ d\mu(h) = ab = c_{ab}(g),$$

as desired.

Denote by 1 the unit of $M(A)$, and by $p$ the function on $G$ which is constantly equal to 1. (Observe that $p = c_1$ when $A$ is unital.) Then $p$ belongs to $M(A \rtimes_\alpha G)$ (and it belongs to $A \rtimes_\alpha G$ if $A$ is unital). It is clear that $p$ is a projection.

Let $a \in A^\alpha$. We claim that $c_a = pc_ap$. Given $g \in G$, we have

$$(c_a * p)(g) = \int_G c_a(h)\alpha_h(p(h^{-1}g))\; d\mu(h) = \int_G a1\; d\mu(h) = a = c_a(g),$$

so $c_a * p = c_a$. A similar computation (or taking adjoints) shows that $p * c_a = c_a$, so the claim is proved. It follows that $c(A^\alpha) \subseteq p(A \rtimes_\alpha G)p$. Let us show the converse inclusion.

Given $f \in C(G, A, \alpha)$ and $g \in G$, we have

$$(f * p)(g) = \int_G f(h)\alpha_s(p(h^{-1}g))\; d\mu(h) = \int_G f(h)\; d\mu(h),$$

and thus

$$(p * f * p)(g) = \int_G p(h)\alpha_h((f * p)(h^{-1}g))\; d\mu(h) = \int_G \int_G \alpha_h(f(k))\; d\mu(k)d\mu(h).$$

Setting $x = \int\limits_G \alpha_h \left( \int\limits_G f(k)\; d\mu(k) \right) d\mu(h)$, we have $x \in A^\alpha$ and $p * f * p = c_x$. It follows that $p(A \rtimes_\alpha G)p \subseteq c(A^\alpha)$. Since uniqueness of $p$ is clear, the proof is complete. □

For future use, we extract some identities from the proof of the previous theorem.

**Remark 6.1.6.** Let the assumptions and notation be as in Theorem 6.1.5, and let $f \in C(G, A)$ and let $g \in G$. Then

$$(f * p)(g) = \int_G f(h)\; d\mu(h), \quad (p * f)(g) = \int_G \alpha_h(f(h^{-1}g))\; d\mu(h)$$

and

$$(f * p * f)(g) = \int\limits_G \alpha_h \left( \int\limits_G f(k)\; d\mu(k) \right) d\mu(h).$$

**Remark 6.1.7.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then there are two canonical ways of embedding $A^\alpha$ into $A \rtimes_\alpha G$: one such embedding is the map $c$ from Theorem 6.1.5, and the other one is the composition $A^\alpha \hookrightarrow A \hookrightarrow A \rtimes_\alpha G$. These embeddings never agree for non-trivial $G$: while the first one is a corner embedding, the second one is unital.

We now apply Theorem 6.1.5 in combination with the theory of Morita equivalence from Section 2.5. For this, we need to know when crossed products and fixed point algebras are $\sigma$-unital, which we do next. The result for the crossed product was proved for general locally compact groups in Proposition 4.1.16, so we treat the case of fixed point algebras of compact group actions here.

Recall that for a locally compact group $G$, its full group algebra $C^*(G)$ is $\sigma$-unital if and only if $G$ is first countable.

**Proposition 6.1.8.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

1. If $(a_j)_{j\in J}$ is an approximate unit for $A$, then $(E(a_j))_{j\in J}$ is an approximate unit for $A$ contained in $A^\alpha$.

2. If $A$ is $\sigma$-unital, then $A^\alpha$ is also $\sigma$-unital.

*Proof.* (1). Let $(a_j)_{j\in J}$ be an approximate identity in $A$. Fix a positive element $a \in A$ and $\varepsilon > 0$. Since $\{\alpha_g(a)\colon g \in G\}$ is norm-compact in $A$, we can find $j_0 \in J$ such that $\|a_j\alpha_g(a) - \alpha_g(a)\| < \varepsilon$ for all $j \geq j_0$ and all $g \in G$. Since $\alpha_g$ is isometric, the previous inequality is equivalent to $\|\alpha_g(a_j)a - a\| < \varepsilon$ for all $j \geq j_0$ and all $g \in G$. By averaging over $G$, we conclude that $\|E(a_j)a - a\| < \varepsilon$ for all $j \geq j_0$, as desired.

(2). This follows immediately from (1). □

**Corollary 6.1.9.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then $A^\alpha$ is Morita equivalent to an ideal in $A \rtimes_\alpha G$. In particular, if $A$ is $\sigma$-unital, then $A^\alpha$ is stably isomorphic to an ideal in $A \rtimes_\alpha G$.

*Proof.* The first claim follows immediately from Theorem 6.1.5 and Example 2.5.2, while the second one follows from Proposition 4.1.16, Proposition 6.1.8 and Corollary 2.5.12. □

The ideal generated by the image of $A^\alpha$ in $A \rtimes_\alpha G$ under the map $c$ from Definition 6.1.3 admits the following natural description.

**Proposition 6.1.10.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. For $a \in A$, let $\widetilde{a} \in C(G, A)$ be given by $\widetilde{a}(g) = \alpha_g(a)$ for all $g \in G$. Then the ideal in $A \rtimes_\alpha G$ generated by $c(A^\alpha)$ agrees with

$$\overline{\mathrm{span}}\{\widetilde{a}^* * \widetilde{b}\colon a, b \in A\}.$$

Moreover, for $a, b \in A$ we have $(\widetilde{a}^* * \widetilde{b})(g) = a^*\alpha_g(b)$ for all $g \in G$.

**Exercise 6.1.11.** Prove Proposition 6.1.10.

## 6.2 Eigenspaces of compact groups actions

In this section, we analyze the structure of a $C^*$-algebra which has a compact group action, through the so-called *eigenspaces* of the action. In the following section, these eigenspaces will be used to define two kinds of spectra for the action, which provide useful information about the ideal structure of the crossed product.

Since this analysis is technically much simpler when the group is abelian, we will restrict to this case throughout. It should be noted, however, that the results in this section admit non-commutative analogs; see ???.

**Definition 6.2.1.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. For a character $\chi \in \widehat{G}$, we define the associated *spectral subspace*, also called *eigenspace* $A(\chi)$ by

$$A(\chi) = \{a \in A\colon \alpha_g(a) = \chi(g)a \text{ for all } g \in G\}.$$

Note that if $\mathrm{t}\colon G \to \{1\}$ denotes the trivial character, then $A_{\mathrm{t}} = A^\alpha$ is the fixed point algebra of $\alpha$.

**Proposition 6.2.2.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

1. For $\chi \in \widehat{G}$, there exists a contractive linear idempotent map $E_\chi\colon A \to A(\chi)$ given by
$$E_\chi(a) = \int_G \overline{\chi(g)}\alpha_g(a)\ d\mu(g)$$
for all $a \in A$. Moreover, $E_\chi \circ E_\tau = 0$ if $\chi \neq \tau$.
2. For all $\chi, \tau \in \widehat{G}$, we have
$$A(\chi)^* = A(\chi^{-1}) \quad \text{and} \quad A(\chi)A(\tau) \subseteq A(\chi\tau).$$
3. Given $a \in A$, we have $E_\chi(a) = 0$ for all $\chi \in \widehat{G}$ if and only if $a = 0$. In particular, $\sum\limits_{\chi \in \widehat{G}} A(\chi)$ is dense in $A$.
4. Given $a \in A$, we have $E_1(a^*a) = 0$ for all $\chi \in \widehat{G}$ if and only if $a = 0$.
5. $\overline{A(\chi)A(\chi)^*}$ is an ideal in $A^\alpha$ for all $\chi \in \widehat{G}$.

*Proof.* (1). Fix $\chi \in \widehat{G}$. For $a \in A$, the element $E_\chi(a)$, as given in the statement, is well-defined because $g \mapsto \alpha(a)$ is continuous and $G$ is compact. Moreover, for $h \in G$ we have

$$\alpha_h(E_\chi(a)) = \int_G \overline{\chi(g)}\alpha_{hg}(a)\ d\mu(g) = \int_G \overline{\chi(h^{-1}g)}\alpha_g(a)\ d\mu(g) = \chi(h)E_\chi(a),$$

so $E_\chi(a) \in A(\chi)$. The resulting map $E_\chi\colon A \to A(\chi)$ is clearly linear and contractive. Given $\tau \in \widehat{G}$ and $a \in a$, we have

$$\begin{aligned}(E_\chi \circ E_\tau)(a) &= \int_G \int_G \chi^{-1}(g)\tau^{-1}(h)\alpha_{gh}(a)\ d\mu(g)d\mu(h)\\ &= \int_G \int_G \chi^{-1}(g)\tau^{-1}(h)\alpha_{gh}(a)\ d\mu(g)d\mu(h)\\ &= \int_G \chi^{-1}(g)\tau(g)d\mu(g) \int_G \tau^{-1}(h)\alpha_h(a)d\mu(h).\end{aligned}$$

When $\tau = \chi$, the above expression equals $E_\chi(a)$, so we deduce that $E_\chi \circ E_\chi = E_\chi$. When $\tau \neq \chi$, then $\int_G \chi^{-1}(g)\tau(g)d\mu(g) = 0$ and hence $E_\chi \circ E_\tau = 0$.

(2). These are straightforward to verify.

(3). Recall that $\overline{\mathrm{span}}^{\|\cdot\|_\infty}\widehat{G} = C(G)$. Then the assumption implies that $\int_G f(g)\alpha_g(a)\ d\mu(g) = 0$ for all $f \in C(G)$. Let $(f_j)_{j\in J}$ be an approximate identity for $L^1(G)$ contained in $C(G)$. Then

$$a = \lim_{j\in J} \int_G f_j(g)\alpha_g(a)\ d\mu(g) = 0,$$

as desired.

(4). This was proved in Proposition 6.1.2.

(5). Fix $\chi \in \widehat{G}$. It follows from (3) that $\overline{A(\chi)A(\chi)^*}$ is contained in $A^\alpha$, so it remains to show that it is an ideal in it. In turn, this follows from the fact that for $a \in A^\alpha$ and $b \in a$, we have $E_\chi(ab) = aE_\chi(b)$ and $E_\chi(ba) = E_\chi(b)a$. ☐

At this point, the reader should notice some similarities between the maps $E_\chi\colon A \to A(\chi)$, for $\chi \in \widehat{G}$ (for $G$ compact) in the previous proposition, and the maps $E_g\colon A \rtimes G \to A$, for $g \in G$ (for $G$ discrete) from Lemma 4.3.3. These are, in some sense, "the same", and the precise relationship will become clear in the next chapter. For now, we give an example in which these maps are really identical.

**Example 6.2.3.** Let $G$ be a compact abelian group, and let $\Gamma$ denote its dual group, which is discrete. Then $C(G)$ can be canonically identified with $C^*(\Gamma)$. Under this identification, the action $\mathtt{Lt}\colon G \to \mathrm{Aut}(C(G))$ is given by $\mathtt{Lt}_g(u_\gamma) = \gamma(g)u_\gamma$ for all $g \in G$ and all $\gamma \in \Gamma$.

For $\gamma \in \Gamma = \widehat{G}$, we have

$$C(G)(\gamma) = \{x \in C^*(\Gamma)\colon \mathtt{Lt}_g(x) = \gamma(g)x \text{ for all } g \in G\} = \mathbb{C}u_\gamma,$$

and the linear idempotent $E_\gamma\colon C(G) \to C(G)(\gamma)$ from Proposition 6.2.2 is precisely the linear idempotent $E_\gamma\colon C^*(\Gamma) \to \mathbb{C}$ from Lemma 4.3.3. To check this, notice first that both maps send $u_\chi$ to $\delta_{\chi,\gamma}u_\gamma$. Therefore they agree on the canonical unitaries of $C^*(\Gamma)$, and thus on all of $C^*(\Gamma)$.

The previous example can be greatly generalized:

**Exercise 6.2.4.** Let $\Gamma$ be a discrete abelian group, and set $G = \widehat{\Gamma}$. Let $\beta\colon \Gamma \to \mathrm{Aut}(B)$ be an action, and set

$$A = B \rtimes_\beta \Gamma \quad \text{and} \quad \alpha = \widehat{\beta}\colon G \to \mathrm{Aut}(A).$$

Show that $A(\chi) = Bu_\chi$ for all $\chi \in \widehat{G} \cong \Gamma$. In particular, this shows that the fixed point algebra of the dual action $\widehat{\beta}$ is $B$.

We make some comments about non-continuous actions.

**Remark 6.2.5.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a not necessarily continuous action. The spectral subspaces $A(\chi)$ can be defined also in this context, and their closed linear span

$A_\alpha = \overline{\sum_{\chi\in\widehat{G}} A(\chi)}$ is a $G$-invariant subalgebra of $A$. It is immediate that the restriction of $\alpha$ to $A_\alpha$ is continuos, and in fact $A_\alpha$ is the largest $G$-invariant subalgebra of $A$ where $\alpha$ is continuous. Indeed, an alternative description of $A_\alpha$ is the following:

$$A_\alpha = \{a \in A\colon \text{ the map } \alpha^a\colon G \to A \text{ given by } g \mapsto \alpha_g(a) \text{ is continuous}\}.$$

Fixed-point algebras of tensor product actions can be computed nicely using spectral subspaces:

**Proposition 6.2.6.** Let $G$ be a compact abelian group, let $A$ and $B$ be $C^*$-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions. Then

$$(A \otimes B)^{\alpha\otimes\beta} = \overline{\sum_{\chi\in\widehat{G}} A(\chi) \otimes B(\chi^{-1})}.$$

In particular, when $\beta = \mathrm{id}_B$, we have $(A\otimes B)^{\alpha\otimes\mathrm{id}_B} = A^\alpha \otimes B$ (and this is valid also for non-abelian groups).

*Proof.* It follows from part (3) of Proposition 6.2.2 that $\sum_{\chi,\tau\in\widehat{G}} A(\chi) \otimes B(\tau)$ is dense in $A \otimes B$. If $E\colon A \otimes B \to (A \otimes B)^{\alpha\otimes\beta}$ denotes the canonical conditional expectation from Proposition 6.1.2, then the image of $\sum_{\chi,\tau\in\widehat{G}} A(\chi)\otimes B(\tau)$ under $E$ is dense in $(A \otimes B)^{\alpha\otimes\beta}$.

Let $\chi, \tau \in \widehat{G}$, let $a \in A(\chi)$ and let $b \in B(\tau)$. We use orthgonality of the characters of $G$ at the last step to get

$$\begin{aligned} E(a \otimes b) &= \int_G \alpha_g(a) \otimes \beta_g(b)\ d\mu(g) \\ &= (a \otimes b) \int_G \chi(g)\tau(g)\ d\mu(g) \\ &= \begin{cases} a \otimes b, & \text{if } \chi = \tau^{-1} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, it follows that $A(\chi) \otimes B(\chi^{-1})$ is contained in $(A \otimes B)^{\alpha\otimes\beta}$ for all $\chi \in \widehat{G}$, and that their span is dense. $\square$

## 6.3 Spectra for compact abelian group actions

In this section, we use the eigenspaces considered in the previous section to define spectra for compact abelian group actions, and show that the ideal structure of the crossed product can be determined in the case of full spectrum.

**Notation 6.3.1.** If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action, we write $\mathrm{Her}_G(A)$ for the set of all $G$-invariant hereditary subalgebras of $A$. For $B \in \mathrm{Her}_G(A)$, we write $\alpha|_B$ for the induced action on $B$.

We now introduce two different spectra for compact abelian group actions.

**Definition 6.3.2.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

1. We define *strong Arveson spectrum* $\widetilde{\mathrm{Sp}}(\alpha)$ to be

$$\widetilde{\mathrm{Sp}}(\alpha) = \{\chi \in \widehat{G} \colon \overline{A(\chi)A(\chi)^*} = A^\alpha\}.$$

2. We define *strong Connes spectrum* $\widetilde{\Gamma}(\alpha)$ to be

$$\widetilde{\Gamma}(\alpha) = \bigcap_{B \in \mathrm{Her}_G(A)} \widetilde{\mathrm{Sp}}(\alpha|_B).$$

There are "weak" versions of these spectra, called respectively the Arveson spectrum $\mathrm{Sp}(\alpha)$ and the Connes spectrum $\Gamma(\alpha)$, that are defined using the condition $A(\chi) \neq 0$, instead of $A(\chi)A(\chi)^*$ being dense in $A^\alpha$. Unlike their strong versions, $\mathrm{Sp}(\alpha)$ and $\Gamma(\alpha)$ are subgroups of $\widehat{G}$. All these spectra are in general different, and the following exercise shows that the "strong" versions are in general different from the regular ones.

**Exercise 6.3.3.** Let $\alpha\colon \{-1, 1\} \to \mathrm{Aut}(C([-1, 1]))$ be the action induced by multiplication by $-1$ on $[-1, 1]$.

1. Describe the spectral subspaces of $\alpha$.
2. Compute $\mathrm{Sp}(\alpha)$, $\Gamma(\alpha)$, $\widetilde{\mathrm{Sp}}(\alpha)$ and $\widetilde{\Gamma}(\alpha)$.

An alternative description of $\widetilde{\mathrm{Sp}}(\alpha)$ is given in the following exercise.

**Exercise 6.3.4.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Show that

$$\widetilde{\mathrm{Sp}}(\alpha) = \{\chi \in \widehat{G} \colon \overline{A(\chi)AA(\chi^{-1})} = A\}.$$

Also, both the Connes spectrum and the strong Connes spectrum can be computed using the dual action.

**Proposition 6.3.5.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then

1. $\Gamma(\alpha) = \{\chi \in \widehat{G} \colon \widehat{\alpha}_\chi(I) \cap I \neq 0 \text{ for all ideals } I \subseteq A \rtimes_\alpha G\}$.
2. $\widetilde{\Gamma}(\alpha) = \{\chi \in \widehat{G} \colon \widehat{\alpha}_\chi(I) \subseteq I \text{ for all ideals } I \subseteq A \rtimes_\alpha G\}$.

*Proof.* Part (1) is proved on pages 391 and 392 of [56]. Part (2) is Lemma 3.4 (but really Lemma 3.2) in [55] – but this is also for locally compact, perhaps there's a better proof for compact $G$? $\square$

The actual spectrum of an action (Arveson's, Connes', or the strong versions), as a subset of $\widehat{G}$, does not provide much information, and the condition that seems to be of interest is fullness of the spectrum (meaning that it is equal to $\widehat{G}$). In this case, it turns out that a lot about the ideal structure of $A \rtimes_\alpha G$, although what exactly can be said depends on what spectrum one is considering.

**Theorem 6.3.6.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

1. The following are equivalent:
   a) $\widetilde{\mathrm{Sp}}(\alpha) = \widehat{G}$;
   b) $c(A^\alpha)$ is a full corner in $A \rtimes_\alpha G$ (see Theorem 6.1.5).
2. The following are equivalent:
   a) $\widetilde{\Gamma}(\alpha) = \widehat{G}$;
   b) For every ideal $J$ in $A \rtimes_\alpha G$ we have $J = (J \cap A) \rtimes_{\alpha_{J\cap A}} G$.

*Proof.* (1). (a) implies (b). Set

$$C = \overline{\mathrm{span}}^{\|\cdot\|_\infty}\{\widetilde{a}^*\widetilde{b} \in C(G, A)\colon a, b \in A\}.$$

We will show that $C = C(G, A)$. Since

$$\|\cdot\|_{A\rtimes_\alpha G} \leq \|\cdot\|_1 \leq \|\cdot\|_\infty$$

on $C(G, A)$, the result will follow from density of $C(G, A)$ in $A\rtimes_\alpha G$. Moreover, since $\overline{\mathrm{span}}^{\|\cdot\|_\infty}\widehat{G} = C(G)$ and

$$\{g \mapsto f(g)a\colon f \in C(G), a \in A\}$$

is dense in $C(G, A)$, it suffices to show that for every $\chi \in \widehat{G}$ and for every $a \in A$, the function $\chi a$ belongs to $C$.

Fix $a \in A$ with $\|a\| = 1$, fix $\chi \in \widehat{G}$ and fix $\varepsilon > 0$. Use part (1) of Proposition 6.1.8 to find $e \in A^\alpha$ such that $\|a - ae\| < \varepsilon/2$. Since $A(\chi)^*A(\chi)$ is dense in $A^\alpha$, there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in A(\chi)$ such that $\left\|e - \sum\limits_{j=1}^n x_j^* y_j\right\| < \varepsilon/2$. Set $f = \sum\limits_{j=1}^n \widetilde{x_j a^*}^* * \widetilde{y}_j$, and note that $f(g) = \chi(g)a \sum\limits_{j=1}^n x_j^* y_j$ for all $g \in G$. Then

$$\|\chi(g)a - f(g)\| = \left\|\chi(g)a - \chi(g)a\sum_{j=1}^n x_j^* y_j\right\| \leq \|a - ae\| + \|a\|\|e - \sum_{j=1}^n x_j^* y_j\| < \varepsilon$$

for all $g \in G$. Since $f$ belongs to $C$, it follows that $\chi(g)a$ belongs to $C$ as well and thus $C = C(G, A)$, as desired.

(b) implies (a). Given $f \in C(G, A)$ and $\tau \in \widehat{G}$, we will denote by $\tau f \in C(G, A)$ the pointwise product of $\tau$ and $f$. Notice that $\|\tau f\|_{A\rtimes_\alpha G} = \|f\|_{A\rtimes_\alpha G}$ for all $f \in C(G, A)$, and that $\tau(f_1 * f_2) = (\tau f_1) * (\tau f_2)$ for all $f_1, f_2 \in C(G, A)$.

Let $\varepsilon > 0$, let $\chi \in \widehat{G}$, and let $x \in A^\alpha$. Then there exist $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ such that $\left\| \chi^{-1} c_x - \sum\limits_{j=1}^{n} \widetilde{a}_j^* \widetilde{b}_j \right\| < \varepsilon$. Recall the definition of the linear contractive idempotent $E_\chi \colon A \to A(\chi)$ from part (1) of Proposition 6.2.2. The proof will be concluded once we show that

$$\left\| x - \sum_{j=1}^{n} E_\chi(a_j^*) E_\chi(b_j) \right\| < \varepsilon.$$

**Claim:** *for $a, b \in A$, and for $p \in M(A \rtimes_\alpha G)$ as in Theorem 6.1.5, we have*

$$p * (\chi^{-1}\widetilde{a}^*) * (\chi^{-1}\widetilde{b}) * p = c_{E_\chi(a)^* E_\chi(b)}.$$

To prove the claim, let $g \in G$. Using Remark 6.1.6, we get

$$\begin{aligned}
(p * (\chi^{-1}\widetilde{a}^*))(g) &= \int_G \alpha_h((\chi^{-1}\widetilde{a}^*)(h^{-1}g)\ d\mu(h) \\
&= \chi^{-1}(g) \int_G \chi(h)\alpha_h(\alpha_{h^{-1}g}(\widetilde{a}(g^{-1}h)^*))\ d\mu(h) \\
&= \chi^{-1}(g) \int_G \chi(h)\alpha_g(\alpha_{g^{-1}h}(a^*))\ d\mu(h) \\
&= \chi^{-1}(g) \left( \int_G \chi^{-1}(h)\alpha_g(\alpha_{g^{-1}h}(a))\ d\mu(h) \right)^* \\
&= \chi^{-1}(g) E_\chi(a)^*.
\end{aligned}$$

Similarly,

$$(\chi^{-1}\widetilde{b} * p)(g) = \int_G \chi^{-1}(h)\widetilde{b}(h)\ d\mu(h) = \int_G \chi^{-1}(h)\alpha_h(b) d\mu(h) = E_\chi(b).$$

Thus,

$$\begin{aligned}
\left( \left[p * (\chi^{-1}\widetilde{a}^*)\right] * \left[(\chi^{-1}\widetilde{b}) * p\right] \right)(g) &= \int_G \chi^{-1}(h) E_\chi(a)^* \alpha_h(E_\chi(b))\ d\mu(h) \\
&= E_\chi(a)^* \int_G \chi^{-1}(h)\alpha_h(E_\chi(b))\ d\mu(h) \\
&= E_\chi(a)^* E_\chi(E_\chi(b)) \\
&= E_\chi(a)^* E_\chi(b),
\end{aligned}$$

as desired. This proves the claim.

Recall that $\chi^{-1}(f_1 * f_2) = (\chi^{-1} f_1) * (\chi^{-1} f_2)$ for all $f_1, f_2 \in C(G, A)$. In the following computation, we use at the first step that $c\colon A^\alpha \to A \rtimes_\alpha G$ is an isometric homomorphism, and at the second step we use the claim and the fact that the image of $c$ is contained in $p(A \rtimes_\alpha G)p$ (see Theorem 6.1.5), to get

$$\begin{aligned}\left\| x - \sum_{j=1}^{n} E_\chi(a_j^*) E_\chi(b_j) \right\| &= \left\| c_x - \sum_{j=1}^{n} c_{E_\chi(a_j^*) E_\chi(b_j)} \right\| \\ &= \left\| p * \left( c_x - \chi^{-1} \sum_{j=1}^{n} \widetilde{a}_j^* * \widetilde{b}_j \right) * p \right\| \\ &\leq \left\| c_x - \chi^{-1} \sum_{j=1}^{n} \widetilde{a}_j^* * \widetilde{b}_j \right\| \\ &= \left\| \chi c_x - \sum_{j=1}^{n} \widetilde{a}_j^* * \widetilde{b}_j \right\|,\end{aligned}$$

as desired. This concludes the proof.

The equivalences in (2) have to be filled in. ☐

There also exist characterizations of fullness of the spectra $\mathrm{Sp}(\alpha)$ and $\Gamma(\alpha)$ in terms of weaker conditions for the ideals in the crossed products. Reference?

In particular, we deduce the following:

**Corollary 6.3.7.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then $A \rtimes_\alpha G$ is simple if and only if $\widetilde{\Gamma}(\alpha) = \widehat{G}$ and $A$ is $\alpha$-simple.

*Proof.* If $A \rtimes_\alpha G$ is simple, then $A^\alpha$, being a corner of it, is also simple. Thus $\widetilde{\Gamma}(\alpha) = \widehat{G}$ and thus $A$ is $\alpha$-simple (since otherwise every nontrivial $G$-invariant ideal would induce a non-trivial ideal in $A \rtimes_\alpha G$). The converse follows immediately from Theorem 6.3.6. ☐

As usual, it is instructive to look at the commutative case to gain intuition.

