RECENT ADVANCES IN THE THEORY OF CONTINUOUS FIELDS OF NUCLEAR C^*-ALGEBRAS

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Warning: little proofreading has been done.

Contents

1. Introduction to $C_0(X)$-algebras and continuous fields

The focus of this series of talks will be C^*-algebras over locally compact second countable Hausdorff spaces. We recall the definition.

Definition 1.1. (Kasparov) Let A be a C^*-algebra and let X be a locally compact second countable Hausdorff space. We say that A is a *$C_0(X)$-algebra* if there exists a homomorphism $\theta\colon C_0(X) \to Z(M(A))$ such that

$$\theta(C_0(X))A = A.$$

The homomorphism θ is called the structure homomorphism. We usually suppress θ from the notation and write fa for $\theta(f)a = a\theta(f)$ for $f \in C_0(X)$ and $a \in A$.

If U is an open subset of X, we write $A(U) = C_0(U)A$ which is an ideal of A, and if F is closed in X, we write $A(F) = A/A(X \setminus F)$. When $F = \{x\}$, we write $A(x)$ for $A(F)$, and call this the *fiber* of A over x.

There is an inclusion

$$A \hookrightarrow \prod_{x\in X} A(x)$$

sending $a \in A$ to $(a(x))_{x\in X}$. It is a fact that for all $a \in A$, the map $X \to \mathbb{R}_{\geq 0}$ given by $x \mapsto \|a(x)\|$ is upper-semicontinuous. (Notice that there is no well-defined map if we omit the norm, since the $a(x)$ live in different C^*-algebras for different $x \in X$.)

Definition 1.2. We say that A is a *continuous $C_0(X)$-algebra* (also called *continuous field C^*-algebra*) if $x \mapsto \|a(x)\|$ is continuous.

It is a fact that if A is separable and $\mathrm{Prim}(A)$ is Hausdorff, then A is a continuous field over $X = \mathrm{Prim}(A)$.

The following is an example of a $C_0(X)$-algebra which is not a continuous field.

Example 1.3. Take $A = \mathbb{C}$ as a $C([0,1])$-algebra, with the action given by $f \cdot a = f(1)a$ for $f \in C([0,1])$ and $a \in \mathbb{C}$. The function $x \mapsto \|a(x)\|$ is given by

$$\|a(x)\| = \begin{cases} 0, & x < 1; \\ |a|, & x = 1. \end{cases}$$

Hence $x \mapsto \|a(x)\|$ is not continuous unless $a = 0$.

For now, we will restrict our attention to separable exact C^*-algebras which are continuous fields. (Exactness is equivalent to sub-nuclear by Kirchberg's theorem.)

Theorem 1.4. (Blanchard-Kirchberg) If A is a separable exact continuous field over a compact space X, then there exists an injective equivariant $C(X)$-linear homomorphism

$$A \hookrightarrow C(X, \mathcal{O}_2) \hookrightarrow C(X, \mathcal{B}(\mathcal{H})).$$

In other words the algebras we are focusing on are generic $C(X)$-invariant subalgebras of $C(X, \mathcal{O}_2)$. If A is as in the theorem and $U, F \subseteq X$ are open and closed respectively, then one can show that the given embedding respects restriction/inclusion in the sense that there is a commutative diagram

$$\begin{tikzcd} A(U) \arrow[d, hook] & C_0(U, \mathcal{O}_2) \arrow[l] \arrow[d, hook] \\ A \arrow[r, hook] \arrow[d, two heads] & C(X, \mathcal{O}_2) \arrow[d, two heads] \\ A(F) \arrow[r, hook] & C(F, \mathcal{O}_2). \end{tikzcd}$$

Definition 1.5. We say that A is *locally trivial* if for all $x \in X$ there exists an open set $U \subseteq X$ containing x such that

$$A(U) \cong C_0(U, D)$$

for some C^*-algebra D.

We will discuss locally triviality of a few examples now.

Example 1.6. Consider

$$A_1 = \{f \in C([0,1], M_3) \colon f(1/2) \in \mathbb{C} \cdot 1_3\}.$$

It is easy to see that A_1 is not locally trivial because the fibers are M_3 for $x \neq 1/2$ and $\mathbb{C}$ for $x = 1/2$. The following examples have isomorphic fibers, so this is no longer an obstruction.

Example 1.7. Set $A_2 = A_1 \otimes \mathcal{K}$. The fibers are $\mathcal{K}$.

Example 1.8. Set $A_3 = A_1 \otimes M_{3^\infty}$. The fibers are M_{3^∞}.

