C^*-DYNAMICS AND CROSSED PRODUCTS
Fourth exercise sheet

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You must hand in a minimum of four exercises by January 9th.

Exercise 1.

(1) Let
$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$
be a short exact sequence of C^*-algebras, and suppose that there is a homomorphism $s\colon B \to A$ satisfying $\pi \circ s = \mathrm{id}_B$. Show that $K_j(A) \cong K_j(I) \oplus K_j(B)$.

(2) Let A be a C^*-algebra. Use the previous part to compute the K-theory of $C(S^1) \otimes A$. (This computation can of course also be done using the Pimsner-Voiculescu 6-term exact sequence, but you are not supposed to use it here.)

(3) Find examples of short exact sequences as in (1) where:
  (a) $K_0(\iota)$ is not injective.
  (b) $K_1(\iota)$ is not injective.
  (c) $K_0(\pi)$ is not surjective.
  (d) $K_1(\pi)$ is not surjective.

Exercise 2. Write a rigorous proof of the fact that $K_0(\mathcal{K}) \cong \mathbb{Z}$ and $K_1(\mathcal{K}) \cong \{0\}$, without using continuity of the K-groups. Be particularly careful with the identification of $V(\widetilde{\mathcal{K}})$.

Exercise 3. Complete the proof of the Pimsner-Voiculescu exact sequence given in class by showing the following.

(1) If α and γ are homotopic automorphisms of a C^*-algebra A, then M_α is isomorphic to M_γ.

(2) If α is an automorphism of a C^*-algebra A, and B is any other C^*-algebra, let $\alpha \otimes \mathrm{id}_B$ denote the associated automorphism of $A \otimes_{\min} B$. Then there is a natural isomorphism
$$M_{\alpha\otimes\mathrm{id}_B} \cong M_\alpha \otimes_{\min} B.$$
How about the maximal tensor product?

Exercise 4. Let A be a C^*-algebra, let $n \in \mathbb{N}$, and let $\alpha \in \mathrm{Aut}(A)$ be an automorphism of order n. We write $\overline{\alpha}\colon \mathbb{Z}_n \to \mathrm{Aut}(A)$ for the induced action. Prove, rigorously, that there is a natural $\mathbb{T}$-equivariant isomorphism
$$\psi\colon (A \rtimes_\alpha \mathbb{Z}, \widehat{\alpha}) \to \left(\mathrm{Ind}_{\mathbb{Z}_n^\perp}^{\mathbb{T}}(A \rtimes_{\overline{\alpha}} \mathbb{Z}_n), \mathrm{Ind}_{\mathbb{Z}_n^\perp}^{\mathbb{T}}(\widehat{\overline{\alpha}})\right),$$
where $\mathbb{Z}_n^\perp = \{z \in \mathbb{T}\colon z^n = 1\}$.

Date: December 18, 2018.

Exercise 5. Let A be a C^*-algebra, let $n \in \mathbb{N}$, and let $\alpha\colon \mathbb{Z}_n \to \mathrm{Aut}(A)$ be an action. Let $\pi\colon A \rtimes_\alpha \mathbb{Z} \to A \rtimes_\alpha \mathbb{Z}_n$ denote the canonical quotient map. Prove that there is an exact sequence

$$\begin{tikzcd}
K_0(A \rtimes_\alpha \mathbb{Z}_n) \arrow[r, "\mathrm{id}-K_0(\widehat{\alpha}_1)"] & K_0(A \rtimes_\alpha \mathbb{Z}_n) \arrow[r] & K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "K_1(\pi)"] \\
K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "K_0(\pi)"] & K_1(A \rtimes_\alpha \mathbb{Z}_n) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{Z}_n). \arrow[l, "\mathrm{id}-K_1(\widehat{\alpha}_1)"]
\end{tikzcd}$$

Exercise 6. Let $n \in \mathbb{N}$ with $n \geq 2$. Recall that the *Cuntz algebra* $\mathcal{O}_n$ is the universal unital C^*-algebra generated by isometries $s_1, \ldots, s_n$ satisfying $\sum\limits_{j=1}^{n} s_j s_j^* = 1$. In this exercise, you may use without proof that this algebra is simple.

(1) Let $e \in M_n$ be the projection $e_{1,1}$. For $m \in \mathbb{N}$, set $D_m = \bigotimes\limits_{m=-k}^{\infty} M_n$, and let $\psi_m\colon D_m \to D_{m+1}$ be given by
$$\psi_m(x) = e \otimes x \in e \otimes D_m \subseteq D_{m+1}$$
for all elementary tensors $x \in D_m$. Denote by D the associated direct limit, with canonical maps $\varphi_m\colon D_m \to D$, for $m \in \mathbb{N}$. Show that D is isomorphic to $M_{n^\infty} \otimes \mathcal{K}$.