**Lemma 6.3.8.** Let $G$ be a compact abelian group, let $X$ be a locally compact Hausdorff space, and let $\alpha\colon G \to \mathrm{Aut}(C_0(X))$ be an action. If $G \curvearrowright X$ is free, then the ideal in $C_0(X) \rtimes_\alpha G$ generated by the image of $C_0(X)^\alpha = C_0(X/G)$, that is

$$\overline{\mathrm{span}}\{(g,x) \mapsto f_1(x) f_2(g^{-1} \cdot x) \colon f_1, f_2 \in C_0(X)\},$$

is equal to $C_0(X) \rtimes_\alpha G$.

*Proof.* Assume that $G \curvearrowright X$ is free. Using the Stone-Weierstrass theorem, it suffices to show that functions of the form given in the statement separate the points of $G \times X$. Let $(g, x), (h, y) \in G \times X$ be distinct points.

If $x \neq y$, then there exists $f_1 \in C_0(X)$ such that $f_1(x) \neq f_1(y)$. Let $f_2 \in C_0(X)$ be any function which is identically 1 on the (compact) orbits of $x$ and $y$. Then

$$f_1(x)f_2(g^{-1} \cdot x) = f_1(x) \neq f_1(y) = f_1(y)f_2(h^{-1} \cdot y).$$

If $x = y$, then $g \neq h$. Since the action is free, we have $g^{-1} \cdot x \neq h^{-1} \cdot x$. Let $f_2 \in C_0(X)$ be any function distinguishing these two points, and let $f_1 \in C_0(X)$ satisfy $f_1(x) \neq 0$. Then

$$f_1(x)f_2(g^{-1} \cdot x) \neq f_1(x) = f_1(y)f_2(h^{-1} \cdot x),$$

as desired. □

For commutative dinamical systems, fullness of the strong spectra is equivalent to freeness:

**Proposition 6.3.9.** Let $G$ be a compact abelian group, let $X$ be a locally compact Hausdorff space, and let $\alpha\colon G \to \mathrm{Aut}(C_0(X))$ be an action. Then the following are equivalent:

1. $\widetilde{\mathrm{Sp}}(\alpha) = \widehat{G}$;
2. $\widetilde{\Gamma}(\alpha) = \widehat{G}$;
3. Every ideal in $C_0(X) \rtimes_\alpha G$ has the form $C_0(U) \rtimes_\alpha G$ for some $G$-invariant open subset $U \subseteq X$;
4. $G \curvearrowright X$ is free.

*Proof.* That (1) implies (4) follows from Lemma 6.3.8 in combination with the first part of Theorem 6.3.6. To prove the converse, suppose that $G \curvearrowright X$ is not free, and find $g \in G \setminus \{1\}$ and $x \in X$ such that $g \cdot x = x$. Let $\chi \in \widehat{G}$ be any character with $\chi(g) \neq 1$. A function $f \in C_0(X)(\chi)$ in particular satisfies $f(x) = f(g \cdot x) = \chi(g)f(x)$, so it must be $f(x) = 0$. Thus $C_0(X)(\chi)^* C_0(X)(\chi)$ consists of functions that vanish on the orbit of $x$, and thus its closure cannot coincide with $C_0(X)^\alpha = C_0(X/G)$. We deduce that $\chi \neq \widetilde{\mathrm{Sp}}(\alpha)$, which is a contradiction.

That (2) implies (1) is a general fact; and the equivalence between (2) and (3) is the content of the second part of Theorem 6.3.6. Finally, since any $G$-invariant hereditary subalgebra of $C_0(X)$ has the form $C_0(U)$ for some $G$-invariant open subset of $X$, and since freeness passes to ideals, another application of Lemma 6.3.8 together with the first part of Theorem 6.3.6 shows that (4) implies (2), thus finishing the proof. □

The case of the gauge action on the irrational rotation algebra $A_\theta$ is a particularly interesting one, that can be analyzed using the tools from this chapter to conclude that $A_\theta$ is simple (although this is of course not the easiest way to prove this fact!).

**Exercise 6.3.10.** Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and consider the irrational rotation algebra $A_\theta$. Let $\gamma \colon S^1 \to \mathrm{Aut}(A_\theta)$ be the gauge action, which is given by

$$\gamma_z(u) = u \quad \text{and} \quad \gamma_z(v) = zv$$

for all $z \in S^1$.

1. Compute the spectral subspaces of $A_\theta$ with respect to $\gamma$.
2. Compute $\widetilde{\Gamma}(\gamma)$.
3. Show, using Takai duality (Theorem 5.2.7) and the characterization of ideals in the crossed product in the case of full strong Connes spectrum (Theorem 6.3.6), that $A_\theta$ is simple.

# Chapter 7

# $K$-theory of crossed products

Many interesting $C^*$-algebras can be described as suitable crossed products, and this presentation is usually used to obtain new information about the internal structure of the algebra in question. There are, for example, a number of tools to study the ideal structure of crossed products, and in particular criteria for deciding when a crossed product is simple. Much less can be said about the structure of projections in crossed products (even for finite group actions, where one can explicitly write down every element in the crossed product). In this context, $K$-theoretic methods are usually very helpful, revealing a great deal of information. Indeed, the computation of the $K$-theories of the irrational rotation algebras, as well as the Cuntz algebras, were originally obtained regarding these objects as crossed products (by the integers, in both cases). In particular, the computation of the $K$-theory for irrational rotation algebras shows that they contain non-trivial projections, a fact that was long believed to be false!

In this chapter, we will study the $K$-theory of crossed products by the reals and the integers (and, as a consequence, by the circle). For the reals, Connes' analog of the Thom isomorphism [14] that the $K$-theory of the crossed product is independent of the action (and hence the same as for the trivial action). For the integers, the result is not so definite but one can obtain a 6-term exact sequence, called the *Pimsner-Voiculescu exact sequence* [77], relating the $K$-groups of the crossed products with those of $A$. The methods presented here also show that the $K$-groups of the crossed product only depend on those of $A$ and the homotopy class of the automorphism.

Both Connes' Thom isomorphism and the Pimsner-Voiculescu exact sequence have by now a number of (independent) proofs. We will here take the shortest path, by deriving the Pimsner-Voiculescu exact sequence from the Thom isomorphism, following an argument of Connes [14].

## 7.1 Connes' Thom isomorphism

Connes' analog of the Thom isomorphism[1] is a generalization of Bott periodicity (Theorem 2.3.18), which is the case of the trivial action. Intuitively speaking, since $\mathbb{R}$ is contractible, any action of it is homotopic to the trivial one, and their crossed products should have the same $K$-theory since $K$-theory is homotopy invariant. This by itself is not enough to prove the theorem (homotopic actions do not in general produce homotopic crossed products), but this general intuition certainly plays a role in the proof.

**Theorem 7.1.1.** Let $A$ be a $C^*$-algebra, and let $\alpha \in \mathrm{Aut}(A)$. Then there are natural isomorphisms

$$K_j(A \rtimes_\alpha \mathbb{R}) \cong K_{1-j}(A) \quad \text{for } j = 0, 1.$$

We will only sketch the idea of the proof of Theorem 7.1.1, leaving most details to the reader. For these, the reader is referred to either [14] or Section 10.9 in [6]. Our goal here is to obtain a description of the isomorphisms that is sufficient for what we do later; that the maps here described do indeed satisfy the properties that we claim, will not be proved.

Let $\gamma$ denote the action of $\mathbb{R}$ on $\mathbb{R} \cup \{+\infty\}$ which fixes $+\infty$ and acts by translation on $\mathbb{R}$. We write $CA$ for the algebra $C_0(\mathbb{R} \cup \{+\infty\}) \otimes A$, and endow it with the action $\gamma \otimes \alpha$. Similarly, we write $SA$ for $C_0(\mathbb{R}) \otimes A$, endowed with the action $\mathtt{Lt} \otimes \alpha$. It is relatively straightforward to check that $SA \rtimes_{\mathtt{Lt}\otimes\alpha} \mathbb{R}$ is naturally isomorphic to $A \otimes \mathcal{K}(L^2(\mathbb{R}))$.

Observe that there is a short exact sequence

$$0 \to SA \rtimes \mathbb{R} \to CA \rtimes \mathbb{R} \to A \rtimes_\alpha \mathbb{R} \to 0,$$

whose associated 6-term exact sequence in $K$-theory (Theorem 2.3.20) becomes

$$\begin{tikzcd}
K_0(A) \arrow[r] & K_0(CA \rtimes \mathbb{R}) \arrow[r] & K_0(A \rtimes_\alpha \mathbb{R}) \arrow[d, "\delta_0"] \\
K_1(A \rtimes_\alpha \mathbb{R}) \arrow[u, "\delta_1"] & K_1(CA \rtimes \mathbb{R}) \arrow[l] & K_1(A). \arrow[l]
\end{tikzcd}$$

The rest of the proof consists in showing that $\delta_0$ and $\delta_1$ are isomorphisms. Next, we show that to prove this, it suffices to show that just one of the maps above is zero.

**Lemma 7.1.2.** The maps $\delta_0$ and $\delta_1$ are always isomorphisms if and only if the natural map $K_0(A) \to K_0(CA \rtimes \mathbb{R})$ is always zero.

[1] The Thom isomorphism is a result relating the $K$-homology groups of a bundle over a space with the $K$-homology groups of the space itself. For one-dimensional bundles, the outcome is a shift by one in the indices of the homology groups.

*Proof.* By exactness, if $\delta_1$ is an isomorphism, then $K_0(A) \to K_0(CA \rtimes \mathbb{R})$ is zero. Conversely, if $K_0(A) \to K_0(CA \rtimes \mathbb{R})$ is always zero, then by taking suspensions (with trivial $\mathbb{R}$-action), and using that the suspension commutes with the cone, we deduce that $K_1(A) \to K_1(CA \rtimes \mathbb{R})$ is always zero as well. In particular, $\delta_0$ and $\delta_1$ are always surjective.

It follows that if $K_j(A \rtimes_\alpha \mathbb{R}) = 0$, then $K_{1-j}(A) = 0$. Applying this to the algebra $A \rtimes_\alpha \mathbb{R}$ and the action $\widehat{\alpha}$, we deduce that if $K_j(A \rtimes_\alpha \mathbb{R} \rtimes_{\widehat{\alpha}} \mathbb{R}) = 0$, then $K_{1-j}(A \rtimes_\alpha \mathbb{R}) = 0$. Note that $A \rtimes_\alpha \mathbb{R} \rtimes_{\widehat{\alpha}} \mathbb{R}$ is isomorphic to $A \otimes \mathcal{K}(L^2(\mathbb{R}))$ by Takai duality.

Putting these things together, we deduce that $K_j(A \rtimes_\alpha \mathbb{R}) = 0$ if and only if $K_{1-j}(A) = 0$. Applied to the algebra $CA$, which is contractible and hence has trivial $K$-groups, we deduce that $K_j(CA \rtimes \mathbb{R}) = 0$ for all algebras $A$ and all actions $\alpha$. Now the exact sequence above implies that $\delta_0$ and $\delta_1$ are isomorphisms, as desired. □

Observe that the natural map $K_0(A) \to K_0(CA \rtimes \mathbb{R})$ is induced by the composition

$$\phi \colon A \hookrightarrow A \otimes \mathcal{K}(L^2(\mathbb{R})) \cong SA \rtimes \mathbb{R} \hookrightarrow CA \rtimes \mathbb{R}.$$

The rest of the proof uses a detailed analysis of this map, which we proceed to sketch. The arguments presented here are not the original ones used by Connes in [14]; instead, we follow arguments of Pimsner and Voiculescu, and specifically Blackadar's presentation in Section 10.9 of [6].

We assume throughout that $A$ is unital, and treat the nonunital case later. Observe that when $A$ is unital, there is a canonical embedding $C_0(\mathbb{R}) \rtimes_{\mathtt{Lt}} \mathbb{R} \to CA \rtimes \mathbb{R}$. Denote by $\chi$ the characteristic function of $(0, \infty)$, and let $p, f \in L^1(\mathbb{R}^2)$ be given by

$$p(x,y) = e^{x/2}e^{-y}\chi(y)\chi(y-x) \quad \text{and} \quad f(x,y) = e^{-x/2}\chi(x)\chi(y-x)$$

for all $x, y \in \mathbb{R}$.

**Lemma 7.1.3.** The functions $p$ and $f$ define elements in $C_0(\mathbb{R}) \rtimes_{\mathtt{Lt}} \mathbb{R}$.

*Proof.* If $p$ and $f$ were continuous on both variables, then they would belong to $C_c(\mathbb{R}^2)$, which is a subalgebra of the crossed product. Although $p$ and $f$ are not continuous on $y$, they can be approximated in norm by elements of $L^1(\mathbb{R}, C_c(\mathbb{R})$ as follows: for $\varepsilon > 0$, let $g_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ be the continuous function that agrees with $\chi$ except on $[0, \varepsilon]$, where it is linear. Set

$$p_\varepsilon(x,y) = e^{x/2}e^{-y}g_\varepsilon(y)g_\varepsilon(y-x) \quad \text{and} \quad f_\varepsilon(x,y) = e^{-x/2}g_\varepsilon(x)g_\varepsilon(y-x)$$

for all $x, y \in \mathbb{R}$. Then $p_\varepsilon$ and $f_\varepsilon$ belong to $C_c(\mathbb{R}^2)$, and $\lim\limits_{\varepsilon \to 0} p_\varepsilon = p$ and $\lim\limits_{\varepsilon \to 0} f_\varepsilon = f$ in the $L^1$-norm. □

A routine computation shows that $p$ is a projection in $C_0(\mathbb{R}) \rtimes_{\mathtt{Lt}} \mathbb{R}$, and hence also in $CA \rtimes \mathbb{R}$. Similarly, one shows that $s = 1 - f$ is an isometry with $ss^* = 1 - p$ in $M(CA \rtimes \mathbb{R})$.

It can also be proved that $\phi$ can be identified with the homomorphism determined by

$$\phi(a)(x, y) = \alpha_x(a)p(x, y)$$

for all $a \in A$ and all $x, y \in \mathbb{R}$. In particular, the image of $\phi$ is contained in the corner $p(CA \rtimes \mathbb{R})p$.

Let $\mu\colon A \to M(CA \rtimes \mathbb{R})$ be the canonical unital embedding as constant functions, and let $B$ be the subalgebra of $M(CA \rtimes \mathbb{R})$ generated by $CA \rtimes \mathbb{R}$ and $\mu(A)$. Then there is a split extension

$$0 \longrightarrow CA \rtimes \mathbb{R} \xrightarrow{\iota} B \longrightarrow A \longrightarrow 0$$

and hence the map $K_0(\iota)\colon K_0(CA \rtimes \mathbb{R}) \to K_0(B)$ is injective. Set $\psi = \iota \circ \phi\colon A \to B$. It thus suffices to show that $K_0(\psi)$ is the zero map.

For $\varepsilon > 0$, set $q_\varepsilon(x, y) = \frac{1}{\varepsilon}p(x/\varepsilon, y/\varepsilon)$ and $t_\varepsilon = \frac{1}{\varepsilon}f(x/\varepsilon, y/\varepsilon)$ for all $x, y \in \mathbb{R}$. Then $q_\varepsilon, t_\varepsilon \in CA \rtimes \mathbb{R}$. We also set $s_\varepsilon = 1 - t_\varepsilon \in M(CA \rtimes \mathbb{R})$.

**Exercise 7.1.4.** For $\varepsilon > 0$, show that $s_\varepsilon$ is an isometry in $M(CA \rtimes \mathbb{R})$ and that $s_\varepsilon s_\varepsilon = 1 - q_\varepsilon$.

For $\varepsilon > 0$, let

$$\omega_\varepsilon\colon A \to M(CA \rtimes \mathbb{R}) \quad \text{and} \quad \phi_\varepsilon\colon A \to CA \rtimes \mathbb{R}$$

be the homomorphisms given by $\omega_\varepsilon(a) = s_\varepsilon\mu(a)s_\varepsilon^*$ and $\phi_\varepsilon(a)(x, y) = \alpha_x(a)q_\varepsilon(x, y)$ for all $a \in A$, and for all $x, y \in \mathbb{R}$. We set $\psi_\varepsilon = \iota \circ \phi_\varepsilon\colon A \to B$. Observe that $\omega_\varepsilon(A) \subseteq (1 - q_\varepsilon)B(1 - q_\varepsilon)$ and $\psi_\varepsilon(A) \subseteq q_\varepsilon B q_\varepsilon$. In particular, $\omega_\varepsilon$ and $\psi_\varepsilon$ have orthogonal ranges and hence their sum $\mu_\varepsilon = \omega_\varepsilon + \psi_\varepsilon$ is a homomorphism $A \to B$.

**Exercise 7.1.5.** Show that $K_0(\omega_\varepsilon) = K_0(\mu)$ for all $\varepsilon > 0$.

Note that the assignment $\varepsilon \mapsto \psi_\varepsilon$ is norm-continuous, and in particular $K_0(\psi_\varepsilon)$ is independent of $\varepsilon$ (and equal to $K_0(\psi)$). One can also show (although it takes somewhat more work), that $\varepsilon \mapsto \mu_\varepsilon$ is also norm-continuous, and thus $K_0(\mu_\varepsilon) = K_0(\mu)$ for all $\varepsilon > 0$. Combining these facts with the previous exercise, we get

$$K_0(\mu) = K_0(\mu_\varepsilon) = K_0(\omega_\varepsilon) + K_0(\psi_\varepsilon) = K_0(\mu) + K_0(\psi),$$

which shows that $K_0(\psi) = 0$ and thus concludes the proof of Theorem 7.1.1.

## 7.2 The Pimsner-Voiculescu exact sequence

In this section, we explain how one can obtain the Pimsner-Voiculescu exact sequence from Theorem 7.1.1. This is not the original argument of Pimsner and Voiculescu from [77], but rather Connes' proof from [14]. In this treatment, the use of the mapping torus is crucial, so we define it next.

**Definition 7.2.1.** Let $A$ be a $C^*$-algebra and let $\alpha \in \mathrm{Aut}(A)$. We define its *mapping torus* $M_\alpha$ by

$$M_\alpha = \{f \in C([0,1], A) \colon f(1) = \alpha(f(0))\}.$$

**Lemma 7.2.2.** Let $A$ be a $C^*$-algebra and let $\alpha \in \mathrm{Aut}(A)$.

1. There is a canonical isomorphism
$$M_\alpha \cong \{f \in C_b(\mathbb{R}, A) \colon f(x+1) = \alpha(f(x)) \text{ for all } x \in \mathbb{R}\}.$$
2. The action $\widetilde{\alpha} \colon \mathbb{R} \to \mathrm{Aut}(C_b(\mathbb{R}, A))$ defined by
$$\widetilde{\alpha}_x(f)(t) = f(t-x)$$
for $x, t \in \mathbb{R}$ and $f \in C_b(\mathbb{R}, A)$ restricts to an action $\widetilde{\alpha} \colon \mathbb{R} \to \mathrm{Aut}(M_\alpha)$.
3. There is a short exact sequence
$$0 \longrightarrow C_0((0,1)) \otimes A \longrightarrow M_\alpha \longrightarrow A \longrightarrow 0.$$

*Proof.* Part (1) is obvious. For part (2), it suffices to notice that $M_\alpha$ (using the presentation from part (1)) is invariant under $\widetilde{\alpha}$.

For part (3), note that a function $f \in C_0((0,1), A)$ naturally belongs to $M_\alpha$ (using the presentation from Definition 7.2.1), and that $C_0((0,1), A)$ is an ideal in $M_\alpha$. Then the quotient map $M_\alpha \to A$ is given by evaluation at 0. $\square$

We will need the following lemma, which resembles Corollary 6.1.4.

**Proposition 7.2.3.** Let $G$ be a locally compact, abelian group, let $A$ be a $C^*$-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Let $C$ be a $C^*$-algebra, and let $\gamma \colon \widehat{G} \to C$ be an action. Suppose that

$$\varphi \colon (A \rtimes_\alpha G, \widehat{\alpha}) \to (C, \gamma)$$

is an equivariant, surjective homomorphism, and let $M(\varphi)$ denote the extension to the multiplier algebras. Then $\varphi$ is injective (and hence an isomorphism) if and only if $M(\varphi)|_A \colon A \subseteq M(A \rtimes_\alpha G) \to M(C)$ is injective.

*Proof.* Since the "only if" implication is obvious, we assume that $M(\varphi)|_A$ is injective. Denote by $J$ the kernel of $\varphi$, which is a $\widehat{G}$-invariant ideal of $A \rtimes_\alpha G$. Using Theorem 5.2.7, we denote by

$$\widehat{\varphi} \colon (A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \mathrm{Ad}(\lambda)) \to (C \rtimes_\gamma \widehat{G}, \widehat{\gamma})$$

the induced $G$-equivariant homomorphism, whose kernel is $J \rtimes_{\widehat{\alpha}} \widehat{G}$. Let $I$ be the unique $G$-invariant ideal such that

$$J \rtimes_{\widehat{\alpha}} \widehat{G} = I \otimes \mathcal{K}(L^2(G));$$

see Proposition 5.2.9. Then $I \rtimes_\alpha G$ is contained in the kernel of $M(\varphi)$, which implies that $I = \{0\}$. We deduce that $\widehat{\varphi}$ is injective, and thus $\varphi$ is injective as well. $\square$

**Proposition 7.2.4.** Let $B$ be a $C^*$-algebra, and let $\beta\colon \mathbb{R} \to \mathrm{Aut}(B)$ be an action which is trivial on $\mathbb{Z} \leq \mathbb{R}$. Denote also by $\beta$ the induced action of $\mathbb{T}$ on $B$, and let $\widehat{\beta} \in \mathrm{Aut}(B \rtimes_\beta \mathbb{T})$ be the dual automorphism. Then $B \rtimes_\beta \mathbb{R}$ is canonically isomorphic to $M_{\widehat{\beta}}$. In particular, $K_j(M_{\widehat{\beta}})$ is canonically isomorphic to $K_{1-j}(B)$.

*Proof.* We use the universal picture of crossed products given in Theorem 2.3.4. Denote by

$$\iota_B\colon B \to M(B \rtimes_\beta \mathbb{T}) \ \text{ and } \ \iota_{\mathbb{T}}\colon \mathbb{T} \to \mathcal{U}(M(B \rtimes_\beta \mathbb{T}))$$

the universal covariant pair for $(\mathbb{T}, B, \beta)$. Given $s \in \mathbb{R}$, we denote by $\chi_s \in \widehat{\mathbb{R}} (\cong \mathbb{R})$ the character given by $\chi_s(t) = e^{ist}$ for all $t \in \mathbb{R}$. We also write $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ for the canonical quotient map We define a covariant pair $(j_B, j_{\mathbb{R}})$ for $(\mathbb{R}, B, \beta)$ on $M_{\widehat{\beta}}$ by

$$j_B(b) = \iota_B(b) \quad \text{and} \quad j_{\mathbb{R}}(t)(s) = \chi_s(t)\iota_{\mathbb{T}}(\pi(t))$$

for all $b \in B$ and all $t, s \in \mathbb{R}$. One checks that $(j_B, j_{\mathbb{R}})$ is indeed a covariant pair on $M_{\widehat{\beta}}$. By Theorem 2.3.4, there is an induced homomorphism $\varphi\colon B \rtimes_\beta \mathbb{R} \to M_{\widehat{\beta}}$. This homomorphism can be seen to be surjective, and it is also $\widehat{\mathbb{R}}$-equivariant, where $M_{\widehat{\beta}}$ carries the real action described in part (2) of Lemma 7.2.2. Since $M(\varphi)$ agrees with $\iota_B$ on $B$, it follows from Proposition 7.2.3 that $\varphi$ is injective, and hence an (equivariant) isomorphism.

The statement about the $K$-theory of the mapping torus now follows from Theorem 7.1.1. ☐

The general structure result for mapping tori given in the previous proposition is just a particular case of a much more general result for induced algebras, which we proceed to describe (more details can be found in [37] and [68].

**Definition 7.2.5.** Let $G$ be a locally compact abelian group, and let $H$ be a closed subgroup in $G$. Given an action $\alpha\colon H \to \mathrm{Aut}(A)$ of $H$ on a $C^*$-algebra $A$, we define the *induced $G$-algebra* $(G, \mathrm{Ind}_H^G(A), \mathrm{Ind}_H^G(\alpha))$ by

$$\mathrm{Ind}_H^G(A) = \{f \in C_b(G, A)\colon f(g+h) = \alpha_h(f(g)) \text{ for all } g \in G, h \in H\},$$

and we let $\mathrm{Ind}_H^G(\alpha)$ be the restriction of the translation action of $G$ on $C_b(G, A)$ to $\mathrm{Ind}_H^G(A)$.

Observe that if $\alpha$ is an automorphism of a $C^*$-algebra $A$, then $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A)$ is just $M_\alpha$ and the induced action $\mathrm{Ind}_H^G(\alpha)$ is the action $\widetilde{\alpha}$ defined in part (2) of Lemma 7.2.2. The following is the general form of Proposition 7.2.4 for abelian groups.

**Theorem 7.2.6.** Let $G$ be a locally compact abelian group, let $H$ be a closed subgroup, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action that is trivial on $H$. We write $\overline{\alpha}\colon G/H \to \mathrm{Aut}(A)$ for the induced action. Then there is a natural $\widehat{G}$-equivariant isomorphism

$$\psi\colon (A \rtimes_\alpha G, \widehat{\alpha}) \to \left(\mathrm{Ind}_{H^\perp}^{\widehat{G}}(A \rtimes_{\overline{\alpha}} G/H), \mathrm{Ind}_{H^\perp}^{\widehat{G}}(\widehat{\overline{\alpha}})\right)$$

The proof of this theorem is beyond the scope of these notes, so we omit it. Some particular cases are much easier to obtain, and we leave the following as an exercise:

**Exercise 7.2.7.** Give a proof of Theorem 7.2.6 for $A$ is unital when $G = \mathbb{Z}$ and $H = n\mathbb{Z}$ for $n \in \mathbb{N}$.

We now turn to the Pimsner-Voiculescu exact sequence.