Example 1.9. Set $A_4 = A_1 \otimes \mathcal{K} \otimes \mathcal{O}_3$. The fibers are $\mathcal{K} \otimes \mathcal{O}_3$.

Example 1.10. Set $A_5 = A_1 \otimes \mathcal{K} \otimes \mathcal{O}_4$. The fibers are $\mathcal{K} \otimes \mathcal{O}_4$.

We will use K-theory to explore these examples. We start with Example 1.6 again. Consider the extension

$$0 \to A([0,1] \setminus \{1/2\}) \to A \to A(1/2) \to 0$$

which is just

$$0 \to C_0([0,1/2)) \otimes M_3 \oplus C_0((1/2,1]) \otimes M_3 \to A \to \mathbb{C} \to 0.$$

Since cones are contractible, they are trivial on K-theory, and we get an isomorphism $K_0(A) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$, with generator given by $[1_A]$. On the other hand, the map $K_0(A) \cong \mathbb{Z}[1_A] \to K_0(A(0)) \cong K_0(M_3)$, is multiplication by 3, which is not an isomorphism. Local triviality would imply that $A \to A(x)$ induces an isomorphism on K-theory for all $x \in X$. Hence Example 1.6 is not locally trivial (we already knew this, though).

Since tensoring with $\mathcal{K}$ does not change the K-theory, it follows that Example 1.7 is not locally trivial either.

We will deal with Example 1.8, Example 1.9 and Example 1.10 at the same time and in a more abstract framework.

Let D be a C^*-algebra and let $\gamma \colon D \to D$ be an injective homomorphism. Hence $\gamma(D)$ is isomorphic to D abstractly, but it is concretely a different C^*-algebra. Consider the continuous field

$$A_\gamma = \{f \in C([0,1], D) \colon f(1/2) \in \gamma(D)\},$$

and look at the extension

$$0 \to CD \oplus CD \to A \to \gamma(D) \cong (D) \to 0$$

where the map $A \to \gamma(D)$ is evaluation at $1/2$. Sice cones are contractible, the above extension induces an isomorphism $K_0(A) \cong K_0(D)$. Moreover, for $x \neq 1/2$ we have

$$\begin{tikzcd} K_0(A) \arrow[r, "(\pi_x)_*"] \arrow[d, "\cong"'] & K_0(A(x)) \arrow[d, "\cong"] \\ K_0(D) \arrow[r, "\gamma_*"'] & K_0(D). \end{tikzcd}$$

It follows that if γ_* is not an isomorphism, then A is not locally trivial.

In Example 1.10, the K-theory of the fibers is $K_0(\mathcal{O}_4) \cong \mathbb{Z}_3$ and the map γ_* is multiplication by 3, which is the zero map on this group. Hence it is not locally trivial. In Example 1.8 and Example 1.9, the maps γ_* are, respectively:

$$\mathbb{Z}[1/3] \xrightarrow{\cdot 3} \mathbb{Z}[1/3] \qquad\qquad \mathbb{Z}_2 \xrightarrow{\cdot 3} \mathbb{Z}_2 \,.$$

Since both these maps are isomorphisms, we do not get an obstruction to local triviality. It turns out that both these examples are locally trivial, but this requires a proof.

Remark 1.11. Since $[0,1]$ is contractible, a locally trivial continuous field over $[0,1]$ is the same as a trivial continuous field, this is, one of the form $C([0,1],D)$ for some C^*-algebra D.

Theorem 1.12. There is an isomorphism $A_\gamma \cong C([0,1],D)$ if and only if there exists a path $[0,1/2] \to \mathrm{End}(D)$ such that $\theta_t \in Aut(D)$ for all $t < 1/2$ and $\theta_{1/2} = \gamma$.

Proof. Notice that one may use the same path on the other half of the interval. The isomorphism is then $C([0,1/2],D) \to A([0,1/2])$ given by $f(t) = \theta_t(f(t))$. □

Corollary 1.13. If A_γ is trivial, then γ is a KK-equivalence since it is homotopic to an automorphism. The converse is true if D is a stable Kirchberg algebra, since a result by Kirchberg and Phillips states that a full KK-trivial endomorphism $D \to D$ is homotopic to an automorphism.

Let A be a continuous field over $[0,1]$ and assume that $A([0,1)) \cong C_0([0,1),D)$ for some unital C^*-algebra D. We would like to show that A ha the form A_γ for some γ, except that now we cannot expect γ to be an endomorphism, just an asymptotic homomorphism.