(2) For $m \in \mathbb{N}$, let $\theta_m\colon D_m \to D_{m-1}$ be an isomorphism (for example, just by reindexing the tensor factors) and let $\alpha_m\colon D_m \to D_m$ be given by
$$\alpha_m(x) = e \otimes \theta_m(x) \in 1 \otimes D_{m-1} \subseteq D_m$$
for all elementary tensors $x \in D_m$. Show that there is an automorphism $\alpha \in \mathrm{Aut}(D)$ such
$$\alpha \circ \varphi_m = \varphi_m \circ \alpha_m$$
for all $m \in \mathbb{N}$.

(3) Compute $K_0(\alpha)$ and $K_1(\alpha)$.

(4) Show that $D \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_n \otimes \mathcal{K}$ as follows:
  (a) Denote by $p \in D_0$ the unit of $D_0 \cong M_{n^\infty}$, and denote by $u \in M(D \rtimes_\alpha \mathbb{Z})$ the canonical unitary. Set $s = up$. Then $D_0 = pDp$ and $p(D \rtimes_\alpha \mathbb{Z})p$ is generated by D_0 and s.
  (b) For $j = 1, \ldots, n$, set $s_j = (e_{j,1} \otimes p)s \in D \rtimes_\alpha \mathbb{Z}$. Then $s_j^* s_j = p$ and $\sum\limits_{j=1}^{n} s_j s_j^* = p$.
  (c) Show that $p(D \rtimes_\alpha \mathbb{Z})p \cong \mathcal{O}_n$.
  (d) For $m \in \mathbb{N}$, let $p_m \in D_m$ be the unit. Then $D \rtimes_\alpha \mathbb{Z}$ is isomorphic to the inductive limit of $p_m(D \rtimes_\alpha \mathbb{Z})p_m$, and $p_{m-1}(D \rtimes_\alpha \mathbb{Z})p_{m-1}$ is generated by $p_m(A \rtimes_\alpha \mathbb{Z})p_m$ and
$$\{e_{j,k} \otimes p_m \colon 1 \leq j, k \leq n\} \subseteq M_n \otimes D_m = D_{m-1}.$$
  (e) Conclude that $D \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_n \otimes \mathcal{K}$.

(5) Compute $K_0(\mathcal{O}_n)$ and $K_1(\mathcal{O}_n)$. Deduce that $\mathcal{O}_n \cong \mathcal{O}_m$ if and only if $n = m$.

Exercise 7.

(1) Let $\alpha\colon \mathbb{Z}_n \to \mathrm{Aut}(\mathcal{O}_2)$ be an action such that $\widehat{\alpha}_1$ is approximately inner. Show that
$$K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n) \cong K_1(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n) \cong \{0\}.$$
(2) Let $\gamma\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action on a (nonzero) AF-algebra A. Show that the dual automorphism $\widehat{\gamma}$ of $A \rtimes_\gamma \mathbb{T}$ is not approximately inner.
(3) What happens in the item above if A is not required to be AF?

Exercise 8. Let A be an AF-algebra and let $\alpha \in \mathrm{Aut}(A)$.

(1) Let $p \in A$ be a nonzero projection. Show that $[p]_0$ is not the trivial element in $K_0(A)$.
(2) If A is unital, show that $A \rtimes_\alpha \mathbb{Z}$ is not AF.
(3) If A is not unital, show with an example that $A \rtimes_\alpha \mathbb{Z}$ may be AF.

Exercise 9. Let G be a discrete group, let A be a unital C^*-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action, and let $w\colon G \to \mathcal{U}(A)$ be an α-cocycle.

(1) Show that the map $\alpha^w\colon G \to \mathrm{Aut}(A)$ given by $\alpha^w_g = \mathrm{Ad}(w_g) \circ \alpha_g$ for all $g \in G$, is an action.
(2) Show that $A \rtimes_\alpha G$ and $A \rtimes_{\alpha^w} G$ are canonically isomorphic.

Suppose now that G is finite. For $g \in G$, let $u_g \in A \rtimes_\alpha G$ denote the canonical unitary implementing α_g. Set
$$p = \frac{1}{|G|}\sum_{g\in G} w_g u_g \quad \text{and} \quad q = \frac{1}{|G|}\sum_{g\in G} u_g.$$
(3) Show that p and q are projections.
(4) Show that w is a coboundary if and only if $p \sim_{\mathrm{M-vN}} q$.

Exercise 9. Let $n \in \mathbb{N}$.

(1) Let A be a C^*-algebra, and let $p_1, \ldots, p_n$ be projections in A that are Murray-von Neumann equivalent. Show that A contains a subalgebra isomorphic to M_n.
(2) If M is a II_1-factor and $n \in \mathbb{N}$, then M contains a subalgebra isomorphic to M_n.

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