**Theorem 7.2.8.** Let $A$ be a $C^*$-algebra, and let $\alpha \in \mathrm{Aut}(A)$, and denote by $\iota\colon A \to A \rtimes_\alpha \mathbb{Z}$ the canonical inclusion. Then there is a natural 6-term short exact sequence

$$\begin{tikzcd}
K_0(A) \arrow[r, "\mathrm{id}-K_0(\alpha)"] & K_0(A) \arrow[r, "K_0(\iota)"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l, "K_1(\iota)"] & K_1(A). \arrow[l, "1-K_1(\alpha)"]
\end{tikzcd}$$

Moreover, $K_j(A \rtimes_\alpha \mathbb{Z}) \cong K_{1-j}(M_\alpha)$ for $j = 0, 1$, and thus the $K$-theory of the crossed product only depends on the homotopy class of $\alpha$ in $\mathrm{Aut}(A)$.

*Proof.* Set $B = A \rtimes_\alpha \mathbb{Z}$ and let $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$, regarded as an action of $\mathbb{R}$ which is trivial on $\mathbb{Z}$. Then $B \rtimes_\beta \mathbb{T} \cong A \otimes \mathcal{K}$ by Takai duality (Theorem 5.2.7), and $B \rtimes_\beta \mathbb{R} \cong M_{\widehat{\beta}}$ by Proposition 7.2.4. Thus, there is a short exact sequence

$$0 \longrightarrow C_0((0,1)) \otimes A \otimes \mathcal{K} \longrightarrow B \rtimes_\beta \mathbb{R} \longrightarrow A \otimes \mathcal{K} \longrightarrow 0;$$

see part (3) of Lemma 7.2.2.

Note that the $K$-theory of $A \otimes \mathcal{K}$ is isomorphic to that of $A$, and that the $K$-theory of $C_0((0,1)) \otimes A \otimes \mathcal{K}$ is isomorphic to that of $A$ with a degree-one shift (by the comments at the end of Section 2.3 and Theorem 2.3.18).

Since $K_j(B \rtimes_\beta \mathbb{R}) \cong K_{1-j}(B)$ by Theorem 7.1.1, the 6-term exact sequence for $K$-theory (Theorem 2.3.20) associated to the short exact sequence above gives the exact sequence in the statement (rotating one place clock-wise). The identification of the maps is left to the reader. (It involves, among others, having a sufficiently good description of the exponential and index maps in Theorem 2.3.20, which we have omitted.)

We turn to the last claim. Note that $M_{\widehat{\beta}}$ is just the mapping torus associated to the automorphism $\widehat{\widehat{\alpha}} \cong \alpha \otimes \mathrm{Ad}(\lambda)$ of $A \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$. Now, since $\lambda$ is homotopic to 1 in $\mathcal{B}(\ell^2(\mathbb{Z}))$ (because the unitary group of $\ell^2(\mathbb{Z})$ is connected), we have a homotopy

$$\alpha \otimes \mathrm{Ad}(\lambda) \sim_h \alpha \otimes \mathrm{id}_{\mathcal{K}(\ell^2(\mathbb{Z}))}.$$

Since homotopic automorphisms give rise to isomorphic mapping tori (see Exercise 7.2.9), we deduce that $M_{\widehat{\beta}}$ is isomorphic to $M_{\alpha\otimes \mathrm{id}_{\mathcal{K}(\ell^2(\mathbb{Z}))}}$. However, it is immediate to see that there is a natural isomorphism

$$M_{\alpha\otimes \mathrm{id}_{\mathcal{K}(\ell^2(\mathbb{Z}))}} \cong M_\alpha \otimes \mathcal{K}(\ell^2(\mathbb{Z})).$$

The result thus follows by combining these facts with the isomorphism $K_j(A \rtimes_\alpha \mathbb{Z}) \cong K_{1-j}(M_{\widehat{\beta}})$ obtained above. □

**Exercise 7.2.9.** Complete the proof of Theorem 7.2.8 by showing the following.

1. If $\alpha$ and $\gamma$ are homotopic automorphisms of a $C^*$-algebra $A$, then $M_\alpha$ is isomorphic to $M_\gamma$.
2. If $\alpha$ is an automorphism of a $C^*$-algebra $A$, then there is a natural isomorphism
$$M_{\alpha\otimes \mathrm{id}_{\mathcal{K}(\ell^2(\mathbb{Z}))}} \cong M_\alpha \otimes \mathcal{K}(\ell^2(\mathbb{Z})).$$

**Exercise 7.2.10.** Let $A$ be an AF-algebra and let $\alpha \in \mathrm{Aut}(A)$.

1. If $A$ is unital, show that $A \rtimes_\alpha \mathbb{Z}$ is not AF.
2. If $A$ is not unital, show with an example that $A \rtimes_\alpha \mathbb{Z}$ may be AF.

### An alternative approach using Toeplitz extensions

In this subsection, we describe a different proof of Theorem 7.2.8, not using the Thom isomorphism Theorem 7.1.1. This approach is necessarily more difficult than the one presented above, and for the sake of brevity we will not prove most of the claims we make. This proof uses Topelitz extensions and has the advantage that the maps $K_0(A \rtimes_\alpha \mathbb{Z}) \to K_1(A)$ and $K_1(A \rtimes_\alpha \mathbb{Z}) \to K_0(A)$ can be described in a satisfactory way, unlike in the proof given before.

Recall that the Toeplitz algebra $\mathcal{T}$ is the universal unital $C^*$-algebra generated by an isometry $s$. Given an automorphism $\alpha \in \mathrm{Aut}(A)$ of a $C^*$-algebra $A$, we write $\mathcal{T}_\alpha$ for the subalgebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{T}$ generated by $A \otimes 1_{\mathcal{T}}$ and $u\otimes s$. We abbreviate $t = u\otimes s$, which is clearly an isometry, and set $p = 1-tt^*$, which is a projection in $\mathcal{T}_\alpha$. We identify $A$ with a subalgebra of $\mathcal{T}_\alpha$ canonically, and write $a$ instead of $a \otimes 1$. Define matrix units in $\mathcal{T}_\alpha$ with values in $A$ by $e_{j,k}(a) = \alpha^j(a)t^j p(t^*)^k$ for $j, k \in \mathbb{N}$. It is relatively straightforward to show that these generate a subalgebra of $\mathcal{T}_\alpha$ isomorphic to $A\otimes\mathcal{K}$, and we denote by $\varphi\colon A \otimes \mathcal{K} \to \mathcal{T}_\alpha$ the resulting map.

**Lemma 7.2.11.** Let the notation be as in the discussion above. There exists a unique homomorphism $\psi\colon \mathcal{T}_\alpha \to A \rtimes_\alpha \mathbb{Z}$ satisfying $\psi(a) = a$ for all $a \in A$ and $\psi(u) = t$.

*Proof.* We define a homomorphism $\psi\colon \mathcal{T}_\alpha \to A \rtimes_\alpha \mathbb{Z}$ as follows. First, by the universal property of $\mathcal{T}$, there exists a (unique) homomorphism $\psi_0\colon \mathcal{T} \to A \rtimes_\alpha \mathbb{Z}$ satisfying $\psi_0(s) = 1$. Since $A \rtimes_\alpha \mathbb{Z}$ commutes with $\psi_0(1)$ in $A \rtimes_\alpha \mathbb{Z}$, there is a well defined homomorphism $\mathrm{id}_{A \rtimes_\alpha \mathbb{Z}} \otimes \psi_0\colon (A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{T} \to A \rtimes_\alpha \mathbb{Z}$, and its restriction to $\mathcal{T}_\alpha$ is the desired map. $\square$

We will assume the following result without providing a proof, although it can be shown with elementary methods.

**Proposition 7.2.12.** Let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of a $C^*$-algebra $A$. Let $\varphi\colon A \otimes \mathcal{K} \to \mathcal{T}_\alpha$ be the map from the discussion above, and let $\psi\colon \mathcal{T}_\alpha \to A \rtimes_\alpha \mathbb{Z}$ be the homomorphism provided by Lemma 7.2.11. Then the following is an exact sequence:

$$0 \longrightarrow A \otimes \mathcal{K} \xrightarrow{\varphi} \mathcal{T}_\alpha \xrightarrow{\psi} A \rtimes_\alpha \mathbb{Z} \longrightarrow 0.$$

We wish to apply the 6-term exact sequence in $K$-theory (Theorem 2.3.20) to the short exact sequence provided by the previous proposition. For this, we need to identify the $K$-theory of the Toeplitz extension with that of $A$ in a canonical way. Let $\kappa\colon A \to A \otimes \mathcal{K}$ be the embedding as the upper-left corner, which induces isomorphisms of the $K$-groups by parts (2) and (3) of Theorem 2.3.15. Let $\iota\colon A \to \mathcal{T}_\alpha$ denote the canonical inclusion. Then $\iota$ also induces isomorphisms on $K$-theory, although this is much more difficult to prove[2]. Next, we need to identify how $\varphi$ acts on $K$-theory, once the $K$-groups of $A \otimes \mathcal{K}$ and of $\mathcal{T}_\alpha$ are identified with those of $A$.

**Lemma 7.2.13.** Adopt the notation from the discussion above, and let $j = 0, 1$. Then the following diagram is commutative:

$$\begin{tikzcd}
K_j(A) \arrow[r, "\mathrm{id} - K_j(\alpha)"] \arrow[d, "K_j(\kappa)"'] & K_j(A) \arrow[d, "K_j(\iota)"] \\
K_j(A \otimes \mathcal{K}) \arrow[r, "K_j(\varphi)"'] & K_j(\mathcal{T}_\alpha).
\end{tikzcd}$$

Consider now the 6-term exact sequence associated to the short exact sequence in Proposition 7.2.12:

$$\begin{tikzcd}
K_0(A \otimes \mathcal{K}) \arrow[r, "K_0(\varphi)"] & K_0(\mathcal{T}_\alpha) \arrow[r, "K_0(\psi)"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "\delta_0"] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_1(\psi)"] & K_1(A). \arrow[l, "\mathrm{id} - K_1(\alpha)"]
\end{tikzcd}$$

[2]The only way I know to show this is by proving that the quasi-homomorphism $\mathcal{T}_\alpha \to A$ induced by the pair $(\mathrm{id}_{\mathcal{T}_\alpha}, \mathrm{Ad}(1_A \otimes s))\colon \mathcal{T}_\alpha \to \mathcal{T}_\alpha$ is an inverse for $\iota$ at the level of $K$-theory.

The $K$-groups of $A \otimes \mathcal{K}$ are identified with those of $A$ via $\kappa$, while the $K$-groups of $\mathcal{T}_\alpha$ are identified with those of $A$ via $\iota$. Since $\psi \circ \iota$ is the canonical inclusion of $A$ into $A \rtimes_\alpha \mathbb{Z}$, and since $K_j(\iota)^{-1} \circ K_j(\varphi) \circ K_j(\kappa) = \mathrm{id} - K_j(\alpha)$ by Lemma 7.2.13, we obtain again the Pimsner-Voiclescu exact sequence from Theorem 7.2.8, with the extra addition that the boundary maps can be described as follows:

$$\begin{tikzcd}
K_0(A) \arrow[r, "\mathrm{id}-K_0(\alpha)"] & K_0(A) \arrow[r, "K_0(\iota)"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "K_0(\kappa)^{-1}\circ\delta_0"] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "K_1(\kappa)^{-1}\circ\delta_1"] & K_1(A) \arrow[l, "K_1(\iota)"] & K_1(A). \arrow[l, "1-K_1(\alpha)"]
\end{tikzcd}$$

## 7.3 Consequences and applications

### Crossed products by the circle and cyclic groups

As a consequence of the Pimsner-Voiculescu exact sequence, in combination with Takai duality, one can derive a 6-term exact sequence involving the $K$-groups of the crossed product by a circle action. This exact sequence is not as useful as the one in Theorem 7.2.8, since each $K$-group of the crossed product appears twice and not just once, but it nevertheless gives useful information in many cases; see Exercise 7.3.2.

**Theorem 7.3.1.** Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a circle action on a $C^*$-algebra $A$. Then there is an exact sequence

$$\begin{tikzcd}
K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r, "\mathrm{id}-K_0(\widehat{\alpha})"] & K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{T}). \arrow[l, "\mathrm{id}-K_1(\widehat{\alpha})"]
\end{tikzcd}$$

The unlabeled maps can also be completely described:

1. When $K_j(A)$ is identified with $K_j(A \otimes \mathcal{K}(L^2(\mathbb{T})))$ under the canonical corner embedding, the maps $K_j(A \rtimes_\alpha \mathbb{T}) \to K_j(A)$ are induced by the canonical inclusion
$$A \rtimes_\alpha \mathbb{T} \hookrightarrow A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z} \cong A \otimes \mathcal{K}(L^2(\mathbb{T})).$$

2. When $K_j(A)$ is identified with $K_{1-j}(A \rtimes_\alpha \mathbb{R})$ using the Thom isomorphism Theorem 7.1.1, the vertical maps are induced by the canonical quotient map $A \rtimes_\alpha \mathbb{R} \to A \rtimes_\alpha \mathbb{T}$.

*Proof.* Apply Theorem 7.2.8 to $\widehat{\alpha} \in \mathrm{Aut}(A \rtimes_\alpha \mathbb{T})$, and identify $A \rtimes_\alpha \mathbb{T} \rtimes_{\widehat{\alpha}} \mathbb{Z}$ with $A \otimes \mathcal{K}(L^2(\mathbb{T}))$ using Takai duality (Theorem 5.2.7) to obtain the exact sequence in the statement, with the proper identification of all the horizontal maps. The identification of the vertical maps is carried out using the description of the vertical maps of the Pimsner-Voiclescu exact sequence given in Section 7.2, in combination with the description of the Thom isomorphism; we omit the details. □

**Exercise 7.3.2.** Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action on a (nonzero) $C^*$-algebra $A$.

1. If $A$ is AF, show that the dual automorphism $\widehat{\alpha}$ of $A \rtimes_\alpha \mathbb{T}$ is not approximately inner.

2. Find an example where $\widehat{\alpha}$ is approximately inner.

The case of finite cyclic groups, whose proof follows the lines of the Pimsner-Voiculescu theorem, is left as an exercise.

**Exercise 7.3.3.** Let $A$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $\alpha\colon \mathbb{Z}_n \to \mathrm{Aut}(A)$ be an action. Let $\pi\colon A \rtimes_\alpha \mathbb{Z} \to A \rtimes_\alpha \mathbb{Z}_n$ denote the canonical quotient map. Prove that there is an exact sequence

$$\begin{tikzcd}
K_0(A \rtimes_\alpha \mathbb{Z}_n) \arrow[r, "\mathrm{id}-K_0(\widehat{\alpha}_1)"] & K_0(A \rtimes_\alpha \mathbb{Z}_n) \arrow[r] & K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "K_1(\pi)"] \\
K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "K_0(\pi)"] & K_1(A \rtimes_\alpha \mathbb{Z}_n) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{Z}_n). \arrow[l, "\mathrm{id}-K_1(\widehat{\alpha}_1)"]
\end{tikzcd}$$

## Some $K$-theory computations

There are a number of very important applications of Theorem 7.2.8. Two historically important consequences have been the computations of the $K$-groups of rotation algebras and Cuntz algebras.

**Proposition 7.3.4.** Let $\theta \in \mathbb{R} \cap [0, 1)$. Then

$$K_0(A_\theta) \cong \mathbb{Z}^2 \quad \text{and} \quad K_1(A_\theta) \cong \mathbb{Z}^2,$$

with $K_1(A_\theta)$ generated by $[u]_1$ and $[v]_1$. Moreover, there exists a projection $p_\theta \in A_\theta$ with $\tau(p_\theta) = \theta$, and for $\theta \notin \mathbb{Q}$, the group $K_0(A_\theta)$ is generated by $[1]_0$ and $[p]_0$.

*Proof.* □

The case of Cuntz algebras is left as an exercise. The reader may assume, without proof, that Cuntz algebras are simple.

**Exercise 7.3.5.** Let $n \in \mathbb{N}$ with $n \geq 2$.

1. Let $e \in M_n$ be the projection $e_{1,1}$. For $m \in \mathbb{N}$, set $D_m = \bigotimes_{m=-k}^{\infty} M_n$, and let $\psi_m \colon D_m \to D_{m+1}$ be given by
$$\psi_m(x) = e \otimes x \in e \otimes D_m \subseteq D_{m+1}$$
for all elementary tensors $x \in D_m$. Denote by $D$ the associated direct limit, with canonical maps $\varphi_m \colon D_m \to D$, for $m \in \mathbb{N}$. Show that $D$ is isomorphic to $M_{n^\infty} \otimes \mathcal{K}$.

2. For $m \in \mathbb{N}$, let $\theta_m \colon D_m \to D_{m-1}$ be an isomorphism (for example, just by reindexing the tensor factors) and let $\alpha_m \colon D_m \to D_m$ be given by
$$\alpha_m(x) = e \otimes \theta_m(x) \in 1 \otimes D_{m-1} \subseteq D_m$$
for all elementary tensors $x \in D_m$. Show that there is an automorphism $\alpha \in \mathrm{Aut}(D)$ such
$$\alpha \circ \varphi_m = \varphi_m \circ \alpha_m$$
for all $m \in \mathbb{N}$.

3. Compute $K_0(\alpha)$ and $K_1(\alpha)$.

4. Show that $D \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_n \otimes \mathcal{K}$ as follows:

   a) Denote by $p \in D_0$ the unit of $D_0 \cong M_{n^\infty}$, and denote by $u \in M(D \rtimes_\alpha \mathbb{Z})$ the canonical unitary. Set $s = up$. Then $D_0 = pDp$ and $p(D \rtimes_\alpha \mathbb{Z})p$ is generated by $D_0$ and $s$.

   b) For $j = 1, \ldots, n$, set $s_j = (e_{j,1} \otimes p)s \in D \rtimes_\alpha \mathbb{Z}$. Then $s_j^* s_j = p$ and $\sum_{j=1}^{n} s_j s_j^* = p$.

   c) Show that $p(D \rtimes_\alpha \mathbb{Z})p \cong \mathcal{O}_n$.

   d) For $m \in \mathbb{N}$, let $p_m \in D_m$ be the unit. Then $D \rtimes_\alpha \mathbb{Z}$ is isomorphic to the inductive limit of $p_m(D \rtimes_\alpha \mathbb{Z})p_m$, and $p_{m-1}(D \rtimes_\alpha \mathbb{Z})p_{m-1}$ is generated by $p_m(A \rtimes_\alpha \mathbb{Z})p_m$ and
$$\{e_{j,k} \otimes p_m \colon 1 \leq j, k \leq n\} \subseteq M_n \otimes D_m = D_{m-1}.$$

   e) Conclude that $D \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_n \otimes \mathcal{K}$.

5. Compute $K_0(\mathcal{O}_n)$ and $K_1(\mathcal{O}_n)$. Deduce that $\mathcal{O}_n \cong \mathcal{O}_m$ if and only if $n = m$.

**Exercise 7.3.6.** Let $\alpha \colon \mathbb{Z}_n \to \mathrm{Aut}(\mathcal{O}_2)$ be an action such that $\widehat{\alpha}_1$ is approximately inner. Show that
$$K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n) \cong K_1(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n) \cong \{0\}.$$

# Part II

# Rokhlin-type properties for finite group actions

# Chapter 8

# Introduction

By the groundbreaking work of Murray and von Neumann, separably acting von Neumann factors can be divided into three types: type I factors have nonzero minimal projections, type II are those that have no minimal projections but contain a finite projection, and type III factors have only infinite projections. Type II factors are further divided into type $\mathrm{II}_1$, when there is a (normalized) finite trace, and type $\mathrm{II}_\infty$ if there is a semifinite trace. (The other types also have subdivisions, but we will not go into that here.) Since factors of type $\mathrm{II}_\infty$ are all tensor products of type $\mathrm{II}_1$-factors with $\mathcal{B}(\ell^2)$, the study of $\mathrm{II}_1$-factors is in some sense equivalent to the study of type II factors. A remarkable result of Connes asserts that for a $\mathrm{II}_1$-factor, hyperfiniteness is equivalent to injectivity, and moreover there exists a unique such $\mathrm{II}_1$-factor, usually denoted by $\mathcal{R}$. This factor has been extensively studied by a number of authors. A common "regularity" property that a factor $\mathcal{M}$ may satisfy is absorbing $\mathcal{R}$ tensorially (usually known as being *McDuff*). McDuff $\mathrm{II}_1$-factors are much better understood than general $\mathrm{II}_1$-factors. Moreover, if $\mathcal{M}$ is any factor, then $\mathcal{M}\overline{\otimes}\mathcal{R}$ is a McDuff factor (of type $\mathrm{II}_1$ if so is $\mathcal{M}$).

Once the classification of von Neumann factors was completed, the attention quickly shifted to the study of their automorphisms, and, more generally, the study of group actions on them. Automorphisms of the hyperfinite $\mathrm{II}_1$-factor $\mathcal{R}$ which have finite order (that is, actions of a finite cyclic group) were studied by Connes [13]. His work was considerably extended by Jones [48], who studied and classified finite group actions on $\mathcal{R}$. These advances culminated in the remarkable work of Ocneanu [67], who classified general amenable group actions on McDuff factors. In particular, it follows from his work that there exists a unique, up to cocycle equivalence, outer action of any given amenable group on $\mathcal{R}$. We will say more about these results in Chapter 10.

The study of the structure and classification of $C^*$-algebras developed, for quite some time, rather independently from the advances on the side of von Neumann algebras. Matui and Sato [65] were the first ones to import techniques from von Neumann algebras in a systematic way, obtaining groundbreaking results. These methods were further developed by a number of authors, and these contributions are particularly relevant in the verifications of (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) in the Toms-Winter conjecture:

**Conjecture 8.0.1.** (Toms-Winter; see, for example, [23]). Let $A$ be a unital, separable, simple, nuclear, infinite dimensional $C^*$-algebra. Then the following are equivalent:

1. $A$ has finite nuclear dimension.
2. $A$ is $\mathcal{Z}$-stable.
3. $A$ has strict comparison of positive elements.

The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) were shown to hold by Winter [104] and Rørdam [85], respectively. As of (3) $\Rightarrow$ (2), the result is known in the case that $T(A)$ is a Bauer simplex and its extreme boundary is finite dimensional, thanks to the independent works of Matui-Sato [64], Kirchberg-Rørdam [53], and Toms-White-Winter [94]. For $C^*$-algebras with stable rank one and locally finite nuclear dimension, the result was recently shown by Thiel [91]. Finally, the implication (2) $\Rightarrow$ (1) is true whenever $T(A)$ is a Bauer simplex, and this was recently shown by Bosa-Brown-Sato-Tikuisis-White-Winter [10].

Now that the Elliott programme to classify simple, nuclear $C^*$-algebras is almost completed (see Appendix A for a historical account), it is natural to shift our attention to the study of their automorphisms, and, more generally, group actions on them. By comparison, this area is considerably underdeveloped, and there were, until recently, no systematic efforts to study their structure and make attempts at their classification. Until around 10 years ago, only rather restricted classes of group actions have been studied at a time. Izumi's study and classification of finite group actions with the *Rokhlin property* [46] can be described as the first instance of a systematic study, where the actions under consideration are not described by the way in which they are constructed (namely, as direct limit actions of very special form), but rather characterized by an abstract property. Roughly speaking, for a finite group action, the Rokhlin property says that there exists a partition of unity, indexed by the elements of the group, consisting of approximately central projections which are cyclically translated by the group action (more details are given in Chapters 10 and 11). Izumi's work was extended by the author and Santiago [31] to the non-unital case, and also to actions of compact groups [32]. The structure of crossed products by actions with the Rokhlin property has also been the object of a number of works by Osaka-Phillips [69], Hirshberg-Winter [44], Pasnicu-Phillips [71] and the author [25, 28].

Actions with the Rokhlin property are rare, and many algebras o not have any. One obstruction is that the Rokhlin property, at least for finite groups, implies certain divisibility properties on $K$-theory. Attempts to circumvent obstructions of this sort led Phillips to introduce the *tracial Rokhlin property* [75], where the projections are now assumed to have a left over which is small in the tracial sense (more details are given in Chapter 12). Among other applications, the tracial Rokhlin property has been used by Echterhoff-Lück-Phillips-Walters [17] to study fixed point algebras of the irrational rotation algebra $A_\theta$ under certain finite group actions, and it was also used by Phillips to show that any simple higher-dimensional noncommutative tori is an $A\mathbb{T}$-algebra [73]. The main result used in these works is a theorem of Phillips, asserting that the crossed product of a $C^*$-algebra with tracial rank zero by a finite group action with the tracial Rokhlin property again has tracial rank zero.

Even the tracial Rokhlin property does not solve what is arguably the strongest restriction that a $C^*$-algebra can have in order to admit Rokhlin actions: the existence of projections. For example, the Jiang-Su algebra does not admit any action with the tracial Rokhlin property. The need to study weaker versions of these properties was quickly recognized, leading to two further notions. The *weak tracial Rokhlin property*, in which one replaces the projections in the definition of the tracial Rokhlin property with positive elements, has been considered (sometimes under different names) by Archey [1], Hirshberg-Orovitz [42], Sato [86], Matui-Sato [63], and Wang [98], among others. The main application of this notion has been showing that Jiang-Su absorption is preserved by taking crossed products by actions with the weak tracial Rokhlin property. We say more about this property in Sections 3 and 5.

A different approach was taken by Hirshberg-Winter-Zacharias [45], who introduced the notion of *Rokhlin dimension* for automorphisms and actions of finite groups. In this formulation, the partition of unity appearing in the Rokhlin property is replaced by a multi-tower partition of unity consisting of positive elements, each of which is indexed by the group elements and permuted by the group action (see Chapter 13 for more details). It is built into the definition that the lowest value of the Rokhlin dimension (which is zero), is equivalent to the Rokhlin property discussed above. Not requiring the existence of projections, actions with finite Rokhlin dimension are more abundant: for actions on the Jiang-Su algebra, Rokhlin dimension equal to one is in fact generic. Despite it being so seemingly common, finite Rokhlin dimension is a powerful tool to prove bounds of the nuclear dimension of crossed products. An advantage of this approach is that the definition of Rokhlin dimension does not require the $C^*$-algebra to be simple; in particular, the theory can be applied to actions on compact Hausdorff spaces. The works of Hirshberg-Winter-Zacharias for $\mathbb{Z}$-actions, and of Szabo [89] for $\mathbb{Z}^d$-actions, illustrate this fact nicely. Rokhlin dimension has been defined for actions of much more general groups: for residually finite groups by Szabo-Wu-Zacharias [90], for compact groups by the author [27] and [26], and further by the author, Hirshberg and Santiago [30], and for the reals by Hirshberg-Szabo-Winter-Wu [43].