Consider the extension

$$0 \to C_0([0,1),D) \to A \to A(1) \to 0.$$

Its Busby invariant is the map

$$\sigma\colon D \to M(I)/I = C_b(([0,1),D)/C_0([0,1),D).$$

Then σ is an asymptotic homomorphism $(\sigma_t)_{t\in[0,1)}$ from D into itself. One then shows that A is isomorphic to A_σ, where

$$A_\sigma = \{(f,d) \in C_b([0,1),D) \oplus D\colon \lim_{t\to 1} \|f(t) - \theta_t(d)\| = 0\}.$$

In this way, asymptotic morphisms are in one-to-one correspondences with $C([0,1])$-algebras with a singularity at $x=1$ (or $x=0$). The case $\sigma_t = \gamma$ for all $t \in [0,1)$ corresponds to A_γ. This is due to Connes-Higson.

One can easily product finitely many singular points, but how many can one produce? We will show an example of a $C([0,1])$-continuous field where all the fibers are isomorphic yet every point in $[0,1]$ is singular.

Example 1.14. (Dadarlat-Elliott) Let D be a UCT unital Kirchberg algebra such that $K_0(D) \cong \mathbb{Z}\oplus\mathbb{Z}$ with $[1_D] = (1,0)$. Choose a dense sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$. Let γ be a unital endomorphism of D such that

$$K_0(\gamma)\colon \mathbb{Z}^2 \to \mathbb{Z}^2 \quad \text{is given by} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $A_n = \{f \in C([0,1],D)\colon f(x_n) \in \gamma(D)\}$, and set

$$B = \bigotimes_{n=1}^{\infty}{}_{C([0,1])} A_n,$$

where the symbol $\bigotimes_{C([0,1])}$ is the *balanced* tensor product over $[0,1]$, which gives a $C([0,1])$-algebra as opposed to a $C([0,1]^\infty)$-algebra. It is easy to see that $B(x) \cong \otimes_{n=1}^\infty D$, but every point in $[0,1]$ is singular. Indeed, for every open interval $(a,b) \subseteq [0,1]$, there exists $x \in (a,b)$ such that

$$(\pi_x)_*\colon K_0(B) \to K_0(B(x))$$

is not injective. Hence B is nowhere trivial, and all points are singular.

Remark 1.15. Any continuous field with fibers that are constant and finite-dimensional is necessarily locally trivial.

2. Local triviality of continuous fields

Semiprojective C^*-algebras are crucial in describing the structure of continuous fields. Any continuous field with stable Kirchberg algebras as fibers such that moreover K_1 of the fibers is torsion free is an inductive limit of continuous fields with finitely many singular points with semiprojective fibers. One then looks at the asymptotic gluing morphisms to treat these continuous fields.

Application. Any continuous field over $[0,1]$ with fibers $\mathcal{O}_2 \otimes \mathcal{K}$ is trivial.

Proof. Approximate the given continuous field by continuous fields with finitely many singularities. One can arrange that the fibers are again $\mathcal{O}_2 \otimes \mathcal{K}$, so the approximating continuous fields are determined by some asymptotic homomorphisms on $\mathcal{O}_2$. Any such asymptotic homomorphism is necessarily KK-trivial, and hence any continuous field over $[0,1]$ with fibers $\mathcal{O}_2 \otimes \mathcal{K}$ and finitely many singularities is trivial. The result now follows from the fact that a direct limit of trivial continuous fields is again a trivial continuous field. □

Definition 2.1. (n-pullbacks) Let X be compact, and let $Y_0, \dots, Y_n \subseteq X$ closed subsets such that $X = Y_0 \cup \cdots \cup Y_n$. For $j = 0, \dots, n$, let E_j be a locally trivial field over Y_j. Suppose there are $C(Y_j \cap Y_i)$-linear homomorphisms

$$\gamma_{ij} \colon E_i|_{Y_i \cap Y_j} \to E_j|_{Y_i \cap Y_j}$$

such that $(\gamma_{jk}(_x \circ (\gamma_{ij})_x = (\gamma_{ik})_x$ for all $x \in Y_i \cap Y_j \cap Y_k$. We define the *pullback* continuous field A by

$$A = \{(e_0, \dots, e_n) \in E_0 \oplus \cdots E_n \colon e_j(x) = (\gamma_{ij})_x(e_i(x)) \text{ for all } x \in Y_i \cap Y_j, i < j\}.$$

Theorem 2.2. Let A be a separable nuclear continuous field over a compact space X of dimension $\dim(X) < \infty$. Suppose that $A(x)$ is a Kirchberg UCT algebra and that $K_1(A(x))$ is torsion free for all $x \in X$. Then there exist subalgebras $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ such that $\overline{\cup_{n \in \mathbb{N}} A_n} = A$, where the inclusions are $C(X)$-linear, and such that A_k is an n-pullback of locally trivial continuous fields for all $k \in \mathbb{N}$.

