C^*-DYNAMICS AND CROSSED PRODUCTS
Second exercise sheet

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You must hand in a minimum of four exercises: at least one of 1 or 2; at least one of 3 or 4, and at least two of 5, 6, 7, 8, by November 28th.

Exercise 1. Let A be a C^*-algebra. Show in full detail (by completing the proofs given in class) that there exists a unique unital C^*-algebra $M(A)$ containing A as an essential ideal such that whenever A is an ideal in some C^*-algebra B, then there is a unique homomorphism $\varphi\colon B \to M(A)$ extending the identity on A and satisfying $\ker(\varphi) = \{b \in B \colon bA = \{0\}\}$. In other words, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, hook] & M(A) \\ & B. \arrow[u, dashed, "\exists!\varphi"'] \end{tikzcd}$$

Exercise 2. Let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a non-degenerate, injective representation of a C^*-algebra A on a Hilbert space $\mathcal{H}$. Show that $M(A)$ can be canonically identified with

$$\{T \in \mathcal{B}(\mathcal{H}) \colon T\pi(A) \subseteq \pi(A), \pi(A)T \subseteq \pi(A)\}.$$

For Exercise 3, you will need to understand the definition of an A-compact operator on a Hilbert A-module, and use the following lemma.

Lemma. Let A be a C^*-algebra, let $\mathcal{E}$ be a countably generated Hilbert A-module, and let $T \in \mathcal{K}_A(\mathcal{E})$ be a positive element. Then T is stricly positive if and only if it has dense range.

Exercise 3. Let A be a C^*-algebra, and let $\mathcal{E}$ be a countably generated Hilbert A-module. Show that

$$\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A,$$

as follows:

(1) If A^+ denotes the one-dimensional unitization of A, denote by $\mathcal{E}^+$ the Hilbert A^+-module which as a vector space is identical to $\mathcal{E}$, with the obvious extended action and the same A-valued inner product as $\mathcal{E}$. Show that if $\mathcal{E}^+ \oplus \mathcal{H}_{A^+} \cong \mathcal{H}_{A^+}$, then $\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A$. Deduce that it is enough to prove the theorem when A is unital (which we will assume from now on).

(2) Let $(\xi_n)_{n\in\mathbb{N}}$ be an enumeration of a countable generating set for $\mathcal{E}$, with each element repeated an infinite number of times, and let $(\delta_n)_{n\in\mathbb{N}}$ be the canonical orthonormal basis of $\mathcal{H}_A$. Show that there is a well-defined operator $T \in \mathcal{K}_A(\mathcal{H}_A, \mathcal{E} \oplus \mathcal{H}_A)$ that satisfies $T(\delta_n) = (\xi_n/2^n, \delta_n/4^n)$ for all $n \in \mathbb{N}$.

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(3) Show that T is injective and has dense range.
(4) Show that T^*T has dense range, and is hence strictly positive.
(5) Show that there is a unique well-defined operator $U \in \mathcal{L}(\mathcal{H}_A, \mathcal{E} \oplus \mathcal{H}_A)$ satisfying $U((T^*T)^{1/2}\xi) = T(\xi)$ for all $\xi \in \mathcal{E}$.
(6) Show that U is a unitary, concluding the proof of the theorem.

Exercise 4. Let A and B be C^*-algebras. Prove that $A \sim_M B$ if and only if there exist a C^*-algebra C and a projection $p \in M(C)$ with $\overline{CpC} = \overline{C(1-p)C} = C$ such that $pCp \cong A$ and $(1-p)C(1-p) \cong B$. For the "only if" implication, let $\mathcal{E}$ be an imprimitivity bimodule and consider

$$C = \left[\begin{array}{cc} A & \mathcal{E} \\ \mathcal{E}^* & B \end{array} \right] = \left\{ \left(\begin{array}{cc} a & \xi \\ \eta^* & b \end{array} \right) : a \in A, b \in B, \xi, \eta \in \mathcal{E} \right\}.$$