With all these seemingly different Rokhlin-type properties, a natural question arises: when does one of these properties imply another one? Except for the obvious implications, it is not clear what the relationship between them is. This is explored in Chapter 14, where we show that for a large class of simple $C^*$-algebras, the weak tracial Rokhlin property and having Rokhlin dimension at most one are equivalent. The goal of this series of lectures is to familiarize the audience with all these Rokhlin-type properties, as well as giving a sample of the techniques that are used to work with each of them.

Throughout, we will work mostly with separable, unital $C^*$-algebras and finite groups. Removing the unitality and separability assumptions assumption is, for the most part, not difficult, and we omit this issue completely here. (The results in Chapter 14 have really only been proved for separable, unital algebras.) Moving away from finite groups involves more complications. Some results hold in general for compact groups, while others hold for discrete amenable groups, and those concerning Rokhlin dimension require the group to be moreover residually finite. While definitions and proofs will be given for finite groups mostly, we will mention, when appropriate, what generalizations have been obtained in the literature.

# Chapter 9

# Preliminaries

## 9.1 Strongly self-absorbing $C^*$-algebras.

A bit strange to have it before McDuff's result...

# Chapter 10

# Classification of outer actions on the hyperfinite $\mathrm{II}_1$-factor

In [48], Vaughan Jones gave a complete classification, up to conjugacy, of all actions of a finite group on the hyperfinite $\mathrm{II}_1$-factor $\mathcal{R}$, using invariants that are essentially algebraic. His result does not generalize to arbitrary $\mathrm{II}_1$-factors, although there is a version for approximately inner actions on McDuff factors. Jones' classification theorem generalizes previous results of Connes [13] for actions of $\mathbb{Z}_n$, and was subsequently extended by Ocneanu, who obtained a similar classification for amenable group actions.

Since the invariants used to classify actions on $\mathcal{R}$ vanish in the case of outer actions, it follows that there is a unique outer action of any finite (or of any amenable) group on $\mathcal{R}$. This particular case is in fact an important step in the proof of the general theorem, arguably the most difficult one, and in this chapter we will outline the argument to obtain this uniqueness result. The proof consists in showing that outer actions on $\mathcal{R}$ have the so-called *Rokhlin property*, and then proving that two actions with the Rokhlin property are conjugate.

The material contained in this chapter inspired many $C^*$-algebraists to study actions on $C^*$-algebras, trying to obtain results similar to those of Connes, Jones and Ocneanu. The rest of these notes cover some of the latest developments in the study of $C^*$-dynamical systems, particularly in what concerns the $C^*$-algebraic versions of Jones' Rokhlin property for finite group actions on von Neumann algebras. This chapter therefore provides a historical perspective on one of the first uses of the Rokhlin property, specifically in what refers to classification of finite group actions on $\mathcal{R}$. Since these lecture notes are devoted to the study of actions on $C^*$-algebras rather than von Neumann algebras, some of the results in this chapter will be only sketched and many facts about the structure theory of $\mathrm{II}_1$-factors, presented in the first section, will be used without proof.

## 10.1 Preliminaries on von Neumann algebras and factors

A von Neumann algebra is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology. Von Neumann algebras have been abstractly characterized by Sakai as those $C^*$-algebras that admit an isometric predual. Despite the fact that von Neumann algebras are $C^*$-algebras in their own right, it is not usually helpful to think of von Neumann algebras as $C^*$-algebras. It may be illustrative to mention that, while $C^*$-algebras are regarded as noncommutative toplogical spaces, von Neumann algebras are usually regarded as noncommutative *measure spaces*. Behind this philosophy is the fact that if $M$ is an abelian von Neumann algebra, then there exists a measure space $(X, \mu)$ such that $M$ is isomorphic to $L^\infty(X, \mu)$[1].

The foundations for the advancement of the theory of von Neumann algebras were laid by Murray and von Neumann in their groundbreaking works in the early 1940's. Among other fundamental results, they showed that any von Neumann algebra decomposes as a direct integral (a generalization of a direct sum) of von Neumann algebras with trivial center (also called *factors*). Since many problems about von Neumann algebras can be reduced to the case of a factor, it is important to understand the structure of the latter. Factors can be classified into three types, with corresponding subtypes:

- **Type I**: there is a nonzero minimal projection.
  - **Type $\mathbf{I}_n$**, for $n \in \mathbb{N}$: the unit can be written as the sum of $n$ minimal projections.
  - **Type $\mathbf{I}_\infty$**: otherwise.
- **Type II**: there are no minimal projections and there is a finite projection;
  - **Type $\mathbf{II}_1$**: the unit is a finite projection.
  - **Type $\mathbf{II}_\infty$**: the unit is an infinite projection.
- **Type III**: all nonzero projections are infinite;
  - **Type $\mathbf{III}_\lambda$**, for $0 \leq \lambda \leq 1$ depending on the Connes spectrum.

Factors of type I can be completely described: $M_n$ is the unique factor of type $\mathrm{I}_n$, while every factor of type $\mathrm{I}_\infty$ has the form $\mathcal{B}(\mathcal{H})$ for some infinite-dimensional Hilbert space $\mathcal{H}$. Moreover, every factor of type $\mathrm{II}_\infty$ is a tensor product[2] of a factor of type $\mathrm{II}_1$ and $\mathcal{B}(\mathcal{H})$. Further, every factor of type III can be written as the crossed product of a factor of type II with an $\mathbb{R}$-action. In other words, the study of von Neumann factors in some sense can be reduced to the study of $\mathrm{II}_1$-factors.

[1] There is also a topological space $Y$ such that $M$ is isomorphic to $C(Y)$, but this space is from many points of view a very pathological one, and it is not useful when analyzing the structure of $M$.

[2] Tensor products of von Neumann algebras are defined spatially, similarly to how minimal tensor products of $C^*$-algebras are constructed. The von Neumann algebraic tensor product is usually denoted by $\overline{\otimes}$.

### $\mathrm{II}_1$-factors.

There are a number of constructions that give rise to factors of type $\mathrm{II}_1$. For this, we briefly discuss crossed products of von Neumann algebras.

Crossed products of actions on von Neumann algebras are defined analogously as the reduced crossed products of $C^*$-algebras, except that one needs to construct a universal regular covariant representation and construct the crossed product spatially.[3] As in the $C^*$-algebra case, if the group is discrete, then the crossed product can be defined more directly as follows. Take any faithful representation of the von Neumann algebra, consider the associated regular covariant representation, and take its integrated form. Then the crossed product is isomorphic to the von Neumann algebra generated by its image. If $M$ is a von Neumann algebra and a group $G$ acts on it, we denote the associated crossed product by $M\overline{\rtimes}G$ to distinguish it from the $C^*$-algebraic crossed product. Note, however, that $M\overline{\rtimes}_\alpha G$ and $M \rtimes_\alpha G$ coincide when $G$ is finite.

Crossed products can be used to construct a number of examples of $\mathrm{II}_1$-factors.

**Example 10.1.1.** Let $G$ be a discrete group, let $(X, \mu)$ be a probability space, and let $G$ act on $(X, \mu)$ in a measure-preserving way. Then $L^\infty(X) \rtimes G$ has a trace, which is induced by $\mu$. If the action is free and ergodic, and $\mu$ has full support, then $L^\infty(X) \rtimes G$ is a $\mathrm{II}_1$-factor.

**Example 10.1.2.** Let $G$ be a discrete group. Then its group von Neumann algebra $L(G) = \mathbb{C}\rtimes G$, which can be alternatively be defined as the weak closure of $C^*_\lambda(G)$ in $\mathcal{B}(\ell^2(G))$, has a trace. Moreover, $L(G)$ is a factor (necessarily of type $\mathrm{II}_1$) if and only if $G$ has *infinite conjugacy classes* (ICC).

Perhaps the most important example of a $\mathrm{II}_1$-factor is the one constructed in the following[4]:

**Example 10.1.3.** The $\mathrm{II}_1$-factor $\mathcal{R}$ is defined as the weak closure of the UHF-algebra $M_{2^\infty}$ in the GNS representation associated to its unique trace. It is a factor because $\tau$ is an extreme state, and it is of type $\mathrm{II}_1$ because by definition the trace on $M_{2^\infty}$ extends to a trace on $\mathcal{R}$.

Factors of type $\mathrm{II}_1$ enjoy two fundamental properties that are used repeatedly: the existence of a unique trace, and the fact that the order of its projections is determined by the values of this trace. Recall that if $p$ and $q$ are projections in a $C^*$-algebra $A$, we write $p \preceq q$ if there exists a projection $q' \leq q$ such that $p \sim_{\mathrm{M-vN}} q'$.

**Theorem 10.1.4.** Let $M$ be a $\mathrm{II}_1$-factor.

[3] There is no obvious or useful analog of the full crossed product, so this distinction is not incorporated in the terminology or notation.

[4] It also arises in the constructions from Example 10.1.1 and Example 10.1.2 whenever $G$ is amenable.

1. There exists a unique weakly continuous trace $\tau\colon M \to \mathbb{C}$, which is necessarily faithful.
2. Comparison: projections $p, q \in M$, one has $p \preceq q$ if and only if $\tau(p) \leq \tau(q)$. Moreover, $p \sim_{\mathrm{u}} q$ if and only if $\tau(p) = \tau(q)$.
3. For every $t \in \mathbb{R}$ there exists a projection $p \in M$ with $\tau(p) = t$.
4. If $p$ is a projection in $M$, then $pMp$ is also a $\mathrm{II}_1$-factor.

A useful consequence of this fact is given in the following exercise:

**Exercise 10.1.5.** Let $n \in \mathbb{N}$.

1. Let $A$ be a $C^*$-algebra, and let $p_1, \ldots, p_n$ be projections in $A$ that are Murray-von Neumann equivalent. Show that $A$ contains a subalgebra isomorphic to $M_n$.
2. If $M$ is a $\mathrm{II}_1$-factor and $n \in \mathbb{N}$, then $M$ contains a subalgebra isomorphic to $M_n$.

### Hyperfiniteness

A key notion in the study of factors is that of *hyperfiniteness*, which we define next.

**Definition 10.1.6.** A von Neumann algebra is said to be *hyperfinite* if it contains an increasing net of finite dimensional subalgebras whose union is dense in the weak operator topology.

**Example 10.1.7.** The $\mathrm{II}_1$-factor $\mathcal{R}$ from Example 10.1.3 is hyperfinite by construction.

A groundbreaking result of Connes [12] asserts that for a separably acting von Neumann factor, hyperfiniteness is equivalent to *injectivity*, and also equivalent to *amenability*.

Major breakthroughs by Murray and von Neumann, Connes, Haagerup, Krieger, and Popa culminated in the classification of hyperfinite factors:

**Theorem 10.1.8.** There is a unique hyperfinite factor of type $\mathrm{I}_n$, for $n \in \mathbb{N}$, $\mathrm{I}_\infty$, $\mathrm{II}_1$, $\mathrm{II}_\infty$, and $\mathrm{III}_\lambda$, for $0 < \lambda \leq 1$. On the other hand, the hyperfinite factors of type $\mathrm{III}_0$ correspond to certain ergodic flows.

It may be interesting to mention that the study of group actions on von Neumann factors was instrumental in obtaining the results above. Concretely, the classification of outer automorphisms on the hyperfinite $\mathrm{II}_1$-factor $\mathcal{R}$ was key in Connes' award-winning proof of the uniqueness of hyperfinite factors of type $\mathrm{III}_\lambda$, for $0 < \lambda < 1$. Indeed, he showed that if $M$ is a hyperfinite $\mathrm{III}_\lambda$-factor, then there is an outer automorphism $\theta$ of $\mathcal{R}\overline{\otimes}\mathcal{B}(\ell^2)$ which scales the trace by $\lambda$, and such that $M$ is isomorphic to the crossed product by $\theta$. In order to show that any two hyperfinite $\mathrm{III}_\lambda$-factors are isomorphic, it therefore suffices to show that any two outer automorphisms of $\mathcal{R}\overline{\otimes}\mathcal{B}(\ell^2)$ which scale the trace by $\lambda$, are cocycle conjugate[5]. In this way, classification of actions on von Neumann algebras arose as an area with fundamental applications to the structure of factors.

Here, we give a brief proof of the uniqueness of $\mathcal{R}$. In its proof, we will need the trace (semi-)norm $\|\cdots\|_2$ associated to a trace $\tau$ in a von Neumann algebra $M$, which is defined by $\|a\|_2 = \tau(a^*a)^{1/2}$ for $a \in M$. It is an easy consequence of Kaplansky's density theorem that, for a $\mathrm{II}_1$-factor $M$, the topology on the norm-unit ball of $M$ generated by $\|\cdot\|_2$ agrees with the weak topology.

**Theorem 10.1.9.** The $\mathrm{II}_1$-factor $\mathcal{R}$ from Example 10.1.3 is the unique separable hyperfinite $\mathrm{II}_1$-factor.

*Proof.* Let $M$ be a hyperfinite $\mathrm{II}_1$-factor with trace $\tau_M$.

**Claim 1:** *given $\varepsilon > 0$ and a finite-dimensional subalgebra $N \subseteq M$, there exist $n \in \mathbb{N}$ and an embedding $M_{2^n} \to M$ such that $\mathrm{d}_{\|\cdot\|_2}(M_{2^n}, x) \leq \varepsilon\|x\|$ for all $x \in N$.* To simplify the argument, we will assume that $N$ is a matrix subalgebra of $M$, say $N \cong M_r$ for some $r \geq 2$. Let $\{\widetilde{e}_{j,k} \colon 1 \leq j, k \leq r\}$ be a system of matrix units in $M$ generating $N$. Find a rational of the form $m/2^n$ satisfying $\left|\tau(\widetilde{e}_{1,1}) - \frac{m}{2^n}\right| < \frac{\varepsilon}{r^2}$, and use Exercise 10.1.5 to find a projection $p \leq \widetilde{e}^{(r)}_{1,1}$ with trace $m/2^n$. For $j, k = 1, \ldots, r$, set $e^{(r)}_{j,k} = \widetilde{e}^{(r)}_{j,1} p \widetilde{e}^{(r)}_{1,k} \in M$. Then $\{e^{(r)}_{j,k} \colon 1 \leq j, k \leq r\}$ is a system of matrix units in $M$ (it may not be unital), and the algebra $P$ that they generate satisfies $\mathrm{d}_{\|\cdot\|_2}(P, x) \leq \varepsilon\|x\|$ for all $x \in N$.

Set $q = 1 - 1_P$. Then both $p$ and $q$ have dyadic rational traces, so there exist $s, t \in \mathbb{N}$ and $n \in \mathbb{N}$ with $\tau(p) = s/2^n$ and $\tau(q) = t/2^n$. Use Theorem 10.1.4 to find unital copies

$$M_s \hookrightarrow pMp \quad \text{and} \quad M_t \hookrightarrow qMq,$$

and choose the corresponding matrix units $\{f^{(s)}_{j,k} \colon 1 \leq j, k \leq s\}$ and $\{f^{(t)}_{j,k} \colon 1 \leq j, k \leq t\}$ in $M$. Now, the projections

$$f^{(s)}_{1,1}, \ldots, f^{(s)}_{s,s}, e^{(r)}_{2,1} f^{(s)}_{1,1} e^{(r)}_{1,2}, \ldots, e^{(r)}_{2,1} f^{(s)}_{s,s} e^{(r)}_{1,2}, \ldots, e^{(r)}_{r,1} f^{(s)}_{1,1} e^{(r)}_{1,r}, \ldots, e^{(r)}_{r,1} f^{(s)}_{s,s} e^{(r)}_{1,r},$$
$$f^{(t)}_{1,1}, \ldots, f^{(t)}_{t,t}$$

are orthogonal, add up to the unit of $M$, and have all the same trace, which equals $\frac{1}{sr+t} = \frac{1}{2^n}$. The algebra they generate, which is isomorphic to $M_{2^n}$, contains $P$, and hence satisfies the conditions in the claim.

**Claim 2:** *$M$ contains $M_{2^\infty}$ as a weakly-dense subalgebra.*

**Claim 3:** *$M$ is isomorphic to $\mathcal{R}$.* Consider the GNS-representation of $M$ associated to $\tau_M$. By the previous claim, $M$ contains $M_{2^\infty}$ as a weakly-dense subalgebra. Since the restriction of $\tau_M$ to $M_{2^\infty}$ is the unique trace of $M_{2^\infty}$,

[5] Two automorphisms $\varphi$ and $\psi$ are cocycle conjugate if and only if there are a unitary $u$ and an automorphism $\theta$ such that $\theta \circ \mathrm{Ad}(u) \circ \varphi \circ \theta^{-1} = \psi$. Once $\varphi$ and $\psi$ are regarded as $\mathbb{Z}$-actions, this means that there is a cocycle perturbation of one of them (in the sense of Definition 10.2.2) which is conjugate to the other one.

this representation restricts to the GNS representation of $M_{2^\infty}$, and hence the von Neumann algebra generated by its image, which is isomorphic to $M$ by weak density, must agree with $\mathcal{R}$.

Finish the proof? □

### Central sequences and McDuff's theorem

Sequences algebras and relative commutants (also called central sequence algebras) are fundamental tools not only in the study of the structure of operator algebras, but also in the study and classification of group actions on them.

The first use of central sequence algebras can be traced to the work of McDuff [66], who characterized those $\mathrm{II}_1$-factors that tensorially absorption of $\mathcal{R}$ in terms of the existence of a unital embedding of $\mathcal{R}$ into the central sequence algebra of the factor. Later on, Jones used central sequence algebras in order to classify outer actions of finite groups on $\mathcal{R}$.

In this subsection, we define the central sequence algebra of a $\mathrm{II}_1$-factor. We begin with a discussion about (free) ultrafilters on $\mathbb{N}$.

**Definition 10.1.10.** A *ultrafilter* over $\mathbb{N}$ is a set $\omega$ of subsets of $\mathbb{N}$ satisfying

1. $\emptyset \neq \omega$;
2. If $S, T \in \omega$, then $S \cap T \in \omega$;
3. For every $S \subseteq \mathbb{N}$, either $S \in \omega$ or $\mathbb{N} \setminus S \in \omega$.

An ultrafilter gives a notion of largeness in $\mathbb{N}$, where one regards a subset of $\mathbb{N}$ to be large if it belongs to the ultrafilter. Limits along ultrafilters are defined naturally, as follows:

**Definition 10.1.11.** Let $X$ be a topological space, let $(x_n)_{n\in\mathbb{N}}$ be sequence in $X$, let $x \in X$, and let $\omega$ be an ultrafilter over $\mathbb{N}$. We say that $(x_n)_{n\in\mathbb{N}}$ *converges to $x$ along* $\omega$ if for every open set $U \subseteq X$ containing $x$, the set $\{n \in \mathbb{N} \colon x_n \in U\}$ belongs to $\omega$. In this case, we write $\lim\limits_{n\to\omega} x_n = x$.

A remarkable properties of ultrafilters is that bounded sequences always converge along an ultrafilter.

**Lemma 10.1.12.** Let $X$ be a compact Hausdorff space, let $(x_n)_{n\in\mathbb{N}}$ be sequence in $X$, and let $\omega$ be an ultrafilter over $\mathbb{N}$. Then there exists a unique $x \in X$ such that $(x_n)_{n\in\mathbb{N}}$ converges to $x$ along $\omega$.

*Proof.* Uniqueness of $x$ is clear since $X$ is Hausdorff. For $S \in \omega$, set

$$X_S = \overline{\{x_n \colon n \in S\}} \subseteq X,$$

which is closed in $X$. Note that the family $\{X_S \colon S \in \omega\}$ has the finite intersection property, by condition (2) in Definition 10.1.11. By compactness of $X$, the intersection of this family is nonempty, so let $x$ be an element in $\bigcap\limits_{S\in\omega} X_S$.

We will show that $(x_n)_{n\in\mathbb{N}}$ converges to $x$ along $\omega$. Let $U \subseteq X$ be an open subset containing $x$. Arguing by contradiction, assume that $\{n \in \mathbb{N} \colon x_n \in U\}$ does not belong to $\omega$. Hence there exists $S \in \omega$ such that $X_S \subseteq X \setminus U$. However, this contradicts the fact that $x$ belongs to $X_S$, thus proving the claim and the lemma. □

We can now define the (central) sequence algebra of a $\mathrm{II}_1$-factor.

**Definition 10.1.13.** Let $M$ be a $\mathrm{II}_1$-factor, and let $\omega$ be an ultrafilter over $\mathbb{N}$. We denote by $\ell^\infty(\mathbb{N}, M)$ the von Neumann algebra of all bounded sequences with values on $M$ (endowed with the supremum norm)[6]. Set

$$J_\omega = \{x \in \ell^\infty(\mathbb{N}, M) \colon \lim_{n\to\omega} \|x_n\|_2 = 0\},$$

which is a weakly-closed two sided ideal in $\ell^\infty(\mathbb{N}, M)$. The quotient is denote by $M^\omega$, and called the *sequence algebra of* $M$. We denote the canonical quotient map by $\kappa_M^\omega \colon \ell^\infty(\mathbb{N}, M) \to M^\omega$, or just $\kappa$ if no confusion is likely to arise.

The sequence algebra $M^\omega$ is endowed with the canonical trace $\tau_\omega \colon M^\omega \to \mathbb{C}$ given by $\tau_\omega(\kappa(x)) = \lim_{n\to\omega} \tau(x_n)$ for all $x \in \ell^\infty(\mathbb{N}, M)$.

The factor $M$ can be identified with the subalgebra of $\ell^\infty(\mathbb{N}, M)$ consisting of the constant sequences, and with a subalgebra of $M^\omega$ via $\kappa$. We define the *central sequence algebra of* $M$ to be the relative commutant $M^\omega \cap M'$.

Note that by definition, every representing sequence $(a_n)_{n\in\mathbb{N}}$ in $M$ of an element $a \in M^\omega \cap M'$ satisfies $\lim_{n\to\omega} \|a_n x - x a_n\|_2 = 0$ for all $x \in M$.

The construction of $M^\omega \cap M'$ is sufficiently functorial that any automorphism $\varphi$ of $M$ induces an automorphism $\varphi^\omega$ of $M^\omega \cap M'$.

Definition of free ultrafilter.

A special feature of the hyperfinite $\mathrm{II}_1$-factor $\mathcal{R}$, is that its central sequence algebra (associated to a free ultrafilter) is again a $\mathrm{II}_1$-factor, and that an automorphism of $\mathcal{R}$ is outer if and only if it induces an outer automorphism of the central sequence algebra.

**Theorem 10.1.14.** Let $\omega$ be a free ultrafilter. Then $\mathcal{R}^\omega \cap \mathcal{R}'$ is a $\mathrm{II}_1$-factor. Moreover, an automorphism $\varphi$ of $\mathcal{R}$ is outer if and only if $\varphi^\omega \in \mathrm{Aut}(\mathcal{R}^\omega \cap \mathcal{R}')$ is outer.

We close this subsection with McDuff's characterization of absorption of $\mathcal{R}$.

**Theorem 10.1.15.** Let $M$ be a $\mathrm{II}_1$-factor, and let $\omega$ be a free ultrafilter. Then the following are equivalent

1. There is an isomorphism $M \cong M \overline{\otimes} \mathcal{R}$;
2. There is a unital embedding $\mathcal{R} \to M^\omega \cap M'$;
3. For some (any) $n \geq 2$, there is a unital embedding $M_n \to M^\omega \cap M'$.

[6] If $M$ is concretely represented on the Hilbert space $\mathcal{H}$, then the algebra $\ell^\infty(\mathbb{N}, M)$ can be represented concretely on $\ell^2(\mathbb{N}) \otimes \mathcal{H}$ in a canonical way.

## 10.2 Outer actions on $\mathrm{II}_1$-factors

Recall that an automorphism $\varphi$ of a $C^*$-algebra $A$ is said to be *outer* if there does not exist a unitary $u \in \mathcal{M}(A)$ with $\varphi = \mathrm{Ad}(u)$. Moreover, an action $\alpha\colon G \to \mathrm{Aut}(M)$ is said to be *outer* if $\alpha_g$ is outer for all $g \in G \setminus \{1\}$.

**Proposition 10.2.1.** Let $G$ be a discrete group, let $M$ be a factor, and let $\alpha\colon G \to \mathrm{Aut}(M)$ be an outer action. Then $M\overline{\rtimes}_\alpha G$ is a factor. If $M$ is of type $\mathrm{II}_1$, then so is $M\overline{\rtimes}_\alpha G$. Finally, if $G$ is finite, then the same conclusions apply to the fixed point algebra $M^G$.

*Proof.* Set $N = M\overline{\rtimes}_\alpha G$. We begin by showing that $N$ is a factor. Let $x \in N$ be a central contraction. We will show that $x$ belongs to $M$, in which case it belongs to the center of $M$ and hence it is a scalar. Denote by $E\colon N \to M$ the canonical conditional expectation. We argue by contradiction, so we assume that there exists $g \in G$ such that $E(xu_g)$ is not zero. (Such a group element exists by faithfulness of $E$, which can be proved analogously to Theorem 4.3.4.) We claim that $\alpha_g$ is inner.

Given $a \in M$, we have

$$E(xu_g)a = E(xu_ga) = E(x\alpha_g(a)u_g) = E(\alpha_g(a)xu_g) = \alpha_g(a)E(xu_g).$$

Now set $z = E(xu_g)$, which is an element in $M$ satisfying $za = \alpha_g(a)z$ for all $a \in M$. By taking adjoints, we also get $az^* = z^*\alpha_g(a)$ for all $a \in M$. Hence

$$z^*za = z^*\alpha_g(a)z = az^*z,$$

and thus $z^*z$ belongs to the center of $M$. Take the polar decomposition $z = v|z|$ of $z$ in $M$, where $v$ is a unitary in $M$ and $|z| = (z^*z)^{1/2}$. Then $|z|$ is a (nonzero) scalar, and thus the identity $za = \alpha_g(a)z$ gives $va = \alpha_g(a)v$ for all $a \in M$. Hence $v \in M$ implements $\alpha_g$, as desired. This is a contradiction, which shows that $N$ is a factor.

Assume that $M$ is of type $\mathrm{II}_1$. Then the factor $N = M\overline{\rtimes}_\alpha G$ cannot be of type $I$ since it contains the $\mathrm{II}_1$-factor $M$, and it admits a (normalized) trace given by $\tau \circ E$. Hence $M$ is of type $\mathrm{II}_1$.