Corollary 2.3. If A is a separable nuclear continuous field such that each fiber satisfied the UCT, then A itself satisfies the UCT.

Proof. Show that every such field is KK_X-equivalent to a Kirchberg field and approximate this one by using the Theorem above. □

Theorem 2.4. Let X be a compact Hausdorff space of finite dimension. Suppose that A is a continuous field over X such that $A(x) \cong \mathcal{O}_2 \otimes \mathcal{K}$ for all $x \in X$. Then $A \cong C(X) \otimes \mathcal{O}_2 \otimes \mathcal{K}$.

The point is that the gluing maps $\gamma_{ij} \colon Y_i \cap Y_j \to \mathrm{End}(\mathcal{O}_2 \otimes \mathcal{K})$ are KK-equivalent to automorphisms, and thus each of the pullbacks is trivial.

There exist proofs of this result that use the classification developed by Kirchberg. See work by Hirshberg-Rørdam-Winter where they show that over finite dimensional spaces, if all of the fibers absorb $\mathcal{O}_2$ (or any strongly self-absorbing C^*-algebra), then the continuous field itself absorbs $\mathcal{O}_2$.

Definition 2.5. Let A be a C^*-algebra. A sequence $(A_k)_{k \in \mathbb{N}}$ of sub-C^*-algebras of A is said to be *exhaustive* in A if for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $F \subseteq_\varepsilon A_k$.

Example 2.6. If $A = \varinjlim A_k$, then $(\iota_{\infty,k}(A_k))_{k \in \mathbb{N}}$ is exhaustive in A.

Theorem 2.7. Let A be a separable continuous field over a compact metrizable space X with $\dim(X) = n < \infty$. Suppose that each $A(x)$ is a UCT Kirchberg algebra. Then A admits an exhaustive sequence $(A_k)_{k \in \mathbb{N}}$ where each A_k is an n-pullback.

If moreover $K_1(A(x))$ is torsion-free for every $x \in X$, then one can arrange that $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$.

Remark 2.8. If $A(x) \cong \mathcal{O}_2 \otimes \mathcal{K}$ for all $x \in X$, then one can arrange that $A_k(x) \cong \mathcal{O}_2 \otimes \mathcal{K}$ for all $k \in \mathbb{N}$ and all $x \in X$ as well. Since A_k has finitely many singularities, it follows that $A_k \cong C(X, \mathcal{O}_2 \otimes \mathcal{K})$ for all $k \in \mathbb{N}$ and hence $A \cong C(X, \mathcal{O}_2 \otimes \mathcal{K})$.

In particular, we obtain the following result.

Corollary 2.9. All $\mathcal{O}_2 \otimes \mathcal{K}$ continuous fields over finite dimensional spaces are trivial. Similarly, if A is unital and $A(x) \cong \mathcal{O}_2$ for all $x \in X$, then $A \cong C(X, \mathcal{O}_2)$.

The case $X = *$ in the following theorem is the well-known fact proved by Kirchberg that any separable nuclear C^*-algebra is KK-equivalent to a Kirchberg algebra.

Theorem 2.10. Any separable nuclear continuous field is KK_X-equivalent to a Kirchberg field.

Proof. Assume that A is unital. Find a $C(X)$-linear unital embedding $\alpha \colon C(X) \otimes \mathcal{O}_2 \to A$. By Blanchard-Kirchberg, there is a $C(X)$-linear unital embedding $\beta \colon A \to C(X) \otimes \mathcal{O}_2$. Consider the composition

$$A \xrightarrow{\beta} C(X) \otimes \mathcal{O}_2 \xrightarrow{\alpha} A,$$

and denote it by θ. Let $s_1, s_2 \in A$ be the images of the canonical generators of $\mathcal{O}_2$ via α, and define $\varphi \colon A \to A$ by

$$\varphi(a) = s_1 a s_1^* + s_2 \theta(a) s_2^*$$

for $a \in A$. The inductive limit $A_\sharp = \varinjlim(A, \varphi)$ is a Kirchberg field. Since $KK_X(\varphi) = [\mathrm{id}_A]$, it follows that the inclusion $A \hookrightarrow A_\sharp$ is a KK_X-equivalence. □

We would like to have a useful criterion to determine when an element $\sigma \in KK_X(A, B)$ is invertible.

Theorem 2.11. Let A and B be separable nuclear continuous fields over a finite dimensional space X. An element $\sigma \in KK_X(A, B)$ is invertible if and only if it is pointwise invertible, this is, if $\sigma_x \in KK(A(x), B(x))^{-1}$ for all $x \in X$.