Define a canonical matrix-type product and involution on C. Let C act on $\mathcal{E} \oplus B$ by

$$\left(\begin{array}{cc} a & \xi \\ \eta^* & b \end{array} \right) \left(\begin{array}{c} \zeta \\ c \end{array} \right) = \left(\begin{array}{c} a \cdot \zeta + \xi \cdot c \\ \langle \eta, \zeta \rangle_B + bc \end{array} \right),$$

(1) Prove that C is a C^*-algebra with the induced operator norm. This C^*-algebra is called the *linking algebra* associated to $\mathcal{E}$.
(2) Show that C and $p = \left(\begin{array}{cc} 1_{M(A)} & 0 \\ 0 & 0 \end{array} \right)$ satisfy the conclusion of the theorem.
(3) Where is fullness of $\mathcal{E}$ used?

Exercise 5. Let G be a compact group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. For $a \in A$, let $\widetilde{a} \in C(G, A)$ be given by $\widetilde{a}(g) = \alpha_g(a)$ for all $g \in G$. Prove that the ideal in $A \rtimes_\alpha G$ generated by $c(A^\alpha)$ agrees with

$$\overline{\mathrm{span}}\{\widetilde{a}^* * \widetilde{b}\colon a, b \in A\}.$$

Moreover, show that for $a, b \in A$ we have $(\widetilde{a}^* * \widetilde{b})(g) = a^* \alpha_g(b)$ for all $g \in G$.

Exercise 6. Let G be a compact abelian group, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Show that

$$\widetilde{\mathrm{Sp}}(\alpha) = \{\chi \in \widehat{G}\colon \overline{A(\chi)AA(\chi^{-1})} = A\}.$$

Exercise 7. Let G be a compact group abelian, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. For $a, b \in A$, for $\chi \in \widehat{G}$, and for $p \in M(A \rtimes_\alpha G)$ the projection associated to the corner-embedding $c\colon A^\alpha \to A \rtimes_\alpha G$, show that

$$p * (\chi^{-1}\widetilde{a}^*) * (\chi^{-1}\widetilde{b}) * p = c_{E_\chi(a)^* E_\chi(b)}.$$

Exercise 8. Let G be a compact group abelian, let A be a C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. The goal of this exercise is to show that if every ideal in $A \rtimes_\alpha G$ is a *crossed product ideal*, that it, it has the form $I \rtimes_\alpha G$ for some G-invariant ideal I in A, then $\widetilde{\Gamma}(\alpha) = \widehat{G}$.

(1) Let $\chi \in \widehat{G}$. Regard $\chi \in C(G) \subseteq M(A \rtimes_\alpha G)$ and show that
$$\overline{(A \rtimes_\alpha G)\chi(A \rtimes_\alpha G)} = A \rtimes_\alpha G.$$
(2) Deduce that $\widetilde{\mathrm{Sp}}(\alpha) = \widehat{G}$. Conclude that in order to show that $\widetilde{\Gamma}(\alpha) = \widehat{G}$, it suffices to show that for every $B \in \mathrm{Her}_G(A)$, all ideals in $B \rtimes_\alpha G$ are crossed product ideals.

(3) Let $B \in \mathrm{Her}_G(A)$. Show that $B \rtimes_\alpha G$ is a hereditary subalgebra of $A \rtimes_\alpha G$, by showing that
$$C(G,B,\alpha)C(G,A,\alpha)C(G,B,\alpha) \subseteq C(G,B,\alpha).$$
(4) Let J be an ideal in $B \rtimes_\alpha G$. Show that there exists a G-invariant ideal I of A such that
$$J = (B \rtimes_\alpha G) \cap (I \rtimes_\alpha G).$$
(5) Prove that $(B \rtimes_\alpha G) \cap (I \rtimes_\alpha G) = (B \cap I) \rtimes_\alpha G$, and deduce that $\widetilde{\Gamma}(\alpha) = \widehat{G}$.

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