When $G$ is finite, one can show as in Theorem 6.1.5 that the element $p = \frac{1}{|G|}\sum\limits_{g\in G} u_g$ is a projection in $N$ and that $pNp$ is isomorphic to $M^\alpha$. Since a corner of a factor (of type $\mathrm{II}_1$) is again a factor (of type $\mathrm{II}_1$), the result follows. $\square$

We will need the notion of a 1-cocycle for a group action.

**Definition 10.2.2.** Let $G$ be a discrete group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. A function $w\colon G \to \mathcal{U}(A)$ is said to be an *$\alpha$-cocycle* if

$$w_{gh} = w_g\alpha_g(w_h)$$

for all $g,h \in G$. We moreover say that $w$ is a *coboundary* if there exists $v \in \mathcal{U}(A)$ such that $w_g = v\alpha_g(v^*)$.

Given an $\alpha$-cocycle $w$, we denote by $\alpha^w \colon G \to \mathrm{Aut}(A)$ the action given by $\alpha^w_g = \mathrm{Ad}(w_g) \circ \alpha_g$ for all $g \in G$.

For finite group actions, deciding whether a cocycle is a coboundary or not amounts to comparing two projections in the crossed product.

**Proposition 10.2.3.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action, and let $w \colon G \to \mathcal{U}(A)$ be an $\alpha$-cocycle. For $g \in G$, let $u_g \in A \rtimes_\alpha G$ denote the canonical unitary implementing $\alpha_g$. Then

$$p = \frac{1}{|G|}\sum_{g\in G} w_g u_g \quad \text{and} \quad q = \frac{1}{|G|}\sum_{g\in G} u_g$$

are projections, and $w$ is a coboundary if and only if $p \sim_{\mathrm{M-vN}} q$.

Since the proof is straightforward, we leave it as an exercise, together with other claims made in Definition 10.2.2.

**Exercise 10.2.4.** Let $G$ be a discrete group, let $A$ be a unital $C^*$-algebra, let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action, and let $w \colon G \to \mathcal{U}(A)$ be an $\alpha$-cocycle.

1. Show that the map $\alpha^w \colon G \to \mathrm{Aut}(A)$ given by $\alpha^w_g = \mathrm{Ad}(w_g) \circ \alpha_g$ for all $g \in G$, is an action.
2. Show that $A \rtimes_\alpha G$ and $A \rtimes_{\alpha^w} G$ are canonically isomorphic.

Suppose now that $G$ is finite. For $g \in G$, let $u_g \in A \rtimes_\alpha G$ denote the canonical unitary implementing $\alpha_g$. Set

$$p = \frac{1}{|G|}\sum_{g\in G} w_g u_g \quad \text{and} \quad q = \frac{1}{|G|}\sum_{g\in G} u_g.$$

3. Show that $p$ and $q$ are projections.
4. Show that $w$ is a coboundary if and only if $p \sim_{\mathrm{M-vN}} q$.

**Theorem 10.2.5.** Let $M$ be a $\mathrm{II}_1$-factor, let $G$ be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(M)$ be an outer action. Then every $\alpha$-cocycle is a coboundary.

*Proof.* Let $w \colon G \to \mathcal{U}(M)$ be an $\alpha$-cocycle, and let $p, q \in M$ be as in Proposition 10.2.3. Denote by $\tau_M$ the unique trace of $M$. By Proposition 10.2.1, the crossed product $M \rtimes_\alpha G$ is a $\mathrm{II}_1$-factor with trace given by $\tau(\sum_{g\in G} a_g u_g) = \tau_M(a_1)$ for all $\sum_{g\in G} a_g u_g \in M \rtimes_\alpha G$. Since $\mathrm{II}_1$-factors have comparison by Theorem 10.1.4, it suffices to show that $\tau(p) = \tau(q)$. It is clear that $\tau(p) = w_1$ and $\tau(q) = 1$, so we shall only prove that $w_1 = 1$. However, this follows from the following identity

$$w_1 = w_{1^2} = w_1\alpha_1(w_1) = w_1 w_1.$$

$\square$

## 10.3 Classification of outer actions on $\mathcal{R}$

In this section, we will sketch Jones' proof that there exists a unique outer action of any finite group on $\mathcal{R}$. Roughly speaking, the argument has two main steps:

- Any outer action on $\mathcal{R}$ has the Rokhlin property.
- Two actions on $\mathcal{R}$ with the Rokhlin property are conjugate.

We will concentrate mostly on the first part, for two reasons. First, because this part is really very special to the hyperfinite $\mathrm{II}_1$-factor, and nothing like this is true in the $C^*$-algebraic context. And second, because the second part of the argument can be proved using techniques similar to those we will present in chapter 11, and there the von Neumann algebraic techniques are not as crucial.

We therefore begin by defining the Rokhlin property for finite group actions on $\mathrm{II}_1$-factors (which we will call the $W^*$-Rokhlin property, to distinguish it from the property we will study in chapter 11).

**Definition 10.3.1.** Let $G$ be a finite group, let $M$ be a $\mathrm{II}_1$-factor, and let $\alpha\colon G \to \mathrm{Aut}(M)$ be an action. We say that $\alpha$ has the *$W^*$-Rokhlin property* if for every finite subset $F \subseteq M$ and every $\varepsilon > 0$, there exist mutually orthogonal projections $p_g \in M$, for $g \in G$, satisfying

1. $\|\alpha_g(p_h) - p_{gh}\|_2 < \varepsilon$ for all $g, h \in G$;
2. $\sum\limits_{g\in G} p_g = 1$;
3. $\|p_g a - a p_g\|_2 < \varepsilon$ for all $g \in G$ and all $a \in F$.

A family of (necessarily orthogonal) projections that add up to the unit is also referred to as a *partition of unity*.

**Remark 10.3.2.** Let $\omega$ be a free ultrafilter. When $M$ is separable, an action $\alpha\colon G \to \mathrm{Aut}(M)$ as above has the $W^*$-Rokhlin property if and only if there exists a unital equivariant embedding

$$(L^\infty(G), \mathtt{Lt}) \to (M^\omega \cap M', \alpha^\omega).$$

The following is the canonical example of a Rokhlin action.

**Example 10.3.3.** Let $G$ be a nontrivial finite group. Consider the conjugation action $\mathrm{Ad}(\lambda)\colon G \to \mathrm{Aut}(\mathcal{B}(\ell^2(G)))$. There is an infinite tensor product action $\delta = \bigotimes\limits_{n\in\mathbb{N}} \mathrm{Ad}(\lambda)$ of $G$ on the UHF-algebra of type $M_{|G|^\infty}$. Since the unique trace of $M_{|G|^\infty}$ is invariant under $\delta$, it follows that there is a well-defined action $\mu\colon G \to \mathrm{Aut}(\mathcal{R})$ which extends $\delta$. We claim that this action has the $W^*$-Rokhlin property. Given a finite subset $F \subseteq \mathcal{R}$ and $\varepsilon > 0$, there is a unital equivariant embedding $\varphi\colon (\mathcal{B}(\ell^2(G)), \mathrm{Ad}(\lambda)) \to (\mathcal{R}, \mu)$ that satisfies $\|\varphi(x)a -$

$a\varphi(x)\|_2 < \varepsilon\|x\|$ for all $x \in \mathcal{B}(\ell^2(G))$ and all $a \in F$[7]. If $\{e_{g,h} : g, h \in G\}$ denotes the matrix units of $\mathcal{B}(\ell^2(G))$, we set $p_g = e_{g,g}$ for all $g \in G$. It is then easy to verify that these projections satisfy conditions 1, 2 and 3 in Definition 10.3.1.

Let $G$ be a finite group and let $X$ be a finite $G$-space. We endow $\mathcal{B}(\ell^2(X))$ with the $G$-action $\gamma\colon G \to \mathrm{Aut}(\mathcal{B}(\ell^2(X)))$ given by $\gamma_g(e_{x,y}) = e_{g\cdot x, g\cdot y}$ for all $x, y \in X$ and all $g \in G$. When $X = G$ with the translation action, we obtain the action of conjugation by the left regular representation.

**Lemma 10.3.4.** Let $M$ be a $\mathrm{II}_1$-factor, let $G$ be a finite group, and let $X$ be a finite $G$-space, and let $\alpha\colon G \to \mathrm{Aut}(M)$ be an outer action. Then there exists a unital, $G$-equivariant embedding

$$(\mathcal{B}(\ell^2(X)), \gamma) \to (M, \alpha).$$

*Proof.* Since $M^G$ is a $\mathrm{II}_1$-factor by Proposition 10.2.1, we use Exercise 10.1.5 to find matrix units $f_{x,y} \in M^G$, for $x, y \in X$. For $g \in G$, set $w_g = \sum_{x\in X} f_{g\cdot x, x}$, which is a unitary in $M^G$. For $g, h \in G$, we have

$$w_g w_h = \sum_{x,y\in X} f_{g\cdot x,x} f_{h\cdot y,y} = \sum_{x\in X} f_{g\cdot x, h^{-1}x} = \sum_{x\in X} f_{(gh)\cdot x, x},$$

as well as $w_g^* = w_{g^{-1}}$. It follows that $w$ is a unitary representation of $G$ with values in $M^G$, and hence it is an $\alpha$-cocycle when regarded as a map $w\colon G \to \mathcal{U}(M)$. By Theorem 10.2.5, there exists $v \in \mathcal{U}(M)$ such that $w_g = v^*\alpha_g(v)$ for all $g \in G$. For $x, y \in X$, set $e_{x,y} = v f_{x,y} v^* \in M$. Clearly these are matrix units generating a copy of $\mathcal{B}(\ell^2(X))$. Moreover,

$$\alpha_g(e_{x,y}) = \alpha_g(v) f_{g,h} \alpha_g(v^*) = v w_g f_{x,y} w_g^* v^* = v f_{g\cdot x, g\cdot y} v^* = e_{g\cdot x, g\cdot y},$$

for all $g \in G$ and all $x, y \in X$. We conclude that the induced unital embedding $\mathcal{B}(\ell^2(X)) \to M$ is equivariant. ☐

We can now prove that outer actions on $\mathcal{R}$ have the $W^*$-Rokhlin property.

**Theorem 10.3.5.** Let $G$ be a finite group, let $\alpha\colon G \to \mathrm{Aut}(M)$ be an action, and let $\omega$ be a free ultrafilter over $\mathbb{N}$. Then the following are equivalent:

1. $\alpha$ is outer;
2. $\alpha$ has the *$W^*$-Rokhlin property*;
3. There is a unital equivariant embedding

$$\psi\colon (\mathcal{B}(\ell^2(G)), \mathrm{Ad}(\lambda)) \to (\mathcal{R}^\omega \cap \mathcal{R}', \alpha^\omega).$$

[7] This follows by identifying $\mathcal{R}$ with the weak closure of the infinite tensor product of $\mathcal{B}(\ell^2(G))$, approximating $F$ in the seminorm $\|\cdot\|_2$ by elements in the UHF-algebra $M_{|G|^\infty}$, then further by elements in some finite tensor product of $\mathcal{B}(\ell^2(G))$ in the $C^*$-norm, and finally choosing a tensor copy of $\mathcal{B}(\ell^2(G))$ that commutes with those approximations.

*Proof.* The implication (2) $\Rightarrow$ (1) is completely general for actions on a $\mathrm{II}_1$-factor $M$: suppose there exist $g_0 \in G$ and $u \in \mathcal{U}(M)$ such that $\alpha_{g_0} = \mathrm{Ad}(u)$. Choose projections $p_g \in M$, for $g \in G$, satisfying the conditions of Definition 10.3.1 for $F = \{u\}$ and $\varepsilon = 1/2$. Complete the proof!

To prove (3) $\Rightarrow$ (2), observe that $(\ell^\infty(G), \mathtt{Lt})$ embeds into $(\mathcal{B}(\ell^2(G)), \mathrm{Ad}(\lambda))$ canonically as multiplication operators. Hence the restriction of $\psi$ to $\ell^\infty(G)$ is the desired homomorphism as in Remark 10.3.2.

It therefore remains to prove (1) $\Rightarrow$ (3), so suppose that $\alpha$ is outer. By Theorem 10.1.14, $\alpha$ induces an outer action on the $\mathrm{II}_1$-factor $\mathcal{R}^\omega \cap \mathcal{R}'$. By Lemma 10.3.4, there exists a unital equivariant embedding

$$\psi\colon (\mathcal{B}(\ell^2(G)), \mathrm{Ad}(\lambda)) \to (\mathcal{R}^\omega \cap \mathcal{R}', \alpha^\omega).$$

□

The above result is quite remarkable, since a very strong *global* property (the $W^*$-Rokhlin property) is obtained from an a priori much weaker *pointwise* property (outerness)

Once the Rokhlin property for outer actions has been established, there are at least two ways of proving the classification of outer actions. The first proof, which is outlined in the rest of this chapter, one of the main tools is that, consists in "splitting off" a tensorial copy of the model action $\mu$ from Example 10.3.3, and then showing that what is left over is the trivial action on the relative commutant. The second proof uses arguments that are much closer in spirit to those used in $C^*$-algebras (specifically, the Evans-Kishimoto intertwining argument), and does not depend on the model action $\mu$. This alternative proof is not explicitly presented in these notes, but it can be easily reconstructed from the classification of actions with the $C^*$-Rokhlin property given in Section 11.2.

We proceed to sketch the first of the proofs described above. For this, we need an equivariant version of McDuff's result Theorem 10.1.15, in which the conclusion is that a given action absorbs an action on $\mathcal{R}$ tensorially. We cannot allow arbitrary actions on $\mathcal{R}$, and the class of actions that are suitable for our purposes is what we call *McDuff actions*.

**Definition 10.3.6.** Let $G$ be a finite group and let $\delta\colon G \to \mathrm{Aut}(\mathcal{R})$ be an action. We say that $\gamma$ is a *McDuff action* if there are an equivariant isomorphism $\theta\colon (\mathcal{R}\overline{\otimes}\mathcal{R}, \delta\otimes\delta) \to (\mathcal{R}, \delta)$ and unitaries $u_n \in (\mathcal{R}\overline{\otimes}\mathcal{R})^G$, for $n \in \mathbb{N}$, satisfying

$$\lim_{n\to\infty} \|u_n\theta(x)u_n^* - x \otimes 1_{\mathcal{R}}\|_2 = 0$$

for all $x \in \mathcal{R}$.

The next exercise contains some examples of McDuff actions.

**Exercise 10.3.7.** Let $G$ be a finite group, and let $X$ be a finite $G$-space. Denote by $\gamma_X : G \to \mathrm{Aut}(\mathcal{B}(\ell^2(G)))$ the induced action, and by $\delta_X : G \to \mathrm{Aut}(\mathcal{R})$

the action given by $\delta = \bigotimes_{n\in\mathbb{N}} \gamma_X$, using the identification $\mathcal{R} \cong \bigotimes_{n\in\mathbb{N}} \mathcal{B}(\ell^2(X))$. Prove that $\delta_X$ is McDuff. Deduce that the trivial action of $G$ on $\mathcal{R}$ is McDuff, and also that the model action $\mu\colon G \to \mathrm{Aut}(\mathcal{R})$ with the $W^*$-Rokhlin property (see Example 10.3.3) is McDuff.

Virtually the same proof as in Theorem 10.1.15 allows one to prove the desired equivariant McDuff result (see [29] for a more general version):

**Theorem 10.3.8.** Let $M$ be a separable II$_1$-factor, let $G$ be a finite group, let $\alpha\colon G \to \mathrm{Aut}(M)$ be an action, let $\delta\colon G \to \mathrm{Aut}(\mathcal{R})$ be a McDuff action, and let $\omega$ be a free ultrafilter. Then the following are equivalent

1. There is an equivariant isomorphism $(M, \alpha) \cong (M\overline{\otimes}\mathcal{R}, \alpha \otimes \delta)$;
2. There is a unital equivariant embedding $(\mathcal{R}, \delta) \to (M^\omega \cap M', \alpha^\omega)$.

Moreover, if there exists a finite $G$-space $X$ such that $\delta = \delta_X$ as in Exercise 10.3.7, then the above are also equivalent to

3. There is a unital equivariant embedding $(\mathcal{B}(\ell^2(X)), \gamma_X) \to (M^\omega \cap M', \alpha^\omega)$.

Combining Theorem 10.3.5 and Theorem 10.3.8, we immediately deduce the following.

**Corollary 10.3.9.** Let $G$ be a finite group and let $\alpha\colon G \to \mathrm{Aut}(M)$ be an outer action. Then there are equivariant isomorphisms

$$(\mathcal{R}\overline{\otimes}\mathcal{R}, \alpha \otimes \mu) \cong (\mathcal{R}, \alpha) \cong (\mathcal{R}\overline{\otimes}\mathcal{R}, \alpha \otimes \mathrm{id}_{\mathcal{R}}).$$

Classification, once the Rokhlin property is established, can be proved along the exact same lines of classification of $C^*$-Rokhlin actions; see next chapter. We don't do it for vNa.

# Chapter 11

# The Rokhlin property

The works of Connes [13] and Jones [48] motivated the search for analogs of the results from chapter 10 in the context of finite group actions on $C^*$-algebras, particularly in what refers to their classification. Early studies came in the works of Fack-Marechal [24] and Herman-Jones [40], who studied what we now call the Rokhlin property for $\mathbb{Z}_n$-actions on UHF-algebras. A few decades later, Izumi's groundbreaking work [46] laid the foundations for a systematic study of the Rokhlin property. There, he gave a complete classification of finite group actions with the Rokhlin property in terms of the approximate unitary equivalence classes of the individual automorphisms. Later works by Hirshberg-Winter [44] Osaka-Phillips [69], and the author [25], focused on the structure of the crossed product as well as the internal properties of Rokhlin actions.

In this chapter, we give an overview of the results concerning finite group actions with the Rokhlin property, including the most prominent uses: their classification (including existence and uniqueness results for homomorphisms) and the structure of their crossed products. We restrict the analysis to unital $C^*$-algebras throughout, in order to avoid unnecessary technicalities.

## 11.1 Finite group actions with the Rokhlin property

In this section, we introduce the Rokhlin property for actions of finite groups and study some basic properties and examples.

We begin by introducing the Rokhlin property for finite group actions on unital $C^*$-algebras.

**Definition 11.1.1.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that $\alpha$ has the *Rokhlin property* if for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, there exist mutually orthogonal projections $p_g \in A$, for $g \in G$, satisfying

1. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;

2. $\sum\limits_{g\in G} p_g = 1$;
3. $\|p_g a - a p_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

The Rokhlin property is clearly invariant under equivariant isomorphism. In the next exercise it is shown that it is also invariant under cocycle conjugacy.

**Example 11.1.2.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action, and let $w\colon G \to \mathcal{U}(A)$ be an $\alpha$-cocycle.

1. Show that $\alpha$ has the Rokhlin property if and only if $\alpha^w$ does.
2. Suppose that $\alpha$ has the Rokhlin property. Show that there exists $v \in \mathcal{U}(A)$ with $w_g = v\alpha_g(v^*)$ for all $g \in G$. (Hint: use Proposition 10.2.3 together with an approximation argument.)

As in the case of von Neumann algebras, an equivalent characterization when $A$ is separable can be given in terms of central sequence algebras. We denote by $A_\infty$ the quotient of $\ell^\infty(\mathbb{N}, A)$ by the ideal $c_0(\mathbb{N}, A)$, and call it the *sequence algebra* of $A$. We write $A_\infty \cap A'$ for the relative commutant of $A$ in the sequence algebra, and call it the *central sequence algebra* of $A$. Note that $C^*$-algebraic sequence algebras are decorated with subscripts, while von Neumann algebraic sequence algebras are decorated with superscripts. We do so because for a $\mathrm{II}_1$-factor $M$, its von Neumann algebraic central sequence and its $C^*$-algebraic central sequence never agree.

**Proposition 11.1.3.** Let $G$ be a finite group, let $A$ be a separable unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then $\alpha$ has the Rokhlin property if and only if there is a unital, equivariant homomorphism $\varphi\colon (C(G), \mathtt{Lt}) \to (A_\infty \cap A', \alpha_\infty)$.

**Exercise 11.1.4.** Give a proof of Proposition 11.1.3.

In order to solve the above exercise, one needs to represent a projection in the (central) sequence algebra with a sequence of projections, which is an easy application of functional calculus. The sum of these projections will then be very close to the unit of the algebra, and a further perturbation argument needs to be performed to find nearby projections that add up exactly to the unit.

Arguments of this nature involve, either explicitly or implicitly, the notion of *weak semiprojectivity.* For algebras that admit a finite presentation in terms of generators and relations (see Section 2.1), this notion can be expressed very nicely[1]. Suppose that $A = C^*(\mathcal{G} : \mathcal{R})$ is the universal $C^*$-algebra on the finite set of generators $\mathcal{G}$ subject to the finite set of relations $\mathcal{R}$. Then $A$ is *weakly semiprojective* if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $B$ is a $C^*$-algebra and $b_x \in B$, for $x \in \mathcal{G}$, are elements satisfying

[1] A more thorough treatment of the subject can be found in [62].

all the relations from $\mathcal{R}$ up to $\delta$, then there exist $c_x \in B$, for $x \in \mathcal{G}$, which satisfy the relations from $\mathcal{R}$ exactly and moreover $\|b_x - c_x\| < \varepsilon$ for all $x \in \mathcal{G}$. Examples of weakly semiprojective algebras include $\mathbb{C}^n$ and $C(S^1)$. In the context of Proposition 11.1.3, weak semiprojectivity of $C(G) \cong \mathbb{C}^{|G|}$ allows one to replace projections that add up to almost one by projections that add up to exactly one.

There is an equivariant version of weak semiprojectivity, called *equivariant weak semiprojectivity*, that has been studied in [74] and [76]. We will not discuss this notion in these notes, but we do want to mention one consequence of the results in [74]. In Theorem 2.5 there, it is shown that for a finite group $G$, the translation action on $C(G)$ is equivariantly weakly semiprojective (this also appears, with an easier proof and for abelian $G$, in Section 5 of [28]), meaning that projections that are "almost" translated by $G$ can be perturbed to find projections that are *exactly* translated. In particular, this leads to the following strengthening of the Rokhlin property.

**Remark 11.1.5.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that $\alpha$ has the Rokhlin property if and only if for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, there exist mutually orthogonal projections $p_g \in A$, for $g \in G$, satisfying

1. $\alpha_g(p_h) = p_{gh}$ for all $g, h \in G$;
2. $\sum\limits_{g \in G} p_g = 1$;
3. $\|p_g a - a p_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

The difference between the conditions above and those listed in Definition 11.1.1 is that in condition (1) in Remark 11.1.5, the projections are exactly permuted by the group action. This strengthening makes a number of arguments much shorter and conceptually clearer, and will be used in a number of proofs in this chapter. This simplification is not strictly necessary, since all the results presented here can be proved using projections as in Definition 11.1.1.

The following is the canonical example of a Rokhlin action; see also Example 10.3.3.

**Example 11.1.6.** Let $G$ be a nontrivial finite group. Consider the conjugation action $\mathrm{Ad}(\lambda)\colon G \to \mathrm{Aut}(\mathcal{B}(\ell^2(G)))$. Then the infinite tensor product action $\delta = \bigotimes\limits_{n \in \mathbb{N}} \mathrm{Ad}(\lambda)$ of $G$ on the UHF-algebra of type $M_{|G|^\infty}$ has the Rokhlin property.

We collect some preservation properties for the Rokhlin property that will be needed later. Their proofs are easy, and are mostly left to the reader.

**Proposition 11.1.7.** Let $G$ be a finite group, let $(A_n, \iota_n)_{n \in \mathbb{N}}$ be a direct system of unital $C^*$-algebras with unital connecting maps, and for each $n \in \mathbb{N}$, let $\alpha^{(n)}\colon G \to \mathrm{Aut}(A_n)$ be an action such that $\iota_n \circ \alpha_g^{(n)} = \alpha_g^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $g \in G$. Set $A = \varinjlim A_n$ and $\alpha = \varinjlim \alpha^{(n)}$. If $\alpha^{(n)}$ has the Rokhlin property for infinitely many values of $n$, then $\alpha$ has the Rokhlin property as well.

*Proof.* Let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$. Find $n \in \mathbb{N}$ and a finite subset $F_0 \subseteq A_n$ such that $\mathrm{dist}(F, \iota_{n,\infty}(F_0)) < \varepsilon/2$. By increasing $n$, we may assume that $\alpha^{(n)}$ has the Rokhlin property. Find projections $q_g \in A_n$, for $g \in G$, satisfying the conditions in Definition 11.1.1 for $\alpha^{(n)}$, $\varepsilon/2$ and $F_0$. One easily checks that the projections $p_g = \iota_{n,\infty}(q_g) \in A$, for $g \in G$, satisfy the conditions in Definition 11.1.1 for $\alpha$, $\varepsilon$ and $F$, as desired. □

**Proposition 11.1.8.** Let $A$ be a unital $C^*$-algebra, let $G$ be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

1. Let $B$ be a unital $C^*$-algebra, and let $\beta\colon G \to \mathrm{Aut}(B)$ be an action of $G$ on $B$. Let $A \otimes B$ be any $C^*$-algebra completion of the algebraic tensor product of $A$ and $B$ for which the tensor product action $\alpha \otimes \beta$ is defined[2]. Then $\alpha \otimes \beta$ has the Rokhlin property.

2. Let $I$ be an $\alpha$-invariant ideal in $A$, and denote by $\overline{\alpha}\colon G \to \mathrm{Aut}(A/I)$ the induced action on $A/I$. Then $\overline{\alpha}$ has the Rokhlin property.

3. Let $p$ be an $\alpha$-invariant projection in $A$. Set $B = pAp$ and denote by $\beta\colon G \to \mathrm{Aut}(B)$ the compressed action of $G$. Then $\beta$ has the Rokhlin property.

**Exercise 11.1.9.** Prove Proposition 11.1.8.

Using Proposition 11.1.7 and Proposition 11.1.8, we can construct more actions with the Rokhlin property.

**Example 11.1.10.** Let $G$ be a finite group, and let $\delta$ be the Rokhlin action of $G$ on $M_{|G|^\infty}$ from Example 11.1.6, and consider the tensor product action

$$\delta \otimes \mathrm{id}_{\mathcal{O}_2}\colon G \to \mathrm{Aut}(M_{|G|^\infty} \otimes \mathcal{O}_2).$$

This action has the Rokhlin property by part (1) of Proposition 11.1.8. Since $M_{|G|^\infty} \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$, we have thus constructed an action of $G$ on $\mathcal{O}_2$ with the Rokhlin property.