A priori, it is not clear that such restrictions preserve fibers. We explain how this is the case when A and B are stable Kirchberg fields. The class $\sigma \in KK_X(A, B)$ contains a $C(X)$-linear homomorphism, and these preserve fibers. The general case uses Theorem 2.10.

Proof. We may assume that A and B are Kirchberg fields and that $\sigma = KK_X(\varphi)$ for some $C(X)$-linear homomorphism $\varphi\colon A \to B$. The cone $C_\varphi \to A \to B$ is KK_X-contractible if and only if φ is a KK_X-equivalence. Notice that $(C_\varphi)_x = C_{\varphi_x}$ for all $x \in X$, and φ_x is a KK-equivalence by assumption. One has $C_\varphi \sim_{KK_X} (C_\varphi)_\sharp$. Moreover, the fibers satisfy

$$((C_\varphi)_\sharp \otimes \mathcal{K})_x \cong \mathcal{O}_2 \otimes \mathcal{K}$$

for all $x \in X$, which by a previous theorem implies that $(C_\varphi)_\sharp \otimes \mathcal{K}$ is trivial itself, hence KK_X contractible. The result now follows. $\square$

Notice that finite dimensionality of X was used only to conclude that $(C_\varphi)_\sharp \otimes \mathcal{K}$ is trivial.

Corollary 2.12. (Criterion for local triviality) Let A be a Kirchberg field over a finite dimensional space X. Suppose that there exists a Kirchberg algebra D and $\sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. Then

$$A \otimes \mathcal{K} \cong C(X, D \otimes \mathcal{K}).$$

Proof. The map $KK_X(C(X) \otimes D, A) \to KK(D, A)$ is a bijection. $\square$

Corollary 2.13. Let A be a separable unital field over a metrizable compact finite dimensional space X. Suppose that $A(x) \cong \mathcal{O}_n$ for all $x \in X$, with $n \in \{2, \dots, \infty\}$. Then A is locally trivial. Moreover, it is trivial if and only if either $n = \infty$ or $(n-1)[1_A] = 0$ in $K_0(A)$.

In particular, any separable unital field over X with fibers $\mathcal{O}_2$ or $\mathcal{O}_\infty$ is trivial.

Proof. For $k \in \mathbb{N}$, denote by $V_k = B(x, 1/k) \subseteq X$ the ball of radius $1/k$ centered at x in X. Using that $A(x) = \varinjlim A(V_k)$ and the fact that $\mathcal{O}_n$ is semiprojective, there exists $k \in \mathbb{N}$ and a unital homomorphism $\sigma\colon \mathcal{O}_n \to A(V_k)$ making the following diagram commute

$$\begin{tikzcd} & A(V_k) \arrow[d] \\ \mathcal{O}_n \arrow[ur, "\sigma"] \arrow[r] & A(x). \end{tikzcd}$$

Hence the map $\sigma_x\colon \mathcal{O}_n \to \mathcal{O}_n$ send $[1_n]$ to $[1_n]$ and is thus a KK-equivalence by the UCT. Since it is a fiber-wise equivalence, it is an equivalence itself. $\square$

We will construct an example of a unital continuous field A over the Hilbert cube $X = [0,1]^\infty$ such that $A(x) \cong \mathcal{O}_2$ for every $x \in X$ but $A \not\cong C(X, \mathcal{O}_2)$. The example is a modification of an example by Hirshberg-Rørdam-Winter which shows that absorption of a strongly self-absorbing C^*-algebra by all fibers does not pass to the continuous field if the underlying space is infinite dimensional.

Example 2.14. Let $e \in M_2(C(S^2))$ be the Bott projection (with rank one), and let $p = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \in M_3(C(S^2))$. Note that p has rank 2. Set $B = \bigotimes_{n=1}^\infty pM_3(C(S^2))p$, so that $B(y) = M_{2^\infty}$ for all $y \in Y = \prod_{n\in\mathbb{N}} S^2$. Notice that B is a continuous field over Y. We compute its K-theory as follows. The map $\mathbb{C} \oplus \mathbb{C} \to pM_3(C(S^2))p$ determined by

$$(1,0) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0_2 \end{pmatrix} \quad \text{and} \quad (0,1) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$$

is a KK-equivalence, so the map $\bigotimes_{n\in\mathbb{N}} \mathbb{C}^2 \cong C(K) \to B$ is a KK-equivalence as well. (Here K denotes de Cantor set.) Thus,

