C^*-DYNAMICS AND CROSSED PRODUCTS
First exercise sheet

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You must hand in a minimum of three exercises (at least two from the second page) by November 14th.

Exercise 1. Set

$$u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(1) Prove that there is a well-defined action $\alpha\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(M_2)$ determined by $\alpha_{(1,0)} = \mathrm{Ad}(u)$ and $\alpha_{(0,1)} = \mathrm{Ad}(v)$.
(2) Prove that this is not an inner action, although $\alpha_g \in \mathrm{Inn}(M_2)$ for all $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise 2. Define a continuous function $u\colon S^1 \to M_2$ by

$$u_\zeta = \begin{pmatrix} \zeta+1 & i(\zeta-1) \\ i(\zeta-1) & -\zeta-1 \end{pmatrix}$$

for all $\zeta \in S^1$.

(1) Prove that u_ζ is a unitary for all $\zeta \in S^1$.
(2) Show that there is a well-defined action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(C(S^1, M_2))$ whose nontrivial automorphism is given by conjugation by u.
(3) Prove that this is not an inner action, although $\mathrm{Ad}(u)$ is an inner automorphism.

Exercise 3. Let A_0 be a C^*-algebra such that $A_0 \otimes_{\max} A_0$ is not isomorphic to $A_0 \otimes_{\min} A_0$. (One could take, for example, A_0 to be the reduced group C^*-algebra of $\mathbb{F}_2$.) Set $A = A_0 \oplus A_0$, and let $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ be the flip action. Denote by $\otimes_\gamma$ the C^*-norm on the algebraic tensor product $A \odot A$ satisfying

$$A \otimes_\gamma A = (A_0 \otimes_{\max} A_0) \oplus (A_0 \otimes_{\min} A_0).$$

Show that $\alpha_1 \odot \alpha_1$ does not extend to an isomorphism of $A \otimes_\gamma A$.

For the following exercise, recall the definition of the operations in $C_c(G, A, \alpha)$, using α-twisted convolution and α-twisted involution.

Exercise 4. Let G be a discrete group, let A be a unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. For $g \in G$, let $\delta_g \in C_c(G, A, \alpha)$ be the Kronecker delta.

(1) Prove that δ_1 is the unit of $C_c(G, A, \alpha)$ (and hence of $A \rtimes_\alpha G$ and $A \rtimes_{\lambda,\alpha} G$).
(2) Prove that the assignment $\delta\colon G \to A \rtimes_\alpha G$, given by $g \mapsto \delta_g$, is a unitary representation.
(3) Prove that $\delta_g a \delta_g^* = \alpha_g(a)$ for all $g \in G$ and all $a \in A$.

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Exercise 5. Let G be a discrete group, let H be a subgroup, and let G act on G/H by translation. Prove that there are canonical isomorphisms

$$c_0(G/H) \rtimes_{\mathtt{Lt}} G \cong C^*(H) \otimes \mathcal{K}(\ell^2(G/H))$$

and

$$c_0(G/H) \rtimes_{\lambda,\mathtt{Lt}} G \cong C^*_\lambda(H) \otimes \mathcal{K}(\ell^2(G/H)).$$

Exercise 6. Let G be a locally compact group, let A and B be C^*-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

(1) Show that there are natural isomorphisms

$$(A \otimes_{\max} B) \rtimes_{\alpha\otimes_{\max}\mathrm{id}_B} G \cong (A \rtimes_\alpha G) \otimes_{\max} B$$

and

$$(A \otimes_{\min} B) \rtimes_{\lambda,\alpha\otimes_{\min}\mathrm{id}_B} G \cong (A \rtimes_{\lambda,\alpha} G) \otimes_{\min} B.$$

(2) Can the previous item be generalized to the case when G acts non-trivially on B?

Exercise 7. Let $\mathbb{Z}_2$ act on S^1 via conjugation, and denote by $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(C(S^1))$ the induced action. Show, in full detail, that

$$C(S^1) \rtimes_\alpha \mathbb{Z}_2 \cong \{f \in C([-1,1], M_2)\colon f(1), f(-1) \text{ are diagonal}\}.$$

Exercise 8. Let $\mathbb{Z}_2$ act on $[-1,1]$ via multiplication by -1, and denote by $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(C([-1,1]))$ the induced action. Compute $C([-1,1]) \rtimes_\alpha \mathbb{Z}_2$.

Exercise 9. Let G be a discrete group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Let $\varphi\colon A \to \mathcal{B}(\mathcal{H})$ be a representation and let $(\mathcal{H}^G, \lambda^{\mathcal{H}}, \varphi^G)$ be its associated regular covariant representation. Let $F \subseteq G$ be a finite set, let $a_g \in A$, for $g \in F$, and set

$$a = \sum_{g\in F} a_g u_g \in C_c(G, A, \alpha) \subseteq A \rtimes_{\lambda,\alpha} G.$$

(1) For $\xi \in \mathcal{H}^G$ and $g \in G$, show that

$$((\varphi^G \rtimes \lambda^{\mathcal{H}})(a)\xi)(g) = \sum_{h\in G} \varphi(\alpha_{g^{-1}}(a_h)(\xi(h^{-1}g)).$$

(2) For $g \in G$, let $s_g \in \mathcal{B}(\mathcal{H}, \mathcal{H}^G)$ be the isometry which sends ξ to $\xi\delta_g$, for all $\xi \in \mathcal{H}$. For all $g, h \in G$, show that

$$s_g^*(\varphi^G \rtimes \lambda^{\mathcal{H}})(a)s_h = \varphi(\alpha_{g^{-1}}(a_{gh^{-1}})).$$

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