**Proposition 11.1.11.** Let $G$ be a finite group, and let $m \in \mathbb{N}$. Then there is action of $G$ on $M_{m^\infty}$ with the Rokhlin property if and only if $|G|$ divides $m$.

*Proof.* Suppose that $|G|$ divides $m$. Then $M_{m^\infty} \cong M_{m^\infty} \otimes M_{|G|^\infty}$, and hence a tensor product construction similar to the one given in Example 11.1.10 shows that there is an action of $G$ on $M_{m^\infty}$ with the Rokhlin property.

Conversely, suppose that there is an action $\alpha\colon G \to \mathrm{Aut}(M_{m^\infty})$ with the Rokhlin property. In particular, there is a projection $p = p_1 \in M_{m^\infty}$ that

[2]See Exercise 3.3.7, and observe that the action $\alpha$ described in that exercise has the Rokhlin property.

satisfies $\sum\limits_{g\in G} \alpha_g(p) = 1$. Since $\alpha_g$ is approximately inner for all $g \in G$, we deduce that $[\alpha_g(p)]_0 = [p]_0$ for all $g \in G$; see Exercise 2.3.17. It follows that $[1]_0 = |G||p]_0$ in $K_0(M_{m^\infty}) \cong \mathbb{Z}\left[\frac{1}{m}\right]$. This is only possible if $|G|$ divides $m$, as desired. $\square$

**Exercise 11.1.12.** Let $\theta \in \mathbb{R}\setminus\mathbb{Q}$, and let $G$ be a nontrivial finite group. Show that there is no action of $G$ on $A_\theta$ with the Rokhlin property.

**Exercise 11.1.13.** Let $G$ be a finite group and let $n \in \{2, \ldots, \infty\}$. Show that there is an action of $G$ on $\mathcal{O}_n$ with the Rokhlin property if and only if $|G|$ divides $n-1$.

## 11.2 Classification of equivariant homomorphisms

The main result of this section asserts that finite group actions with the Rokhlin property are classified, up to conjugation by an approximately inner automorphism, by the approximate equivalence classes of the individual automorphisms; see Corollary 11.2.6. This result is due to Izumi in the unital case [46], and to the author and Santiago in the nonunital case [31]. We only treat the unital case here, but we follow the arguments from [31], since they allow us to prove a more general result regarding equivariant homomorphisms.

We begin by introducing a useful definition.

**Definition 11.2.1.** Let $A$ and $B$ be unital $C^*$-algebras, let $G$ be a finite group, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions, and let

$$\varphi, \psi\colon (A, \alpha) \to (B, \beta)$$

be equivariant homomorphisms. We say that $\varphi$ and $\psi$ are *$G$-approximately unitarily equivalent*, written $\varphi \underset{G\text{-aue}}{\sim} \psi$, if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exists a $G$-invariant unitary $u \in B^\beta$ satisfying $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in F$.

Note that when $\alpha$ and $\beta$ are trivial, then $G$-approximate unitary equivalence is just the usual approximate unitary equivalence, which we denote by $\underset{\text{aue}}{\sim}$.

In the proof of Proposition 11.2.3, we will repeatedly use the fact that, whose proof is left as an exercies.

**Exercise 11.2.2.** Let $p_1, \ldots, p_n$ be orthogonal projections in a $C^*$-algebra $A$, and let $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$. Prove that

$$\left\| \sum_{j=1}^{n} p_j x_j - \sum_{j=1}^{n} p_j y_j \right\| \leq \max_{j=1,\ldots,n} \|x_j - y_j\|.$$

Without orthogonality of the projections, the best bound is $\sum\limits_{j=1}^{n} \|x_j - y_j\|$.

**Proposition 11.2.3.** Let $A$ and $B$ be unital $C^*$-algebras, let $G$ be a finite group, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions, and let $\varphi, \psi\colon (A, \alpha) \to (B, \beta)$ be equivariant homomorphisms. Suppose that $\beta$ has the Rokhlin property. Then $\varphi \underset{\mathrm{aue}}{\sim} \psi$ if and only if $\varphi \underset{G\text{-aue}}{\sim} \psi$.

*Proof.* We only need to prove the "only if" implication, so let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$. Without loss of generality, we assume that $\|a\| \le 1$ for all $a \in F$. Find $\delta > 0$ such that whenever $z$ is an element in a unital $C^*$-algebra satisfying $\|zz^* - 1\| < \delta$ and $\|z^*z - 1\| < \delta$, then there exists a unitary $u$ with $\|u - z\| < \varepsilon/6$.

Set $F' = \bigcup\limits_{g\in G} \alpha_g(F)$, which is a $G$-invariant finite subset of $A$. Since $\varphi \sim_{\mathrm{aue}} \psi$, there exists $v \in \mathcal{U}(B)$ such that

$$\|v\varphi(a')v^* - \psi(a')\| < \varepsilon/6$$

for all $a' \in F'$.

Fix $g \in G$ and $a \in F$. Then $a' = \alpha_{g^{-1}}(a)$ belongs to $F'$. Using the inequality above and the fact that $\varphi$ and $\psi$ are equivariant, we get

$$\|v\beta_{g^{-1}}(\varphi(a))v^* - \beta_{g^{-1}}(\psi(a))\| < \varepsilon/6.$$

Apply $\beta_g$ to the inequality above to get

$$\|\beta_g(v)\varphi(a)\beta_g(v)^* - \psi(a)\| < \varepsilon/6.$$

Using the Rokhlin property for $\beta$ and Remark 11.1.5, choose a partition of unity of projections $p_g \in B$, for $g \in G$, satisfying $\beta_g(p_h) = p_{gh}$ for all $g, h \in G$, and $\|p_g x - x p_g\| < \min\{\delta/2, \varepsilon/6\}$ for all $x \in \varphi(F) \cup \{\beta_g(v), \beta_g(v^*) \colon g \in G\}$. Set

$$z = \sum_{g\in G} p_g \beta_g(v) \in B.$$

Then

$$zz^* = \sum_{g,h\in G} p_g\beta_g(v)\beta_h(v^*)p_h \underset{\frac{2\delta}{2}}{\approx} \sum_{g,h\in G} p_g p_h \beta_g(v)\beta_h(v^*) = \sum_{g\in G} p_g\beta_g(vv^*) = 1.$$

Similarly, one checks that $\|z^*z - 1\| < \delta$. Moreover, given $g \in G$, one has

$$\beta_g(z) = \sum_{h\in G} \beta_g(p_h)\beta_{gh}(v) = \sum_{h\in G} p_{gh}\beta_{gh}(v) = z,$$

so $z$ belongs to $B^\beta$. By the choice of $\delta$, we can find a unitary $u \in B^\beta$ satisfying $\|u - z\| < \varepsilon/6$.

Given $a \in F$, we have

$$\begin{aligned}
u\varphi(a)u^* \underset{\frac{2\varepsilon}{6}}{\approx} z\varphi(a)z^* &= \sum_{g,h\in G} p_g\beta_g(v)\varphi(a)\beta_h(v^*)p_h \\
&\underset{\frac{3\varepsilon}{6}}{\approx} \sum_{g,h\in G} p_g p_h\beta_g(v)\varphi(a)\beta_h(v^*) \\
&= \sum_{g\in G} p_g\beta_g(v)\varphi(a)\beta_g(v^*) \\
&\underset{\frac{\varepsilon}{6}}{\approx} \psi(a).
\end{aligned}$$

Thus $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in F$. We conclude that $\varphi \underset{G\text{-aue}}{\sim} \psi$, as desired. ☐

In the next proposition, we show that equivariant homomorphisms can be classified if the codomain action has the Rokhlin property.

**Proposition 11.2.4.** Let $A$ and $B$ be unital $C^*$-algebras, let $G$ be a finite group, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions, and let $\varphi\colon A \to B$ be a homomorphism. Suppose that $\beta$ has the Rokhlin property and that $\beta_g \circ \varphi \underset{\text{aue}}{\sim} \varphi \circ \alpha_g$ for all $g \in G$. Then:

1. For any $\varepsilon > 0$ and for any finite set $F \subseteq A$ there exists a unitary $w \in \mathcal{U}(B)$ satisfying the following inequalities for all $g \in G$ and all $a \in F$:

$$\begin{aligned}
&\|(\beta_g \circ \mathrm{Ad}(w) \circ \varphi)(a) - (\mathrm{Ad}(w) \circ \varphi \circ \alpha_g)(a)\| < \varepsilon, \text{ and} \\
&\|(\mathrm{Ad}(w) \circ \varphi)(a) - \varphi(a)\| < \varepsilon + \sup_{h\in G} \|(\beta_h \circ \varphi \circ \alpha_{h^{-1}})(a) - \varphi(a)\|.
\end{aligned}$$

2. When $A$ is separable, there is an equivariant homomorphism $\psi\colon (A,\alpha) \to (B,\beta)$ with $\psi \underset{\text{aue}}{\sim} \varphi$.

*Proof.* (1). Let $F$ be a finite subset of $A$ and let $\varepsilon > 0$. Without loss of generality, we may assume that $F$ consists of contractions. Set $F' = \bigcup_{g\in G} \alpha_g(F)$, which is a finite subset of $A$. Since $\beta_g \circ \varphi \underset{\text{aue}}{\sim} \varphi \circ \alpha_g$ for all $g \in G$, there exist unitaries $u_g \in \mathcal{U}(B)$, for $g \in G$, such that

$$\|(\beta_g \circ \varphi)(b) - (\mathrm{Ad}(u_g) \circ \varphi \circ \alpha_g)(b)\| < \frac{\varepsilon}{10},$$

for all $b \in F'$ and $g \in G$. For $a \in F$ and $g, h \in G$, we have

$$\begin{aligned}
&\|(\mathrm{Ad}(u_g) \circ \varphi \circ \alpha_h)(a) - (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \varphi)(a)\| \\
&\quad = \big\|(\mathrm{Ad}(u_g) \circ \varphi \circ \alpha_g)(\alpha_{g^{-1}h}(a)) - (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \varphi \circ \alpha_{h^{-1}g})(\alpha_{g^{-1}h}(a))\big\| \\
&\quad \leq \big\|(\mathrm{Ad}(u_g) \circ \varphi \circ \alpha_g)(\alpha_{g^{-1}h}(a)) - (\beta_g \circ \varphi)(\alpha_{g^{-1}h}(a))\big\| \\
&\qquad + \big\|(\beta_g \circ \varphi)(\alpha_{g^{-1}h}(a)) - (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \varphi \circ \alpha_{h^{-1}g})(\alpha_{g^{-1}h}(a))\big\| \\
&\quad \leq \frac{\varepsilon}{10} + \frac{\varepsilon}{10} = \frac{\varepsilon}{5}.
\end{aligned}$$

Let $\delta > 0$ such that whenever $x \in B$ satisfies $\|x^*x - 1\| < \delta$ and $\|xx^* - 1\| < \delta$, then there exists a unitary $w \in \mathcal{U}(B)$ with $\|w - x\| < \varepsilon/5$. Using the Rokhlin property for $\beta$ and Remark 11.1.5, choose a partition of unity of projections $p_g \in B$, for $g \in G$, satisfying $\beta_g(p_h) = p_{gh}$ for all $g, h \in G$, and $\|p_g x - x p_g\| < \min\{\delta/2, \varepsilon/5\}$ for all $x \in \varphi(F') \cup \{u_g, u_g^* \colon g \in G\}$. Set $x = \sum\limits_{g \in G} p_g u_g \in B$. Using an argument similar to the one in the proof of Proposition 11.2.3, it is easily seen that $\|x^*x - 1\| < \delta$ and $\|xx^* - 1\| < \delta$. Find $w \in \mathcal{U}(B)$ with $\|w - x\| < \varepsilon/5$. In particular, $\|xbx^* - wbw^*\| < \|b\| 2\varepsilon/5$ for all $b \in B$. Moreover, for $b \in B$, one has

$$xbx^* = \sum_{g \in G} p_g(u_g b u_g^*) = \sum_{g \in G} p_g \mathrm{Ad}(u_g)(b).$$

Fix $a \in F$ and $h \in G$. Then

$$\begin{aligned}(\beta_h \circ \mathrm{Ad}(w) \circ \varphi)(a) &\underset{\frac{2\varepsilon}{5}}{\approx} \sum_{g \in G} p_{hg}(\beta_h \circ \mathrm{Ad}(u_g) \circ \varphi)(a) \\ &= \sum_{g \in G} p_g(\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \varphi)(a),\end{aligned}$$

and

$$(\mathrm{Ad}(w) \circ \varphi \circ \alpha_h)(a) \underset{\frac{2\varepsilon}{5}}{\approx} \sum_{g \in G} p_g(\mathrm{Ad}(u_g) \circ \varphi \circ \alpha_h)(a).$$

Using that the $p_g$ are orthogonal projections, we get

$$\begin{aligned}&\|(\beta_h \circ \mathrm{Ad}(w) \circ \varphi)(a) - (\mathrm{Ad}(w) \circ \varphi \circ \alpha_h)(a)\| \\ &\qquad \leq \frac{4\varepsilon}{5} + \sup_{g \in G} \left\|(\mathrm{Ad}(u_g) \circ \varphi \circ \alpha_h)(a) - (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \varphi)(a)\right\| \\ &\qquad < \frac{5\varepsilon}{5} = \varepsilon.\end{aligned}$$

Thus,

$$\begin{aligned}\|(\mathrm{Ad}(w) \circ \varphi)(a) - \varphi(a)\| &= \left\| \sum_{g \in G} p_g((\mathrm{Ad}(u_g) \circ \varphi)(a) - \varphi(a)) \right\| \\ &\leq \sup_{g \in G} \|(\mathrm{Ad}(u_g) \circ \varphi)(a) - \varphi(a)\| \\ &\leq \sup_{g \in G} \big( \left\|(\mathrm{Ad}(u_g) \circ \varphi \circ \alpha_g)(\alpha_{g^{-1}}(a)) - (\beta_g \circ \varphi)(\alpha_{g^{-1}}(a))\right\| \\ &\qquad + \left\|(\beta_g \circ \varphi \circ \alpha_{g^{-1}})(a) - \varphi(a)\right\| \big) \\ &\leq \varepsilon + \sup_{g \in G} \left\|(\beta_g \circ \varphi \circ \alpha_{g^{-1}})(a) - \varphi(a)\right\|,\end{aligned}$$

as desired.

(2) Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $A$ whose union is a $G$-invariant dense $* - \mathbb{Q}[i]$-subalgebra of $A$[3]. Set $\varphi_1 = \varphi$ and find a unitary $w_1 \in \mathcal{U}(B)$ such that the conclusion of the first part of the proposition is satisfied with $\varphi_1$ and $\varepsilon = 1$. Set $\varphi_2 = \mathrm{Ad}(w_1)\circ\varphi_1$, and find a unitary $w_2 \in \mathcal{U}(B)$ such that the conclusion of the first part of the proposition is satisfied with $\varphi_2$ and $\varepsilon = \frac{1}{2}$. Iterating this process, there exist $*$-homomorphisms $\varphi_n \colon A \to B$, with $\varphi_1 = \varphi$, and unitaries $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{U}(B)$ such that $\varphi_{n+1} = \mathrm{Ad}(w_n) \circ \varphi_n$, for all $n \in \mathbb{N}$, which moreover satisfy

$$\max_{g\in G} \|(\beta_g \circ \varphi_n)(a) - (\varphi_n \circ \alpha_g)(a)\| < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$ and for all $a \in F_n$, and

$$\|\varphi_{n+1}(a) - \varphi_n(a)\| < \frac{3}{2^n}$$

for all $n \in \mathbb{N}$ and for all $a \in F_n$. Then the sequence $(\varphi_k(a))_{k\in\mathbb{N}}$ is Cauchy in $B$ for every $a \in \bigcup\limits_{n\in\mathbb{N}} F_n$, which is a dense $*$-subalgebra of $A$. Denoting by $\psi_0(x)$ its limit, it follows that the map $\psi_0 \colon \bigcup\limits_{n\in\mathbb{N}} F_n \to B$ is well-defined, linear, $*$-multiplicative and contractive. Therefore, it extends by continuity to a map $\psi \colon A \to B$. By construction, each $\varphi_n$ is unitarily equivalent to $\varphi$, so $\varphi$ is approximately unitarily equivalent to $\varphi$. Finally, it is clear that $\psi$ is equivariant. $\square$

By combining Proposition 11.2.4 with an intertwining argument, we will show that Rokhlin actions can be classified up to conjugacy by approximately inner automorphisms; see Corollary 11.2.6. The required intertwining technique, known as the Evans-Kishimoto intertwining, is an equivariant version of Elliott's intertwining argument, which we present next.

**Theorem 11.2.5.** Let $A$ and $B$ be separable unital $C^*$-algebras, let $G$ be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be actions. Suppose that there exist equivariant homomorphisms $\varphi \colon (A, \alpha) \to (B, \beta)$ and $\psi \colon (B, \beta) \to (A, \alpha)$ such that

$$\psi \circ \varphi \underset{G\text{-aue}}{\sim} \mathrm{id}_A \quad \text{and} \quad \varphi \circ \psi \underset{G\text{-aue}}{\sim} \mathrm{id}_B.$$

Then there exists an equivariant isomorphism $\theta \colon (A, \alpha) \to (B, \beta)$ with $\theta \underset{G\text{-aue}}{\sim} \varphi$.

[3]To accomplish this, one starts with any increasing sequence $(\widetilde{F}_n)_{n\in\mathbb{N}}$ of self-adjoint finite subsets of $A$ with dense union. By replacing each $\widetilde{F}_n$ with $\bigcup\limits_{g\in G} \alpha_g(F_n)$, we may assume that these sets are $G$-invariant. Also fix an exhausting sequence $(Q_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{Q}[i]$. One then takes $F_1 = \widetilde{F}_1$. Inductively, set $F_n = (Q_{n-1} \cdot F_{n-1}) \cup (F_{n-1} \cdot F_{n-1}) \cup \widetilde{F}_n$. Then the sequence $(F_n)_{n\in\mathbb{N}}$ satisfies the desired conditions.

*Proof.* Let $(F_n^A)_{n\in\mathbb{N}}$ and $(F_n^B)_{n\in\mathbb{N}}$ be increasing sequences of $G$-invariant finite subsets of $A$ and $B$, respectively, whose unions are $G$-invariant dense $*-\mathbb{Q}[i]$-subalgebras of $A$ and $B$, respectively. Set $\varphi_0 = \varphi$ and $\psi_0 = \psi$. Since $\psi_0 \circ \varphi_0 \underset{G\text{-aue}}{\sim} \mathrm{id}_A$, there exists a $G$-invariant unitary $u_1 \in \mathcal{U}(A^\alpha)$ such that

$$\|(\mathrm{Ad}(u_1)\circ\psi_0\circ\varphi_0)(a) - a\| < \frac{1}{2}$$

for all $a \in F_1^A$. Set $\psi_1 = \mathrm{Ad}(u_1)\circ\psi_0$, which is also equivariant and satisfies $\psi_1 \underset{G\text{-aue}}{\sim} \psi_0$. Since $\varphi_0\circ\psi_1 \underset{G\text{-aue}}{\sim} \varphi_0\circ\psi_0 \underset{G\text{-aue}}{\sim} \mathrm{id}_B$, there exists a $G$-invariant unitary $v_1 \in \mathcal{U}(B^\beta)$ such that

$$\|(\mathrm{Ad}(v_1)\circ\varphi_0\circ\psi_1)(b) - b\| < \frac{1}{2}$$

for all $b \in F_1^B$. Set $\varphi_1 = \mathrm{Ad}(v_1)\circ\varphi_0$, which is also equivariant and satisfies $\varphi_1 \underset{G\text{-aue}}{\sim} \varphi_0$. Proceeding inductively, we find a $G$-invariant unitary $u_n \in \mathcal{U}(A^\alpha)$ such that

$$\|(\mathrm{Ad}(u_n)\circ\psi_{n-1}\circ\varphi_{n-1})(a) - a\| < \frac{1}{2^n}$$

for all $a \in F_n^A$. We then set $\psi_n = \mathrm{Ad}(u_n)\circ\psi_{n-1}$, and continue to find a $G$-invariant unitary $v_n \in \mathcal{U}(B^\beta)$ such that

$$\|(\mathrm{Ad}(v_n)\circ\varphi_{n-1}\circ\psi_n)(b) - b\| < \frac{1}{2^n}$$

for all $b \in F_n^B$. Then we set $\varphi_n = \mathrm{Ad}(v_n)\circ\varphi_{n-1}$, and continue. The result is a so-called *approximate intertwining* diagram, which is a diagram of the form

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_0"'] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_1"] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_2"] & & A \arrow[r] \arrow[dr, "\varphi_3"] & \cdots & \\
& B \arrow[ur, "\psi_0"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "\psi_1"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "\psi_2"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[r] & \cdots ,
\end{tikzcd}$$

where the $n$-th triangle that has $A$ as two of its vertices is commutative within $1/2^n$ on $F_n^A$, and the $n$-th triangle that has $B$ as two of its vertices is commutative within $1/2^n$ on $F_n^B$.

Given $a \in \bigcup\limits_{n\in\mathbb{N}} F_n^A$, the sequence $(\varphi_k(a))_{k\in\mathbb{N}}$ is Cauchy in $B$, and we denote by $\theta_0(a) \in B$ its limit. Similarly, $(\psi_k(b))_{k\in\mathbb{N}}$ is Cauchy in $A$ for all $b \in \bigcup\limits_{n\in\mathbb{N}} F_n^B$, and we denote by $\pi_0(b) \in A$ its limit. The resulting maps

$$\theta_0\colon \bigcup_{n\in\mathbb{N}} F_n^A \to B \text{ and } \pi_0\colon \bigcup_{n\in\mathbb{N}} F_n^B \to A$$

are easily seen to be well-defined, $\mathbb{Q}[i]$-linear, contractive, self-adjoint and equivariant. Thus, they extend to equivariant homomorphisms $\theta\colon (A,\alpha) \to$

$(B,\beta)$ and $\pi\colon (B,\beta) \to (A,\alpha)$, respectively. Since $\pi_0 \circ \theta_0$ is the identity on $\bigcup\limits_{n\in\mathbb{N}} F_n^A$, and $\theta_0 \circ \pi_0$ is the identity on $\bigcup\limits_{n\in\mathbb{N}} F_n^B$, we deduce that $\pi \circ \theta = \mathrm{id}_A$ and $\theta \circ \pi = \mathrm{id}_B$. In other words, $\theta$ is an equivariant isomorphism with inverse $\pi$. Finally, since $\varphi_n \underset{G\text{-aue}}{\sim} \varphi$ for all $n \in \mathbb{N}$, and $\psi_n \underset{G\text{-aue}}{\sim} \psi$ for all $n \in \mathbb{N}$, it is easy to see that $\theta \underset{G\text{-aue}}{\sim} \varphi$ and $\pi \underset{G\text{-aue}}{\sim} \psi$, thus completing the proof. $\square$

We are now ready to prove the main result of this section.

**Corollary 11.2.6.** Let $G$ be a finite group, let $A$ be a separable unital $C^*$-algebra, and let $\alpha, \beta\colon G \to \mathrm{Aut}(A)$ be actions with the Rokhlin property. Then there exists an approximately inner automorphism $\theta \in \overline{\mathrm{Inn}}(A)$ satisfying

$$\theta \circ \alpha_g \circ \theta^{-1} = \beta_g$$

for all $g \in G$, if and only if $\alpha_g \underset{\text{aue}}{\sim} \beta_g$ for all $g \in G$.

*Proof.* We use part (2) of Proposition 11.2.4 for $\mathrm{id}_A\colon A \to A$ to obtain an equivariant homomorphism $\varphi\colon (A,\alpha) \to (B,\beta)$ with $\varphi \underset{\text{aue}}{\sim} \mathrm{id}_A$. Exchanging the roles of $\alpha$ and $\beta$ and applying part (2) of Proposition 11.2.4 again, we obtain an equivariant homomorphism $\psi\colon (A,\beta) \to (B,\alpha)$ with $\psi \underset{\text{aue}}{\sim} \mathrm{id}_A$. In particular, the equivariant homomorphisms

$$\psi \circ \varphi\colon (A,\alpha) \to (A,\alpha) \quad \text{and} \quad \varphi \circ \psi\colon (A,\beta) \to (A,\beta)$$

satisfy $\psi \circ \varphi \underset{\text{aue}}{\sim} \mathrm{id}_A$ and $\varphi \circ \psi \underset{\text{aue}}{\sim} \mathrm{id}_A$. By Proposition 11.2.3, we deduce that $\psi \circ \varphi \underset{G\text{-aue}}{\sim} \mathrm{id}_A$ and $\varphi \circ \psi \underset{G\text{-aue}}{\sim} \mathrm{id}_A$. Finally, by Theorem 11.2.5, we conclude that there exists an equivariant isomorphism $\theta\colon (A,\alpha) \to (B,\beta)$ with $\theta \underset{G\text{-aue}}{\sim} \varphi$. In particular, $\theta \underset{\text{aue}}{\sim} \varphi \underset{\text{aue}}{\sim} \mathrm{id}_A$, so $\theta$ is approximately inner. $\square$

The assumption in the corollary above that both $\alpha$ and $\beta$ have the Rokhlin property cannot be removed. For example, the action $\delta$ from Example 11.1.6 has the Rokhlin property, and it is pointwise approximately unitarily equivalent to the trivial action.

We note some immediate consequences of Corollary 11.2.6.

**Corollary 11.2.7.** Let $G$ be a finite group. Then there is, up to conjugacy, a unique action of $G$ on $\mathcal{O}_2$ with the Rokhlin property.

*Proof.* Since any automorphism of $\mathcal{O}_2$ is approximately inner, Corollary 11.2.6 implies that there is *at most* one Rokhlin action of $G$ on $\mathcal{O}_2$. Hence, it suffices to argue that there exists *some* Rokhlin action, and this follows from Example 11.1.10. $\square$

**Corollary 11.2.8.** Let $G$ be a finite group, and let $m \in \mathbb{N}$.

1. If $|G|$ divides $m$, then there is, up to conjugacy, a unique action of $G$ on $M_{m^\infty}$ with the Rokhlin property.