$$K_0(B) \cong C(K, \mathbb{Z}).$$

Now let $A = B \otimes \mathcal{O}_3$, whose fibers are unital UCT Kirchberg algebras with trivial K-theory. It follows from Kirchberg-Phillips that $A(y) \cong \mathcal{O}_2$ for all $y \in Y$. On the other hand,

$$K_0(A) = K_0(B \otimes \mathcal{O}_3) \cong K_0(B) \otimes K_0(\mathcal{O}_3) \cong C(K, \mathbb{Z}) \otimes \mathbb{Z}_2 \cong C(K, \mathbb{Z}_2) \not\cong 0,$$

so in particular A is not trivial (this is, does not have the form $C(Y) \otimes \mathcal{O}_2$). We now show how we can replace $Y = \prod_{n\in\mathbb{N}} S^2$ with $X = [0,1]^\infty$ to get the desired example. One can embed Y into X by universality of the Hilbert cube. By Blanchard-Kirchberg, there is an embedding $\eta\colon B \hookrightarrow C(Y) \otimes \mathcal{O}_2$. Under this inclusion, one has

$$A = \{f \in C(X, \mathcal{O}_2) \colon f|_Y \in \eta(B)\}.$$

In particular, $A(x) \cong \mathcal{O}_2$ for all $x \in X$ and there is a short exact sequence

$$0 \to C_0(X \setminus Y) \otimes \mathcal{O}_2 \to A \to A(x) \cong \eta(B) \cong B \to 0.$$

Since $C_0(X \setminus Y) \otimes \mathcal{O}_2$ is KK-contractible, one concludes that $K_0(A) \cong K_0(B)$ and thus A is not trivial either.

The same example can e modified to get one such continuous field with fibers $\mathcal{O}_n$ for $2 \le n < \infty$, but not for $n = \infty$ so far. The following natural question remains open, but is expected to be answered positively.

Question 2.15. Is there a continuous field over $[0,1]^\infty$ whose fibers are $\mathcal{O}_\infty$ that is not trivial?

The main issue is detecting local triviality of continuous fields over infinite dimensional spaces. Equivariant E-theory $E_X(A,B)$, as developed by Dadarlat-Meyer, is useful in this context.

Theorem 2.16. Let A be a separable nuclear C^*-algebra with $\mathrm{Prim}(A) = X$ Hausdorff. (Recall that any such algebra is a continuous field over X with simple fibers.) Suppose that $KK(I,I) = 0$ for all ideals I in A. Then

$$A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong C_0(X) \otimes \mathcal{O}_2 \otimes \mathcal{K}.$$

We recall the definition of X-equivariant E-theory.

Definition 2.17. If X is a second countable topological space, a C^*-algebra A is said to be a *C^*-algebra over X* if there is a function $\mathcal{O}(X) \to \mathrm{Ideals}(A)$, denoted by $U \mapsto A(U)$, satisfying

(1) $A(\emptyset) = 0$ and $A(X) = A$,
(2) $A(U_1 \cap U_2) = A(U_1) \cdot A(U_2)$ $(= A(U_1) \cap A(U_2))$ for open sets U_1 and U_2 in X.
(3) $A(\bigcup_{n\in\mathbb{N}} U_n) = \varinjlim A(U_1) + \cdots + A(U_n)$ which in some sense can be interpreted as $\overline{\sum_{n\in\mathbb{N}} A(U_n)}$.

The bivariant functor $E_X(A,B)$ is constructed using asymptotic morphisms $\varphi\colon A \to C_b([0,\infty), B)$ satisfying

$$\varphi(A(U)) \in C_b([0,\infty), B(U)) + C_0([0,\infty), B)$$

for all $U \subseteq X$ open. (It is enough to check this condition on a basis for the topology of X.)

It is a fact that if A is a nuclear continuous field over a Hausdorff space X, then $KK_X(A,B) \cong E_X(A,B)$ for any C^*-algebra B. Nevertheless, E_X-theory is more suitable for some computations.

We illustrate the difference between E_X-theory and KK_X-theory in the following example.

Example 2.18. Consider the following extension of algebras over $[0,1]$:

$$0 \to C_0((0,1]) \to C([0,1]) \to \mathbb{C} \to 0.$$

It induces an element $\delta \in KK^1(\mathbb{C}, C_0((0,1]))$, but it does not induce an element $\delta_{[0,1]} \in KK^1_{[0,1]}(\mathbb{C}, C_0((0,1]))$. It nevertheless induces an element $\delta_{[0,1]} \in E^1_{[0,1]}(\mathbb{C}, C_0((0,1]))$.

