C^*-DYNAMICS AND CROSSED PRODUCTS
Fifth exercise sheet

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You must hand in a minimum of two exercises by January 30th.

Exercise 1. Let A be a unital C^*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

(1) Let B be a unital C^*-algebra, and let $\beta\colon G \to \mathrm{Aut}(B)$ be an action of G on B. Let $A \otimes B$ be any C^*-algebra completion of the algebraic tensor product of A and B for which the tensor product action $\alpha \otimes \beta$ is defined. Show that $\alpha \otimes \beta$ has the Rokhlin property.
(2) Let I be an α-invariant ideal in A, and denote by $\overline{\alpha}\colon G \to \mathrm{Aut}(A/I)$ the induced action on A/I. Show that $\overline{\alpha}$ has the Rokhlin property.
(3) Let p be an α-invariant projection in A. Set $B = pAp$ and denote by $\beta\colon G \to \mathrm{Aut}(B)$ the compressed action of G. Show that β has the Rokhlin property.

Exercise 2. Let G be a finite group, let A be a unital C^*-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action, and let $w\colon G \to \mathcal{U}(A)$ be an α-cocycle.

(1) Show that α has the Rokhlin property if and only if α^w does.
(2) Suppose that α has the Rokhlin property. Show that there exists $v \in \mathcal{U}(A)$ with $w_g = v\alpha_g(v^*)$ for all $g \in G$. (Hint: use Proposition 10.2.3 together with an approximation argument.)

In the following exercise, you may use that $\mathbb{C}^n$ is semiprojective for every $n \in \mathbb{N}$.

Exercise 3. Let G be a finite group, let A be a separable unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Prove that α has the Rokhlin property if and only if there is a unital, equivariant homomorphism $\varphi\colon (C(G), \mathtt{Lt}) \to (A_\infty \cap A', \alpha_\infty)$.

Exercise 4. Let A be a unital C^*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Given a finite subset $F \subseteq A$ and $\varepsilon > 0$, show that there exists a unital map $\psi\colon A \to A^\alpha$ such that

(1) For all $a, b \in F_1$, we have
$$\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon \quad \text{and} \quad \|\psi(a)^* - \psi(a^*)\| < \varepsilon;$$
(2) For all $x \in A^\alpha$, we have $\psi(x) = x$.

Exercise 5. Let G be a finite group and let $n \in \{2, \ldots, \infty\}$. Show that there is an action of G on $\mathcal{O}_n$ with the Rokhlin property if and only if $|G|$ and $n-1$ are relatively prime. (By convention, ∞ is relatively prime with any natural number.) Hint: for the "if" implication, use the existence of an isomorphism $\mathcal{O}_n \otimes M_{|G|^\infty} \cong \mathcal{O}_n$. For the "only if" implication, use an argument similar to the one used with UHF-algebras.

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Exercise 6. Let $\theta \in \mathbb{R}\setminus\mathbb{Q}$, and let G be a nontrivial finite group. Show that there is no action of G on A_θ with the Rokhlin property.

Exercise 7. Let A be a unital C^*-algebra, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Denote by $\iota\colon A^\alpha \to A$ the canonical inclusion.

(1) Show that $K_0(\iota)$ and $K_1(\iota)$ are injective.
(2) Show that $K_0(\iota)$ is an order-embedding, that is, that for all $x, y \in K_0(A^\alpha)$, one has $K_0(\iota)(x) \leq K_0(\iota)(y)$ if and only if $x \leq y$.[1]
(3) Show that
$$K_j(\iota)(K_j(A^\alpha)) = \{x \in K_0(A)\colon K_j(\alpha_g)(x) = x \text{ for all } g \in G\}.$$

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[1] This part requires some additional knowledge of K-theory, namely the order structure on K_0, and may be skipped.