2. If $|G|$ does not divide $m$, then there is no action of $G$ on $M_{m^\infty}$ with the Rokhlin property.

*Proof.* This follows from Proposition 11.1.11, together with the fact that any automorphism of $M_{m^\infty}$ is approximately inner. □

## 11.3 Existence of Rokhlin actions

Corollary 11.2.6 asserts that Rokhlin actions are classified in terms of the approximate unitary equivalence classes of the automorphisms. In this subsection, which is based on results of Barlak-Szabo from [3], we will study the problem of finding Rokhlin actions that induced prescribed approximate unitary equivalence classes. While it is in general difficult to obtain general existence theorems for Rokhlin actions, this satisfactory results can be obtained in the case of UHF-absorbing $C^*$-algebras.

To state this question rigorously, we need to introduce some notation.

**Notation 11.3.1.** Let $A$ be a unital $C^*$-algebra. We denote by $\mathrm{Aut}_{\mathrm{aue}}(A)$ the quotient of $\mathrm{Aut}(A)$ by the relation of approximate unitary equivalence. One readily checks that $\mathrm{Aut}_{\mathrm{aue}}(A)$ is a group, with the operation induced by composition on $\mathrm{Aut}(A)$. For an automorphism $\varphi \in \mathrm{Aut}(A)$, we denote by $[\varphi]_{\mathrm{aue}}$ the associated class in $\mathrm{Aut}_{\mathrm{aue}}(A)$.

Corollary 11.2.6 can be restated by saying that $[\cdot]_{\mathrm{aue}}$ is a complete invariant for actions with the Rokhlin property. It is therefore natural to wonder what is the *range* of this invariant. In other words, we wish to answer the following question:

**Question 11.3.2.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, and let $\Theta\colon G \to \mathrm{Aut}_{\mathrm{aue}}(A)$ be a group homomorphism. Does there exist an action $\alpha\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property satisfying $[\alpha]_{\mathrm{aue}} = \Theta$?

The above question does not always have an affirmative answer: for $A = C(\mathbb{Z}_2)$, the (trivial) homomorphism $\Theta\colon \mathbb{Z}_2 \to \mathrm{Aut}_{\mathrm{aue}}(A)$ with $\Theta_{-1} = [\mathrm{id}_A]_{\mathrm{aue}}$ cannot be realized by a Rokhlin action. This is due to the fact that two automorphisms of a commutative $C^*$-algebra are approximately unitarily equivalent if and only they are equal.

**Remark 11.3.3.** Question 11.3.2 has two difficulties: first, given a map $\Theta$ as in the statement, it is not a priori clear that there exists *some* action that implements $\Theta$ (this is in fact often a very difficult problem). But even once this is overcome, a second difficulty is finding a *Rokhlin* action that implements $\Theta$.

A general characterization of those homomorphisms $G \to \mathrm{Aut}_{\mathrm{aue}}(A)$ that can be realized by Rokhlin actions is probably out of reach, but we will see that when $A$ absorbs the UHF-algebra $M_{|G|^\infty}$, then Question 11.3.2 always has an affirmative answer; see Theorem 11.3.5. This result in fact shows that the trivial homomorphism $G \to \mathrm{Aut}_{\mathrm{aue}}(A)$ can be realized by Rokhlin action if and only if $A$ absorbs $M_{|G|^\infty}$.

We need a preparatory lemma.

**Lemma 11.3.4.**

**Theorem 11.3.5.** Let $G$ be a finite group, and let $A$ be a unital $C^*$-algebra. Then the following are equivalent:

1. For every group homomorphism $\Theta\colon G \to \mathrm{Aut}_{\mathrm{aue}}(A)$, there exists an action $\alpha\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property satisfying $[\alpha]_{\mathrm{aue}} = \Theta$.
2. There is an isomorphism $A \otimes M_{|G|^\infty} \cong A$.

*Proof.* □

For a finite group $G$ and a unital $C^*$-algebra $A$, we denote by $\mathrm{Rok}_G(A)$ the set of all Rokhlin actions of $G$ on $A$, and by $\overline{\mathrm{Rok}}_G(A)$ the set of all conjugacy classes of Rokhlin actions using approximately inner automorphisms.

The results of this section imply the following.

**Corollary 11.3.6.** Let $G$ be a finite group, and let $A$ be a unital $C^*$-algebra satisfying $A \otimes M_{|G|^\infty} \cong A$. Then the natural map $\mathrm{Rok}_G(A) \to \mathrm{Hom}(G, \mathrm{Aut}_{\mathrm{aue}}(A))$ is bijective and induces a bijection between $\overline{\mathrm{Rok}}_G(A)$ and $\mathrm{Hom}(G, \mathrm{Aut}_{\mathrm{aue}}(A))$.

*Proof.* This follows immediately from Corollary 11.2.6 and Theorem 11.3.5. □

## 11.4 Duality for Rokhlin actions

## 11.5 Crossed products of Rokhlin actions

In this section, we present a systematic study of the structure of crossed products and fixed point algebras by finite group actions with the Rokhlin property. Our main technical result is the existence of an approximate homomorphism from the algebra to its subalgebra of fixed points, which is a left inverse for the canonical inclusion. Combining this with arguments involving weak semiprojectivity, it is shown that a number of classes, characterized by inductive limit decompositions with weakly semiprojective building blocks, are closed under formation of crossed products by such actions. We will also show that a number of structural properties, which have been thoroughly studied in the classification of simple, nuclear $C^*$-algebras, are also inherited by the crossed product and the fixed point algebra whenever the action has the Rokhlin property.

The results and arguments in this section are based on [25]. Some of the results here presented were also independently obtained in [44], [69], and [2], but we chose to follow [25] since the methods there developed, specifically the existence of a sequence of completely positive contractive maps $A \to A^\alpha$ that asymptotically fix $A^\alpha$, lead to the most general results.

One feature of the approach we follow here, is that the attention is shifted from the crossed product to the fixed point algebra. We have seen in Theorem 6.1.5 that if a finite (or even compact) group $G$ acts on a $C^*$-algebra $A$, then $A^G$ is a corner in $A \rtimes G$. Using this, one can many times obtain information about the fixed point algebra through the crossed product. When this corner is full, then one may also use this result to transfer information from the fixed point algebra to the crossed product. We therefore begin by showing that this is indeed the case whenever the action has the Rokhlin property.

**Proposition 11.5.1.** Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ is Morita equivalent to $A \rtimes_\alpha G$, and hence they are stably isomorphic.

*Proof.* We will use the notation from Proposition 6.1.10. Fix $g \in G$, and denote by $u_g$ the canonical unitary in the crossed product $A \rtimes_\alpha G$ implementing $\alpha_g$. We claim that it is enough to show that $u_g$ is in the closed linear span of the functions $\widetilde{a^*} * \widetilde{b}$, for $a, b \in A$. Indeed, if this is the case, and if $x \in A$, then $xu_g$ also belongs to the closed linear span, and elements of this form span $A \rtimes_\alpha G$.

For $n \in \mathbb{N}$, find projections $p_g^{(n)} \in A$, for $g \in G$, such that

1. $\left\|\alpha_g(p_h^{(n)}) - p_{gh}^{(n)}\right\| < \frac{1}{n}$ for all $g, h \in G$; and
2. $\sum\limits_{g\in G} p_g^{(n)} = 1$.

For $a, b \in A$, the function $\widetilde{a^*} * \widetilde{b} \in A \rtimes_\alpha G$ can be written as a linear combination of the canonical unitaries in the following way:

$$\widetilde{a^*} * \widetilde{b} = \left(\sum_{h\in G} \alpha_h(b) u_h\right).$$

Thus, for $n \in \mathbb{N}$ and $k \in G$, we have

$$\widetilde{p_{gk}^{(n)*}} * \widetilde{p_k^{(n)}} = p_{gk}^{(n)} \left(\sum_{h\in G} \alpha_h(p_k^{(n)}) u_h\right).$$

We use pairwise orthogonality of the projections $p_g^{(n)}$, for $g \in G$, at the third step, to get

$$\begin{aligned}
\left\|\widetilde{p_{gk}^{(n)*}} * \widetilde{p_k^{(n)}} - p_{gk}^{(n)} u_g\right\| &= \left\|p_{gk}^{(n)} \left(\sum_{h\in G} p_{gk}^{(n)} \alpha_h(p_k^{(n)}) u_h\right) - p_{gk}^{(n)} u_g\right\| \\
&\leq \left\|p_{gk}^{(n)} \alpha_g(p_k^{(n)}) u_h - p_{gk}^{(n)} u_h\right\| + \sum_{h\in G, h\neq g} \left\|p_{gk}^{(n)} \alpha_h(p_k^{(n)}) u_h\right\| \\
&< \left\|\alpha_g(p_k^{(n)}) - p_{gk}^{(n)}\right\| + \sum_{h\in G, h\neq g} \left\|\alpha_h(p_k^{(n)}) - p_{hk}^{(n)}\right\| \\
&< \frac{1}{n} + (|G| - 1)\frac{1}{n} = \frac{|G|}{n}.
\end{aligned}$$

It follows from condition (2) above that

$$\limsup_{n\to\infty}\left\|\sum_{k\in G}\widetilde{p_{gk}^{(n)*}}*\widetilde{p_k^{(n)}}-u_g\right\|\leq\limsup_{n\to\infty}\frac{|G|^2}{n}=0.$$

By Proposition 6.1.10, we deduce that $u_g$ belongs to the closed two sided ideal in $A\rtimes_\alpha G$ generated by $A^\alpha$. In particular, $A^\alpha$ is Morita equivalent to $A\rtimes_\alpha G$. Since both algebras are unital, we conclude that they are stably isomorphic by Theorem 2.5.11. □

In view of the previous proposition, in order to understand the structure of the crossed product of an action with the Rokhlin property, it suffices to understand the structure of its fixed point algebra. Indeed, we will show that a number of properties pass from $A$ to $A^\alpha$ and $A\rtimes_\alpha G$. The properties we will consider are all preserved by stable isomorphism, and hence our strategy will be to show first that the property in question passes to the fixed point algebra, and then use Proposition 11.5.1 to deduce the same for the crossed product.

Our main tool to obtain information about the fixed point algebra is the following.

**Theorem 11.5.2.** Let $A$ be a unital $C^*$-algebra, let $G$ be a finite group, and let $\alpha\colon G\to\mathrm{Aut}(A)$ be an action with the Rokhlin property. Given a finite subset $F_1\subseteq A$, a finite subset $F_2\subseteq A^\alpha$ and $\varepsilon>0$, there exists a unital completely positive map $\psi\colon A\to A^\alpha$ such that

1. For all $a,b\in F_1$, we have

$$\|\psi(ab)-\psi(a)\psi(b)\|<\varepsilon;$$

2. For all $a\in F_2$, we have $\|\psi(a)-a\|<\varepsilon$.

In particular, when $A$ is separable, there exists an approximate homomorphism $(\psi_n)_{n\in\mathbb{N}}$ consisting of unital completely positive linear maps $\psi_n\colon A\to A^\alpha$ for $n\in\mathbb{N}$, such that $\lim\limits_{n\to\infty}\|\psi_n(a)-a\|=0$ for all $a\in A^\alpha$.

*Proof.* Set $F=F_1^2\cup F_2$. Without loss of generality, we may assume that $\|a\|\leq 1$ for all $a\in F$. Use the Rokhlin property for $\alpha$, in the form given in Remark 11.1.5, to find projections $p_g\in A$, for $g\in G$, satisfying

(a) $\alpha_g(p_h)=p_{gh}$ for all $g,h\in G$;

(b) $\sum\limits_{g\in G}p_g=1$;

(c) $\|p_ga-ap_g\|<\varepsilon$ for all $g\in G$ and all $a\in F$.

Define a linear map $\psi\colon A \to A^\alpha$ by

$$\psi(a) = \sum_{g\in G} p_g \alpha_g(a) p_g$$

for all $a \in A$. We claim that $\psi$ has the desired properties. We first check that the range of $\psi$ really is contained in $A^\alpha$. For $h \in G$ and $a \in A$, we use condition (a) above at the second step te get

$$\alpha_h(\psi(a)) = \sum_{g\in G} \alpha_h(p_g)\alpha_{hg}(a)\alpha_h(p_g) = \sum_{g\in G} p_{hg}\alpha_{hg}(a)p_{hg} = \psi(a),$$

as desired.

It is also clear that $\psi$ is unital and completely positive. Let $a, b \in F_1$. We use condition (c) at the third step to get

$$\begin{aligned}
\psi(a)\psi(b) &= \sum_{g,h\in G} p_g\alpha_g(a)p_g p_h\alpha_h(a)p_h \\
&= \sum_{g\in G} p_g\alpha_g(a)p_g\alpha_g(b)p_g \\
&\approx_{\varepsilon} \sum_{g\in G} p_g\alpha_g(a)\alpha_g(b)p_g \\
&= \psi(ab).
\end{aligned}$$

Finally, given $a \in F_2 \subseteq A^\alpha$, we use condition (c) at the second step to get

$$\psi(a) = \sum_{g,h\in G} p_g\alpha_g(a)p_g \approx_{\varepsilon} \sum_{g,h\in G} p_g\alpha_g(a) = a,$$

thus completing the proof of the theorem. □

In some concrete applications, it may be more convenient to have a map $\psi\colon A \to A^\alpha$ that is *exactly* the indentity on $A^\alpha$. This is possible, at the cost that the resulting map will not in general be completely positive. Since we will not use this variant of Theorem 11.5.2, we leave it as an exercise.

**Exercise 11.5.3.** Let $A$ be a unital $C^*$-algebra, let $G$ be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Given a finite subset $F \subseteq A$ and $\varepsilon > 0$, show that there exists a unital map $\psi\colon A \to A^\alpha$ such that

1. For all $a, b \in F_1$, we have

$$\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon \quad \text{and} \quad \|\psi(a)^* - \psi(a^*)\| < \varepsilon;$$

2. For all $x \in A^\alpha$, we have $\psi(x) = x$.

Using Theorem 11.5.2, it is possible to show that a variety of structural properties pass from $A$ to $A^\alpha$ and $A\rtimes_\alpha G$. The following result, which combines results from [69], [44] and [25], summarizes some of the most important ones:

**Theorem 11.5.4.** The following classes of unital $C^*$-algebras are closed under formation of crossed products and passage to fixed point algebras by actions of finite groups with the Rokhlin property:

1. $C^*$-algebras that are direct limits of certain weakly semiprojective $C^*$-algebras. This includes UHF-algebras, AF-algebras, AI-algebras, A$\mathbb{T}$-algebras, countable inductive limits of one-dimensional NCCW-complexes, and several other classes;

2. Kirchberg algebras;

3. Simple $C^*$-algebras with tracial rank at most one;

4. Simple, separable, $C^*$-algebras satisfying the Universal Coefficient Theorem;

5. $C^*$-algebras with nuclear dimension at most $n$, for $n \in \mathbb{N}$;

6. $C^*$-algebras with decomposition rank at most $n$, for $n \in \mathbb{N}$;

7. $C^*$-algebras with real rank zero or stable rank one;

8. $C^*$-algebras with strict comparison of positive elements;

9. Separable $\mathcal{D}$-absorbing $C^*$-algebras, for a strongly self-absorbing $C^*$-algebra $\mathcal{D}$;

10. $C^*$-algebras whose $K$-groups are either: trivial, free, torsion-free, torsion, or finitely generated;

11. Weakly semiprojective $C^*$-algebras.

All of the properties listed in the theorem above are stable under Morita equivalence, so it suffices to prove the claims for the fixed point algebra. To do this, one should first prove that if $\iota\colon B \to A$ is a unital inclusion of $C^*$-algebras and there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of unital compeltely positive maps $\psi_n\colon A \to B$ that is asymptotically multiplicative and asymptotically the identity on $B$, then all of the properties listed in Theorem 11.5.4 pass from $A$ to $B$. The proof of the theorem is then completed by applying this observation with $B = A^\alpha$ in combination with Theorem 11.5.2.

As an example, we explain in detail how to prove the part of item (7) that refers to stable rank one[4].

[4] Recall that a unital $C^*$-algebra has *stable rank one* if the invertible elements are dense.

*Proof.* Suppose that $A$ has stable rank one. Let $a \in A^\alpha$ and let $\varepsilon > 0$. Without loss of generality, we may assume that $\varepsilon < 2$. Find $x \in A$ invertible with $\|x - a\| < \varepsilon/2$. Let $\psi\colon A \to A^\alpha$ be a unital completely positive map as in the conclusion of Theorem 11.5.2 for $F_1 = \{x, x^{-1}\}$ and $F_2 = \{a\}$ with tolerance $\varepsilon/2$. Then

$$\|\psi(x)\psi(x^{-1}) - 1\| < \frac{\varepsilon}{2} < 1 \quad \text{and} \quad \|\psi(x^{-1})\psi(x) - 1\| < \frac{\varepsilon}{2} < 1.$$

It follows that $\psi(x)\psi(x^{-1})$ and $\psi(x^{-1})\psi(x)$ are invertible, so there exist $b, c \in A^\alpha$ with $\psi(x)\psi(x^{-1})b = 1$ and $c\psi(x^{-1})\psi(x) = 1$. In particular, $\psi(x)$ is left and right invertible in $A^\alpha$, so it is invertible. Since

$$\|\psi(x) - a\| \leq \frac{\varepsilon}{2} + \|\psi(x) - \psi(a)\| \leq \varepsilon,$$

we conclude that $A^\alpha$ has stable rank one. $\square$

With the exception of preservation of the UCT, the remaining statements in Theorem 11.5.2 can be proved using similar ideas, and we leave the verification as an exercise.

**Exercise 11.5.5.** Complete the proof of Theorem 11.5.2, possibly skiping item (4).

# Chapter 12

# The (weak) tracial Rokhlin property

# Chapter 13

# Rokhlin dimension

The noncommuting tower version is easy to cover: just prove finiteness of nuclear dimension is preserved. Duality theory, ideals in crossed products. It would make sense to have a second subsection with the commuting tower version. There I can add the stuff from GHS, example of difference, the non-existence results of Hirshberg-Phillips and myself. One has to be careful because this will take more than one lecture...

# Chapter 14

# Strongly outer actions on $\mathcal{Z}$-stable $C^*$-algebras

# Appendix A

# Structure and classification of $C^*$-algebras

In this appendix, we give an overview of the historical developments surrounding one of the main driving forces in $C^*$-algebra theory in the last three decades: the Elliott classification programme. The material contained here will help in the understanding and contextualization of the topics covered in the second part of these lecture notes. No proofs will be given here, and we will refer the reader to the literature for further details instead.

## A.1 The beginnings of the Elliott programme

In the early 1990s, George Elliott initiated a programme to classify simple, separable, nuclear $C^*$-algebras in terms of $K$-theory. The evidence available at the time was in retrospect rather limited, and it consisted of Glimm's classification of UHF-algebras [33]; the subsequent work of Elliott on AF-algebras [18]; and Elliott's classification of A$\mathbb{T}$-algebras[1] of real rank zero [19]. The precise form of the invariant saw some changes once it was realized that $K$-theory by itself would not be enough to classify algebras without sufficiently many projections. This invariant is now known as the *Elliott invariant*, and is defined as follows.

**Definition A.1.1.** Let $A$ be a unital $C^*$-algebra. We define its *Elliott invariant* $\mathrm{Ell}(A)$ to be the quadruple

$$\mathrm{Ell}(A) = ((K_0(A), K_0(A)_+, [1_A]_0), K_1(A), T(A), \rho),$$

where $\rho\colon K_0(A) \times T(A) \to \mathbb{R}$ is the canonical pairing determined by $\rho([p]_0, \tau) = \tau(p)$ for all projections $p \in A \otimes \mathcal{K}$ and all traces $\tau \in T(A)$.

There is a natural notion of equivalence of Elliott invariants, where the objects involved are supposed to be isomorphic in such a way that the pairings of the different tuples fit into a commutative diagram.

[1] An *A$\mathbb{T}$-algebra* is a direct limit of algebras of the form $F \otimes C(\mathbb{T})$, where $F$ is a finite dimensional $C^*$-algebra.

Elliott's original conjecture is the following:

**Conjecture A.1.2.** Let $A$ and $B$ be simple, separable, unital, nuclear $C^*$-algebras. Then $A$ and $B$ are isomorphic if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Some words about the conditions in Conjecture A.1.2 are in order. Unitality is the least crucial of all assumptions, and it is in fact not strictly necessary (although the invariant in the non-unital case is slightly different). Simplicity is a very common assumption throughout mathematics, and the classification of simple $C^*$-algebras is a very natural place to start if one wishes to classify more complicated algebras. Finally, separability and nuclearity are included as assumptions because one should not expect to obtain reasonable classification results for $C^*$-algebras whose weak closures in their GNS representations are not classified or even properly understood. Weak closures in GNS constructions are separable (in the sense of tracial von Neumann algebras) if the original $C^*$-algebra is separable, and they are all hyperfinite if and only if the $C^*$-algebra is nuclear. (And if the state is an extreme trace, then the weak closure is a factor, indeed the hyperfinite $\mathrm{II}_1$-factor $\mathcal{R}$.) In particular, separability and nuclearity are conditions that guarantee the classifiability of the weak closures.

In this formulation, Elliott's conjecture saw remarkable success in the 1990's. For example, Elliott's classification of AF-algebras is a confirming example, since for unital AF-algebras the Elliott invariant can be seen to be equivalent to the triple $(K_0(A), K_0(A)^+, [1_A]_0)$. The same is true for Elliott's classification of A$\mathbb{T}$-algebras of real rank zero in terms of $K_0$ and $K_1$.

## A.2 Kirchberg algebras: a major success, modulo the UCT

Perhaps the most outstanding success of this programme came with the classification of purely infinite $C^*$-algebras, so we define this notion next.

**Definition A.2.1.** A simple, unital $C^*$-algebra $A$ is said to be *purely infinite* if for all $a \in A \setminus \{0\}$ there exist $x, y \in A$ such that $xay = 1$.

Examples of purely infinite simple algebras are the Cuntz algebras $\mathcal{O}_n$ (these are also nuclear and separable), as well as the Calkin algebra (which is not nuclear, or even separable). Purely infinite simple $C^*$-algebras do not admit any trace, so their Elliott invariant reduces to $K$-theory.

A highly praised result of Kirchberg and Phillips [51, 72] from the late 1990's confirms Elliott's conjecture in the purely infinite case. This result builds on vast amounts of work of several authors, among which Kirchberg stands out. For this reason, a purely infinite, simple, separable and nuclear $C^*$-algebras is commonly referred to as a *Kirchberg* algebra.

**Theorem A.2.2.** [Kirchberg-Phillips]. Let $A$ and $B$ be unital Kirchberg algebras satisfying the Universal Coefficient Theorem (UCT). Then $A$ and $B$ are isomorphic if and only if $(K_0(A), [1_A]_0) \cong (K_0(B), [1_B]_0)$ and $K_1(A) \cong K_1(B)$.

The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ played a pivotal role in the proof of the above theorem (at least in the approach taken in [72]), thanks to the following facts (known as Kirchberg's Geneva Theorems [52]):

- If $A$ is a Kirchberg algebra, then $A \otimes \mathcal{O}_\infty \cong A$;
- If $A$ is a simple, separable, unital, exact $C^*$-algebra, then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

The reader will note that Theorem A.2.2 has an unexpected assumption: the UCT. Accordingly, we define it next.

**Definition A.2.3.** Let $A$ and $B$ be separable $C^*$-algebras. We say that the pair $(A, B)$ *satisfies the UCT* if the following conditions are satisfied:

1. The natural map $\tau_{A,B} \colon KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$ defined by Kasparov in [50], is surjective.
2. The natural map $\mu_{A,B} \colon \ker(\tau_{A,B}) \to \mathrm{Ext}(K_*(A), K_{*+1}(B))$ is an isomorphism.

If this is the case, by setting $\varepsilon_{A,B} = \mu_{A,B}^{-1} \colon \mathrm{Ext}(K_*(A), K_{*+1}(B)) \to KK(A,B)$, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_*(A), K_{*+1}(B)) \xrightarrow{\varepsilon_{A,B}} KK(A,B) \xrightarrow{\tau_{A,B}} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0,$$

which is natural on both variables because so are $\tau_{A,B}$ and $\mu_{A,B}$.

We further say that $A$ *satisfies the UCT*, if $(A, B)$ satisfies the UCT for every separable $C^*$-algebra $B$.

Phillips also showed that any Kirchberg algebra, satisfying the UCT or not, has the same $K$-theory as a Kirchberg algebra that *does* satisfy the UCT. In particular, if there exist Kirchberg algebras that do not satisfy the UCT, then these algebras will not be able to be classified using the Elliott invariant.

Not being able to prove that it holds automatically in the nuclear case[2], the community began to regard the UCT as an additional condition that ought to be added to the Elliott conjecture. The UCT assumption remains until today as a rather mysterious one, and the problem of whether a nuclear $C^*$-algebra satisfies the UCT is a very relevant one, known as the *UCT problem*.

## A.3 AH-algebras and the power of Elliott's intertwining argument

The arguments used by Glimm in the classification of UHF-algebras [33], and by Elliott both in the classification of AF-algebras [18] and of A$\mathbb{T}$-algebras [19], relied on a very powerful technique of "intertwining" two inductive systems corresponding to different algebras to obtain an isomorphism between them.

[2]There are counterexamples for exact $C^*$-algebras.

This technique, which became known as the *Elliott intertwining*, allows for a surprising level of flexibility, and it was then quickly realized that it could be adapted to deal with much more general cases. In the equivariant setting, the reader will find an application of a basic form of this intertwining argument in Theorem 11.2.5.

Elliott's intertwining argument is particularly powerful whenever the algebras in question can be constructed as inductive limits of algebras in a specified class, which are usually referred to as "building blocks". For AF-algebras, the building blocks are the finite dimensional $C^*$-algebras, while for A$\mathbb{T}$-algebras the building blocks are the circle algebras, namely sums of algebras of the form $M_n(C(\mathbb{T}))$. In this context, one wishes to understand when an isomorphism between the Elliott invariants is induced by homomorphisms between the building blocks, so as to "intertwine" the inductive systems. Once one knows that a certain map between building blocks exists, it is also necessary to know when two such maps are (approximately) unitarily equivalent. The proof of Theorem 11.2.5 showcases how existence and uniqueness results for maps can be used to obtain a classification theorem.