The condition $\varphi(A(U)) \in C_b([0,\infty), B(U)) + C_0([0,\infty), B)$ can be arranged for one (fixed) open set U at a time. Given an asymptotic morphism $\varphi\colon A \to B$, we denote by $\dot\varphi\colon A \to B_\infty$ the induced homomorphism.

Lemma 2.19. Given an X-asymptotic morphism $\varphi = (\varphi_t)_{t\in[0,\infty)}\colon A \to B$ and an open subset $U \subseteq X$, there exists an X-asymptotic morphism $\varphi^U\colon A \to B$ such that $\dot{\varphi^U} = \dot\varphi$ and $\varphi^U(A(U)) \in C_b([0,\infty), B(U))$.

Proof. The asymptotic morphism φ^U is constructed by setting $\varphi^U = \widetilde{S_U} \circ \dot\varphi$ in the following diagram

$$\begin{tikzcd}
C_b([0,\infty), B(U)) \arrow[r] \arrow[d, leftrightarrow, "S_U"] & C_b([0,\infty), B) \arrow[d, leftrightarrow, "\widetilde{S_U}"] \\
B_\infty(U) \arrow[r, "\iota"] & B_\infty(X) \\
 & A. \arrow[u, "\dot\varphi"']
\end{tikzcd}$$

□

Corollary 2.20. For every $U \subseteq X$ open, the inclusion $B(U) \hookrightarrow B$ induces an isomorphism

$$E^*_X(A(U), B(U)) \xrightarrow{\cong} E^*_X(A(U), B)\ .$$

Corollary 2.21. Suppose that X is a finite space. Given $x \in X$, denote by $U_x \subseteq X$ the minimal open subset of X containing x. For $x \in X$ and a C^*-algebra D, let $\iota_x(D)$ be the X-algebra with fiber D over x and $\mathbb{C}$ elsewhere. If B is any other field, then

$$E^*_X(\iota_x(D), B) \cong E(D, B(U_x)).$$

We need a result that approximates $E^*_X(A,B)$ by E-groups over finite spaces. Let $(U_n)_{n\in\mathbb{N}}$ be a basis for the topology of X, and denote by τ_n the topology on X generated by $\{U_1, \ldots, U_n\}$. Let X_n denote the (finite) T_0-quotient of X with the topology τ_n. We may regard A and B as algebras over X_n.

Theorem 2.22. (Dadarlat-Meyer) There is a short exact sequence

$$0 \to \varprojlim^1 E^{*+1}_{X_n}(A,B) \to E^*_X(A,B) \to \varprojlim E^*_{X_n}(A,B) \to 0.$$

In E_{X_n} we have asymptotic morphisms that are X_n-equivariant, this is, equivariant with respect to $\{U_1, \ldots, U_n\}$. The key point is that $\varprojlim^1 E^{*+1}_{X_n}(A,B)$ is nilpotent: the product of any two elements is zero.

Theorem 2.23. Let A and B be separable C^*-algebras over a second countable topological space X. Then $\sigma \in E^*_X(A,B)$ is invertible if and only if $\sigma_U \in E^*(A(U),B(U))$ is invertible for all $U \subseteq X$ open.

Proof. Assume we have proved the theorem for the finite spaces X_n. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim^1 E^{*+1}_{X_n}(A,B) \arrow[r] \arrow[d] & E^*_X(A,B) \arrow[r] & \varprojlim E^*_{X_n}(A,B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim^1 E^{*+1}_{X_n}(A,C) \arrow[r] & E^*_X(A,C) \arrow[r] & \varprojlim E^*_{X_n}(A,C) \arrow[r] & 0
\end{tikzcd}$$

and the 5-Lemma, it follows that the result also holds for X itself. Thus, we may prove the result assuming that X is finite. Suppose moreover that $A \cong \iota_x(D)$ for some $x \in X$ and some C^*-algebra D. Suppose $\sigma_U \in E^*_X(B(U),C(U))$ is invertible for all $U \subseteq X$ open. Then we have

$$\begin{tikzcd}
E_X(\iota_x(D),B) \arrow[r, "\cdot\sigma_U"] \arrow[d, "\cong"] & E_X(\iota_x(D),C) \arrow[d, "\cong"] \\
E_X(D,B(U_x)) \arrow[r, "\cdot\sigma_{U_x}"] & E_X(D,C(U_x)),
\end{tikzcd}$$

and since σ_{U_x} is bijective, the result follows. The general case where $A \ncong \iota_x(D)$ follows from the 5-Lemma and using induction. □

3. Classification of one-parameter continuous fields

We will use E_X to study continuous fields over $[0,1]$. We say that a C^*-algebra D has *rational K-theory* if $K_0(D) \cong K_0(D) \otimes \mathbb{Q}$.

Theorem 3.1. (Bentmann-Dadarlat) Let A and B be separable continuous fields over $[0,1]$ whose fibers are stable UCT Kirchberg algebras with rational K-theory. Then

$$A \cong B \quad \text{if and only if} \quad FK(A) \cong FK(B).$$

We describe $FK(A)$ in this case. It is the set of all 6-term exact sequences

$$\begin{tikzcd}
K_0(A(U)) \arrow[r] & K_0(A(Y)) \arrow[r] & K_0(A(Y \setminus U)) \arrow[d] \\
K_1(A(Y \setminus U)) \arrow[u] & K_1(A(Y)) \arrow[l] & K_1(A(U)) \arrow[l]
\end{tikzcd}$$

associated to extensions of the form $0 \to A(U) \to A(Y) \to A(Y \setminus U) \to 0$, where $U \subseteq X$ is open, $Y \subseteq X$ is closed, and $Y \setminus U$ is connected.

Proof. The idea is to approximate $[0,1]$ by finite spaces of accordion type. Let $(d_n)_{n\in\mathbb{N}}$ be a dense sequence in $(0,1)$. For $n \in \mathbb{N}$, let $U_{2n-1} = [0,d_n)$ and $U_{2n} = (d_n,1]$. Denote by τ_n the topology generated by $\{U_1,\ldots,U_{2n}\}$ and by X_n its T_0-quotient. Then X_n is an accordion space of the form

$$\begin{tikzcd}[column sep=small]
& \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & \\
\bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet.
\end{tikzcd}$$

We want to find a surjection $\varinjlim E_{X_n}(A,B) \to \operatorname{Hom}(FK(A),FK(B))$, because this will imply that an isomorphism of the filtrated K-theory lifts to an invertible element in E_X:

$$E_X(A,B) \twoheadrightarrow \varinjlim E_{X_n}(A,B) \twoheadrightarrow \operatorname{Hom}(FK(A),FK(B))$$

and the result will follow from the previous Theorems. We need some results. □

Theorem 3.2. (Bentmann-Köler) Suppose that X is a finite accordion space, and let A and B be C^*-algebras over X in the Bootstrap category for E_X. Then there is a short exact sequence

$$0 \to \operatorname{Ext}^1_{NT^X}(FK^X(A),FK^X(B)) \to E^*_X(A,B) \to \operatorname{Hom}_{NT^X}(FK^X(A),FK^X(B)) \to 0.$$

We will apply this short exact sequence to the spaces $X = X_n$.

Proof. (Continuation, Theorem 3.1) Notice that $A(x) \cong A(x) \cong \mathcal{Q} \otimes \mathcal{K}$, and since $[0,1]$ is finite dimensional, it follows by Hirshberg-Rørdam-Winter that $A \cong A \otimes \mathcal{Q} \otimes \mathcal{K}$. In particular, $\operatorname{Ext}^1_{NT^{X_n}}(FK^{X_n}(A),FK^{X_n}(B)) = 0$ because the K-theory of A is rational. Thus,

$$E^*_{X_n}(A,B) \xrightarrow{\cong} \operatorname{Hom}_{NT^{X_n}}(FK^{X_n}(A),FK^{X_n}(B)).$$

By continuity of K-theory, the right-hand side is isomorphic to $\mathrm{Hom}(FK(A), FK(B))$. The resulting isomorphism $\varinjlim E_{X_n}(A, B) \to \mathrm{Hom}(FK(A), FK(B))$ is the desired map. $\square$

The following remains as an open problem.

Problem 3.3. Classify all continuous fields of stable UCT Kirchberg algebras over $[0, 1]$.

3.1. **Comments on the UCT for $C_0(X)$-algebras.** For C^*-algebras, the UCT involves a surjection

$$KK(A, B) \twoheadrightarrow \mathrm{Hom}(K_*(A), K_*(B)).$$

For X-algebras with X totally disconnected, one may guess that the surjection would be

$$KK_X(A, B) \twoheadrightarrow \mathrm{Hom}_{C(X,\mathbb{Z})}(K_*(A), K_*(B)) \cong \mathrm{Hom}_{NT^X}(FK^X(A), FK^X(B)).$$

However, this map is in general not surjective. One must instead use K-theory with coefficients:

$$KK_X(A, B) \twoheadrightarrow \mathrm{Hom}_{C(X,\Lambda)}(\underline{K}(A), \underline{K}(B)).$$

The case of 1-dimensional spaces is even more complicated.