Despite the fact that some inductive limit $C^*$-algebras may appear to have a very particular, perhaps even artificial, form, the study of such objects gained significant impetus once it was proved by Elliott and Evans [20] that all irrational rotation algebras are A$\mathbb{T}$-algebras. Furthermore, and at least shortly after Elliott formulated his conjecture, it was surprising that $K$-theory does indeed classify non-trivial classes of $C^*$-algebras, and particular inductive limits were among the first classes for which this could be established. The extent to which this could be pushed is surprising, and we describe next the most advanced result in this direction.

A *homogeneous algebra* is a sum of algebras of the form $pM_n(C(X))p$, where $X$ is a compact, Hausdorff space and $p \in M_n(C(X))$ is a projection. An *AH-algebra* is a direct limit of homogeneous algebras. Finally, an AH-algebra is said to have *very slow dimension growth* if, very roughly speaking, the covering dimensions of the spaces appearing in some AH-decomposition increase much slower than the ranks of the projections in the decomposition.

AH-algebras are automatically separable, nuclear, and satisfy the UCT. In particular, when they are simple and unital, they fall within the class of algebras covered by Elliott's conjecture. Indeed, building on vasts amount of previous work, simple AH-algebras with very slow dimension growth were classified by Elliott-Gong-Li [22] and Gong [35]:

**Theorem A.3.1.** Let $A$ and $B$ be simple, unital AH-algebras of very slow dimension growth. Then $A$ and $B$ are isomorphic if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Later results show that the assumption of very slow dimension growth is equivalent to formally weaker conditions on the growth of the AH-algebra (specifically what is known as *slow* dimension growth); see [104].

`homogeneous is something else, and ASH classification missing.`

## A.4 Classification through tracial approximations

Despite the power of Elliott's intertwining argument, most of the natural examples of simple, nuclear $C^*$-algebras rarely come with canonical inductive limit decompositions using more tractable building blocks. It was thus quickly realized that it was necessary to obtain classification results using assumptions that could be checked in cases where the algebra is not given as a direct limit.

A major breakthrough in this direction was Lin's study and classification of $C^*$-algebras of tracial rank zero. Roughly speaking[3], a simple unital $C^*$-algebra $A$ is said to have *tracial rank zero* if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$ there exist a projection $p \in A$, a finite dimensional $C^*$-algebra $C$, and a unital homomorphism $\varphi\colon C \to pAp$ such that, with $p^\perp = 1 - p$, one has $\|a - pap - p^\perp a p^\perp\| < \varepsilon$ for all $a \in F$ and $\tau(p^\perp) < \varepsilon$ for all $\tau \in T(A)$.

Lin's result from [57] reads as follows:

**Theorem A.4.1.** Let $A$ and $B$ be separable, simple, unital, nuclear $C^*$-algebras of tracial rank zero satisfying the UCT. Then $A$ and $B$ are isomorphic if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

One of the main advantages of the above result is that its main hypothesis, tracial rank zero, can be verified for a wide variety of $C^*$-algebras, including many arising from topological dynamical systems ??, or even noncommutative dynamical systems ??, without having to prove that the algebras in question admit tractable inductive limit decompositions.

Lin's tracial rank zero classification was shortly after extended to algebras of *tracial rank one* [58].

## A.5 Exotic examples: from Jiang-Su to Villadsen, Rørdam and Toms

The mid 1990s saw the construction of simple nuclear $C^*$-algebras which were not believed (or hoped) to exist. These examples would later revolutionize the Elliott programme, since some of these pathological examples unequivocally demonstrated the need to modify Elliott's conjecture.

In his monograph [49], Kaplansky asked whether every simple $C^*$-algebra must contain a nontrivial projection. The consensus at the time seems to be that this would not be the case, and that the most likely counterexample would be either the irrational rotation algebra $A_\theta$ or the reduced group $C^*$-algebra $C^*_\lambda(\mathbb{F}_2)$. The first candidate turned out to have lots of nontrivial projections (see [83]), while the second one was indeed proved to be simple [79] and projectionless [78]. It was then asked whether a *nuclear* simple $C^*$-algebra must contain a nontrivial projection. Even this question has a negative answer, as Blackadar showed in [8] and [9]. A later construction due to Jiang and Su [47] (which, with today's knowledge, is known to be the same as the one constructed

[3] This formulation is only correct whenever $A$ has strict comparison of positive elements by traces.

by Blackadar in [9]) attracted a great deal of attention due to its connections to Elliott's classification programme. Indeed, the algebra $\mathcal{Z}$ constructed in [47], now known as the *Jiang-Su algebra*, is an infinite-dimensional simple, separable, unital, nuclear $C^*$-algebra satisfying the UCT with the same Elliott invariant as the complex numbers. Even more, under a very mild condition on $K_0(A)$ called *weak unperforation*[4], any $C^*$-algebra $A$ as in the assumptions of Conjecture A.1.2 satisfies $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{Z})$; see [36].

It follows that at the level of the Elliott invariant, and when $K_0$ is weakly unperforated, the Jiang-Su algebra $\mathcal{Z}$ acts as a tensorial unit. Hence, if the Elliott Conjecture were to hold, every $C^*$-algebra with weakly unperforated $K_0$-group should be isomorphic to its tensor product with $\mathcal{Z}$. These observations motivated significant efforts to understand the structure of $\mathcal{Z}$-stable $C^*$-algebras, which were regarded as the $C^*$-algebraic analogs of the McDuff factors. Among many other remarkable properties, $\mathcal{Z}$-stable $C^*$-algebras enjoy a formidable dichotomy relative to the structure of their projections: a simple $\mathcal{Z}$-stable $C^*$-algebra is either stably finite or purely infinite; see [36]. In particular, simple $\mathcal{Z}$-stable $C^*$-algebras admit a type-like characterization analogous to von Neumann factors.

The question of whether every infinite-dimensional $C^*$-algebra as in Conjecture A.1.2 has weakly unperforated $K_0$-group was open at the time, and it was settled shortly after by Villadsen [97]. Villadsen's construction of a simple, separable, nuclear $C^*$-algebra whose $K_0$-group is not weakly unperforated answered a long-standing question of Blackadar, the order on projections in this algebra is not determined by their behavior on traces. (This is in stark contrast to the situation for $\mathrm{II}_1$-factors; see Theorem 10.1.4.) The methods and constructions used by Villadsen were later refined and extended to construct a number of "pathological" examples of nuclear $C^*$-algebras, two which we proceed to discuss.

In [84], Rørdam showed that there exists a simple, separable, nuclear $C^*$-algebra containing a finite and an infinite projection; this $C^*$-algebra is in particular infinite but not purely infinite, and hence does not satisfy the dichotomy that $\mathcal{Z}$-stable $C^*$-algebras do. Perhaps most importantly, Rørdam's example showed that Elliott's conjecture, at least in its original form, was not true (even if the UCT is added as an assumption): the $C^*$-algebra he constructed has the same Elliott invariant as a purely infinite algebra (namely, its tensor product with $\mathcal{Z}$), yet it is not itself a Kirchberg algebra. Shortly after, Toms [93] constructed two $C^*$-algebras $A$ and $B$ that could not be distinguished by virtually any "reasonable" functor[5], including the Elliott invariant, the real and stable ranks, and any homotopy-invariant continuous functor, yet were not isomorphic. The construction of these algebras follows ideas of Villadsen and is simpler than related constructions: the algebra $A$ is constructed as a direct limit of algebras of the form $M_k(C([0,1]^n))$, with connecting maps defined us-

[4]An ordered group $G$ is said to be *weakly unperforated* if for any $g \in G$, one has $g \geq 0$ if and only if for some (for all) $n \in \{2, 3, \dots\}$ one has $ng \geq 0$.

[5]One for which it was conceivable that a range result could also be proved.

ing diagonal maps and point evaluations. (Of course, the choice of matrix sizes, dimension of the space, and connecting maps must be done with extreme care.) The $C^*$-algebra $B$ is then the tensor product of $A$ with any UHF-algebra (and it could also be taken to be $A \otimes \mathcal{Z}$).

A natural reaction to the incompleteness of the Elliott invariant is to expand it to include whatever data was used to prove its incompleteness. This is, however, not always a good idea. For once, it is possible that the resulting invariant is still incomplete, thus requiring further modifications. Even worse, one may incorporate so much new information that a result proving its completeness may lack any practical impact. In this sense, not all counterexamples are of the same quality.

While the examples of Rørdam could be distinguished using their stable rank, Tom's examples required a much finer invariant: the Cuntz semigroup. However, since range results for Cuntz semigroups are still until today out of reach, any classification using this invariant would not be accompanied with any result describing the range of the invariant (which all previous results did). Toms' counterexample was therefore regarded as evidence *not* of the fact that the invariant must be enlargened, but rather that the class of $C^*$-algebras one wishes to classify must be restricted. The results from [36] that were discussed above strongly suggest that the class of separable, simple, nuclear, unital $C^*$-algebras that are $\mathcal{Z}$-stable may be particularly tractable, and that the Elliott invariant may not be the right invariant to use outside the $\mathcal{Z}$-stable case. Further evidence of the fact that $\mathcal{Z}$-stable $C^*$-algebras are distinctly tame, was provided by Rørdam in [85], where it is shown that $\mathcal{Z}$-stable $C^*$-algebras always have strict comparison of positive elements by quasitraces[6]; and that they have stable rank one whenever they are finite.

The possibility that $\mathcal{Z}$-stability had to be added to the assumptions in the Elliott conjecture was quickly agreed upon, and this led to the *revised Elliott conjecture*:

**Conjecture A.5.1.** Let $A$ and $B$ be simple, separable, unital, nuclear, $\mathcal{Z}$-stable $C^*$-algebras. Then $A$ and $B$ are isomorphic if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

In retrospect, the necessity of some form of tensorial absorption should not come as a surprise, since tensorial stability is ubiqutous in results about classification and structure of operator algebra. For example, a major step in Connes' groundbreaking work [12] involved showing that an injective $\mathrm{II}_1$-factor absorbs the hyperfinite $\mathrm{II}_1$-factor tensorially; similarly, a highly praised result of Kirchberg [52] asserts that a simple, nuclear, separable $C^*$-algebra is purely infinite if and only if it absorbs the Cuntz algebra $\mathcal{O}_\infty$ tensorially.

Whether the UCT has to be included as an assumption in the above conjecture is still up to debate. In particular, a verification of Conjecture A.5.1 would imply a positive solution to the UCT question. The fact that a simple

[6] This means the following: for positive elements $a, b \in M_\infty(A)$, one has $a \precsim b$ whenever $d_\tau(a) < d_\tau(b)$ for all 2-quasitraces $\tau$ on $A$, where $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$.

$\mathcal{Z}$-stable $C^*$-algebra is either stably finite or purely infinite, implies that Conjecture A.5.1 has two cases: the (purely) infinite and the (stably) finite. Since the infinite case was beautifully settled by the work of Kirchberg and Phillips, all efforts towards confirming the revised version of Elliott's conjecture focused on the stably finite case. In this setting, a combination of very deep results of Cuntz [16], Blackadar-Handelman [7], and Haagerup [38], implies that any stably finite, simple, nuclear, unital $C^*$-algebra has a tracial state. This fact was of enormous significance since it opened the doors to the use of techniques from $\mathrm{II}_1$-factors in the study of stably finite, nuclear $C^*$-algebras, via the GNS construction.

## A.6 The Toms-Winter regularity conjecture

The study of $C^*$-algebra theory is usually referred to as *noncommutative topology*, since Gelfand's theorem establishes a duality between commutative $C^*$-algebras and locally compact Hausdorff spaces. This point of view has been very fruitful, particularly as topological tools, such as $K$-theory, have been successfully adapted to the $C^*$-algebraic setting. In this vein, it is natural to extend other topological notions to the noncommutative world. In the case of covering dimension, several different approaches have been proposed, including the real and stable rank. In the early 2000's, Winter studied new notions of covering dimension for $C^*$-algebras, which are finite only for nuclear $C^*$-algebras; see [100, 102]. Later, two new dimensional notions were introduced, which would play a revolutionary role in the study of the structure of nuclear $C^*$-algebras: the decomposition rank [54], and the nuclear dimension [106]. As in the case of their precursors, these are defined by regarding a completely positive approximation $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ of a nuclear $C^*$-algebra[7] as a noncommutative analog of an open covering, and the imposition of extra conditions on the maps $\varphi_\lambda$ can be used to define a dimensional invariant for the $C^*$-algebra in question. In order to describe what extra conditions are used, we introduce a definition.

**Definition A.6.1.** Let $\varphi\colon A \to B$ be a completely positive map between $C^*$-algebras $A$ and $B$.

1. We say that $\varphi$ has *order zero* if $\varphi(a)\varphi(b) = 0$ whenever $a, b \in A_+$ satisfy $ab = 0$.

2. For $n \in \mathbb{N}$, we say that $\varphi$ is *$n$-decomposable* if $A$ can be written as a direct sum $A = A_0 \oplus \cdots \oplus A_n$ and $\varphi|_{A_j}$ is order zero for all $j = 0, \ldots, n$.

[7] This means that $F_\lambda$ is finite dimensional, the maps $\psi_\lambda\colon A \to F_\lambda$ and $\varphi_\lambda\colon F_\lambda \to A$ are completely positive, with $\psi_\lambda$ contractive, and $\lim_\lambda \|\varphi_\lambda(\psi_\lambda(a)) - a\| = 0$ for all $a \in A$. One obtains an equivalent notion if one requires that the maps $\varphi_\lambda$ be contractive, which is how this notion is usually defined. However, for the purposes of this discussion, a completely positive approximation has bounded maps into the given $C^*$-algebra.

3. We say that $\varphi$ is *decomposable* if there exists $n \in \mathbb{N}$ such that $\varphi$ is $n$-decomposable.

In order to motivate the definitions of decomposition rank and nuclear dimension, we reproduce here a characterization of nuclearity using order zero approximations which was obtained by Hirshberg-Kirchberg-White [41].

**Theorem A.6.2.** Let $A$ be a $C^*$-algebra. Then $A$ is nuclear if and only if there exists a completely positive approximation $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ for $A$, where $\psi_\lambda$ is decomposable for all $\lambda \in \Lambda$.

The notions of nuclear dimension and decomposition rank are refinements of the above result, where one moreover imposes a uniform bound on the number of components of the maps $\psi_\lambda$.

**Definition A.6.3.** Let $A$ be a $C^*$-algebra, and let $n \in \mathbb{N}$.

1. We say that $A$ has *nuclear dimension at most* $n$, and write $\dim_{\mathrm{nuc}}(A) \leq n$, if there exists a completely positive approximation $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ for $A$, where $\psi_\lambda$ is $n$-decomposable for all $\lambda \in \Lambda$.
2. We say that $A$ has *decomposition rank at most* $n$, and write $\mathrm{dr}(A) \leq n$, if there exists a completely positive approximation $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ for $A$, where $\psi_\lambda$ is $n$-decomposable and contractive for all $\lambda \in \Lambda$.

The *nuclear dimension* $\dim_{\mathrm{nuc}}(A)$ and *decomposition rank* $\mathrm{dr}(A)$ of $A$ are defined as the smallest integers for which condition (1) or (2), as appropriate, are satisfied.

These dimensional notions enjoy a number of good properties: they do not increase when passing to ideals, hereditary subalgebras, or quotients; are invariant under Morita equivalence; behave well with respect to extensions and direct limits; and agree with covering dimension of the spectrum in the commutative setting. One clearly always has $\dim_{\mathrm{nuc}}(A) \leq \mathrm{dr}(A)$.

The difference between nuclear dimension and decomposition rank (contractivity of $\psi_\lambda$) seems like a minor one, perhaps even one that could be arranged via rescaling. However, as it turns out, there is a dramatic difference between both notions: while the decomposition rank is only possibly finite for (strongly) quasidiagonal $C^*$-algebras, nuclear dimension is finite for all Kirchberg algebras. However, when the decomposition rank of a simple $C^*$-algebra is finite, its value tends to agree with the nuclear dimension of the algebra[8].

The connections between finiteness of the nuclear dimension or decomposition rank, on the one hand, and classification and structure, on the other, were noticed early on. A first result in this direction draws a connection between finiteness of the decomposition rank and tracial rank zero: a unital, separable, simple $\mathcal{Z}$-stable $C^*$-algebra with finite decomposition rank has tracial rank zero if and only if it has real rank zero [101][9].

[8] This is now known to always be the case when the algebra satisfies the UCT, as we discuss later.

[9] As it was proved later in [103], the assumption of $\mathcal{Z}$-stability is automatic in this context.

In the light of the above result and the counterexamples to Elliott's conjecture mentioned in Section A.5, it is interesting to understand when the nuclear dimension of Villadsen's algebras is finite. This task was carried out by Toms and Winter in [95], where they showed that these algebras, finiteness of the decomposition rank, $\mathcal{Z}$-stability, and strict comparison, are equivalent, and the class of Villadsen algebras that satisfy these conditions is covered by Elliott's conjecture. These conditions make sense for arbitrary $C^*$-algebras not necessarily of Villadsen type, and the suspicion that the equivalence between them may hold in much greater generality began to emerge. This speculation chrystalized in the following regularity conjecture of Toms and Winter:

**Conjecture A.6.4.** Let $A$ be a simple, separable, unital, nuclear $C^*$-algebra. Then the following conditions are equivalent:

1. $\dim_{\mathrm{nuc}}(A) < \infty$.
2. $A$ is $\mathcal{Z}$-stable.
3. $A$ has strict comparison of positive elements.

When $A$ is finite, the above conditions are also equivalent to

4. $\mathrm{dr}(A) < \infty$.

Thus, the Toms-Winter conjecture predicts the equivalence between three seemingly unrelated properties of topological, analytic, and algebraic flavors. Despite their diverse nature, these regularity properties are all satisfied by those classes of $C^*$-algebras which have been successfully classified by the Elliott invariant, and they all fail for the exotic algebras that provide counterexamples to Elliott's conjecture.

The only obvious implication is (4) $\Rightarrow$ (1), while the implication (2) $\Rightarrow$ (3) had been verified by Rørdam [85]. Perhaps one of the major breakthroughs in the first decade of this century was the fact that $\mathcal{Z}$-stability, a necessary assumption to obtain classification by $K$-theory, is *implied* by finiteness of the decomposition rank [103], and even by finiteness of the nuclear dimension [104], at least in the simple, unital, and separable case, thus proving (1) $\Rightarrow$ (2) and hence establishing an expected implication between a topological property (finiteness of a "noncommutative covering dimension") and an algebraic property (tensorial absorption of $\mathcal{Z}$). This result had huge implications, since it is often much easier to verify that the nuclear dimension of a given $C^*$-algebra is finite, than verifying that it is $\mathcal{Z}$-stable. A remarkable example of this situation is given by a result of Toms-Winter [96], stating that minimal crossed product of homeomorphisms of finite dimensional spaces have finite nuclear dimension, and are hence $\mathcal{Z}$-stable.

These developments gave great impetus to the study of the structure of nuclear $C^*$-algebras, which underwent revolutionary and fast-paced progress, verifying several particular cases of Conjecture A.6.4. In one of their celebrated works, Matui and Sato [64] used novel von Neumann algebra methods to prove that (3) $\Rightarrow$ (2) whenever $A$ has finitely many extreme traces. This implication is now known to hold whenever the extreme boundary $\partial_e T(A)$ is compact and finite dimensional; see the independent works [53, 94, 87], which largely build on Ozawa's notion [70] of $W^*$-bundle. Using radically different techniques, Thiel has also shown that this implication holds whenever $A$ has stable rank one and "locally finite nuclear dimension", without any restrictions on the geometry of $T(A)$; see [91].

For a long time, the implication (2) $\Rightarrow$ (1) was only accessible through classification: the strategy was to identify a classifiable $C^*$-algebra with an explicit model which has finite nuclear dimension. Besides depending on vast amounts of technical machinery, this approach is necessarily restricted to $C^*$-algebras satisfying the UCT. In later breakthroughs, Matui and Sato [65], and Sato, White, and Winter [88], proved this implication under the assumption that $A$ has a unique trace. Finally, the implication (2) $\Rightarrow$ (1) has been very recently proved by Castillejos-Evington-Tikuisis-White-Winter in [11]. There, the authors also showed that, in the finite case, the combination of (1) and quasidiagonality of all traces is known to be equivalent to (4), and in this case one even gets $\mathrm{dr}(A) = \dim_{\mathrm{nuc}}(A) \in \{0, 1\}$, the value zero being realized precisely by the AF-algebras. For stably finite algebras satisfying the UCT, quasidiagonality of all of its traces is automatic by the main result of [92].

The proofs of (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1), in most of the situations where they are known to hold, use fair amounts of von Neumann algebra techniques. In this context, the passage from von Neumann algebras to $C^*$-algebras is, however, not completely direct. Among other things, it is necessary to reinterpret notions from von Neumann algebras in the $C^*$-algebraic setting. This rephrasing process is now usually referred to as "coloring"; we elaborate on this idea next.

Let us consider, for example, the notion of hyperfiniteness. If one wants to obtain its $C^*$-algebraic analog, the most naive thing to do is to replace the weak operator topology with the norm topology, and the resulting notion is the property of *being AF*. Such $C^*$-algebras, albeit being very tractable, represent a rather small class, and therefore are not a satisfactory conceptual analog of the hyperfinite von Neumann algebras. A more flexible and useful counterpart is played by those algebras that have finite nuclear dimension. The difference between these two notions is that in the definition of nuclear dimension, instead of having homomorphisms from finite dimensional $C^*$-algebras, one allows for a finite number of maps of order zero. Regarding these maps as having different "colors", we usually say that having finite nuclear dimension is the *colored analog* of hyperfiniteness, while being AF is the *strict analog*. As a further example, the strict analog of the $W^*$-Rokhlin property (10) is the $C^*$-algebraic Rokhlin property (11), while its colored analog is Rokhlin dimension (13).

The above reasoning motivates the principle of *coloring* von Neumann algebraic properties. Intuitively, this amounts to replacing projections with a sum of finitely many positive elements, and homomorphisms with a sum of finitely many order zero maps. The minimum number of positive elements (or order zero maps) needed is used to define a notion of *dimension*, in such a way that dimension zero corresponds to the strict analog of the property in question. For example, nuclear dimension zero is equivalent to being AF. This general philosophy is the guiding principle behind many recent advances, including many recent developments in $C^*$-dynamics.

## A.7 The end of the story

One of the most striking developments in the "new era" of the Elliott programme is what is sometimes referred to as Winter's deformation technique [105], which uses $\mathcal{Z}$-stability in an innovative way. Denote by $\mathcal{Q}$ the universal UHF-algebra, and let $p$ and $q$ be different prime numbers. In this context, the basic idea is to regard $A \otimes \mathcal{Z}$ as a stationary inductive limit of paths in $A \otimes \mathcal{Q}$ with endpoints in $A \otimes M_{p^\infty}$ and $A \otimes M_{q^\infty}$. Assuming that $A \otimes \mathcal{Q}, A \otimes M_{p^\infty}$ and $A \otimes M_{q^\infty}$ are classifiable in terms of their Elliott invariants, and that isomorphisms can be constructed in a continuous way, Winter showed that $A \otimes \mathcal{Z}$ can also be classified in terms of its Elliott invariant.

This breakthrough provided a new framework to obtain positive classification results, but it also demanded much stronger *uniqueness* results for isomorphisms, not only up to approximate unitary equivalence, but up to asymptotic unitary equivalence. In this context, several variants of the Basic Homotopy Lemma played key roles. Once the difficulties associated to this new approach were sorted out, it was proved that the class of simple, separable, nuclear, $\mathcal{Z}$-stable unital $C^*$-algebras which satisfy the UCT and whose tensor products with UHF-algebras have tracial rank at most one, can be classified in terms of their Elliott invariants; see [61, 105, 60, 59]. This class is indeed much larger than those classes that had been previously classified: for example, the Jiang-Su algebra itself has infinite tracial rank, but its tensor product with each UHF-algebra has tracial rank zero. This class was however known not to contain every separable, simple, nuclear, $\mathcal{Z}$-stable unital $C^*$-algebra satisfying the UCT [10], so more general classification theorems were needed.

In terms of proving isomorphism theorems, the final step in this direction was taken by Gong-Lin-Niu in [34], where they introduced the notion of *generalized tracial rank*, which is inspired in the notion of tracial rank but where point-line algebras are used in the tracial approximation. Again with the aid of Winter's deformation technique, the authors showed that the class $\mathcal{B}$ of simple, separable, nuclear, $\mathcal{Z}$-stable unital $C^*$-algebras which satisfy the UCT and whose tensor products with UHF-algebras have generalized tracial rank at most one, can be classified in terms of their Elliott invariants. Moreover, this class of algebras exhausts the possible values of the Elliott invariant, in the sense that if $A$ is a simple, separable, nuclear, $\mathcal{Z}$-stable unital $C^*$-algebra, then there exists a $C^*$-algebra $B$ in $\mathcal{B}$ such that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. This, however, does not on the face of it complete the proof of Conjecture A.5.1, since one does not

[10] The $K_0$-group of any algebra in this class is a simple, rational Riesz group, and there are also some constraints on the pairing map. For example, certain inductive limits of "point-line" algebras (also called Elliott-Thomsen building blocks) do not belong to this class.

a priori know whether one gets an isomorphism $A \cong B$. It does nevertheless show that, in order to complete the proof of the revised Elliott conjecture, it is necessary to understand the structure of simple, nuclear $\mathcal{Z}$-stable $C^*$-algebras, and in particular show that any such algebra automatically belongs to $\mathcal{B}$.

That this is the case follows, under the ever-present assumption of the UCT, by the combination of the works of Elliott-Gong-Lin-Niu [21], Tikuisis-White-Winter [92], and Castillejos-Evington-Tikuisis-White-Winter [11]. Indeed, in [21] it is shown that every finite, separable, simple, unital $C^*$-algebra with finite nuclear dimension (hence nuclear and $\mathcal{Z}$-stable), all of whose traces are quasidiagonal and which satisfies the UCT, belongs to $\mathcal{B}$. By the main result of [92], quasidiagonality of all traces is automatic in the previous setting, and this confirmed the Elliott conjecture for UCT algebras of finite nuclear dimension. Finally, it is shown in [11] that finiteness of the nuclear dimension follows from $\mathcal{Z}$-stability for algebras as in Conjecture A.5.1, and hence this conjecture is verified in the presence of the UCT. Whether this assumption is also automatic remains unknown, and this is one of the main open problems in the structure of nuclear $C^*$-algebras.

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