HIGHER RANK GRAPH ALGEBRAS

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ABSTRACT. These are lecture notes of a course given by **Alex Kumjian** at the *RMMC Summer School* at the University of Wyoming, Laramie, June 1-5, 2015.

Warning: little proofreading has been done.

Contents

1. Introduction

C^*-algebras of higher-rank graphs (or k-graphs) first appeared in work of Kumjian and Pask as generalizations of graph C^*-algebras. Their introduction was inspired by work of Robertson and Steger on C^*-algebras arising from certain group actions on buildings. C^*-algebras of k-graphs form a wide class nuclear C^*-algebras that satisfy the UCT. They provide key examples in noncommutative geometry and have been used to calculate nuclear dimension for Kirchberg algebras. Under mild hypotheses there is a path groupoid $\mathcal{G}_\Lambda$ such that $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$. Using this isomorphism, it was shown that $C^*(\Lambda)$ is simple iff Λ is aperiodic and cofinal. Recently, cohomological properties of k-graphs have been explored. Given a k-graph Λ and a $\mathbb{T}$-valued 2-cocycle c, one may form the twisted k-graph C^*-algebra$C^*(\Lambda, c)$. Examples include all noncommutative tori and crossed products of Cuntz algebras by quasifree automorphisms. When the cocycle is of exponential form, it was shown that $K_*(C^*(\Lambda, c)) \cong K_*(C^*(\Lambda))$. Simplicity of $C^*(\Lambda, c)$ has been characterized, but the ideal structure remains an enigma.

2. Graph algebras

We recall the definition of graph algebras. Let $E = (E^0, E^1, r, s)$ be a directed graph. Suppose that r is onto (E is source free), and finite-to-one (E is row finite). Let $C^*(E)$ denote the universal C^*-algebra generated by a family $\{p_v : v \in E^0\}$ of orthogonal, and a family $\{t_e : e \in E^1\}$ of partial isometries, satisfying

(1) $t_e^* t_e = p_{s(e)}$ for all $e \in E^1$, and
(2) $p_v = \sum\limits_{r(e)=v} t_e t_e^*$ for all $v \in E^0$.

With the convention adopted, the concatenation

$$\bullet \xrightarrow{e} \bullet \xrightarrow{f} \bullet$$

is written fe (thought of as composition of arrows). Some authors use the opposite convention.

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Example 2.1. For $n \geq 1$, denote by O_n the graph with one vertex and n loops around it. For $n = 1$, we have $C^*(O_1) \cong C(\mathbb{T})$. On the other hand, for $n \geq 2$ it is easy to check that $C^*(O_n) \cong \mathcal{O}_n$, the Cuntz algebra on n generators.

Example 2.2. Given $n \geq 1$, let C_n denote the graph with n vertices $\{v_1, \ldots, v_n\}$ and n edges $\{e_1, \ldots, e_n\}$, and with range and source maps given by $r(e_j) = v_j$ and $s(e_j) = e_{j+1}$ for all $j = 1, \ldots, n$, with indices taken modulo n. Then $C^*(C_n) \cong M_n(C(\mathbb{T}))$, with the center being generated by

$$\sum_{j=1}^{n} t_{e_j} \cdots t_{e_n} t_{e_1} \cdots t_{e_{j-1}}.$$

Example 2.3. Let Ω denote the graph with $\Omega^0 = \Omega^1 = \mathbb{N}$, with $s(n) = n + 1$ and $r(n) = n$ for all $n \in \mathbb{N}$.

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots$$

Then $C^*(\Omega) \cong \mathcal{K}(\ell^2(\mathbb{N}))$.

A *small category* is a category $\mathcal{C}$ for which the class $\mathrm{Ob}(\mathcal{C})$ of its objects is a set. In this case, the class $\mathrm{Mor}(\mathcal{C})$ of its morphisms is also a set. Identify $\mathrm{Ob}(\mathcal{C})$ with the subset of $\mathrm{Mor}(\mathcal{C})$ consisting of the identity morphisms. For $\xi \in \mathrm{Mor}(u, v)$, write $s(\xi) = u$ and $r(\xi) = v$. There is a composition map

$$m \colon \mathcal{C} * \mathcal{C} = \{(\xi, \eta) \in \mathcal{C} \times \mathcal{C} \colon s(\xi) = r(\eta)\} \to \mathcal{C}$$

given by $m(\xi, \eta) = \xi\eta$ for $(\xi, \eta) \in \mathcal{C} * \mathcal{C}$. The map m is associative and satisfies $\xi = \mathrm{id}_{r(\xi)}\xi = \xi\mathrm{id}_{s(\xi)}$ for all $\xi \in \mathrm{Mor}(\mathcal{C})$.

Example 2.4. Consider $\mathcal{C} = \mathbb{N}$ with $\mathrm{Ob}(\mathbb{N}) = \{0\}$. Composition is $m(j, k) = j + k$.

Example 2.5. Let $(P, \leq)$ be a partially ordered set. Set $\widetilde{P} = \{(p, q) \colon p \leq q\}$. Then $\mathrm{Ob}(P) = P$, $\mathrm{id}_p = (p, p)$, and $(p, q)(q, r) = (p, r)$ for all $p, q, r \in P$.

Given a directed graph E and $n \geq 2$, let E^n denote the set of n-paths on the graph, and extend r and s to E^n in the obvious way. Set $E^* = \bigcup\limits_{n \in \mathbb{N}} E^n$. Then E^* is a small category, and there is a degree functor $d \colon E^* \to \mathbb{N}$ with the following property. Whenever $m, n \in \mathbb{N}$ and $\lambda \in E^*$ satisfy $d(\lambda) = n + m$, there exist unique $\nu, \mu \in E^*$ with $\lambda = \nu\mu$, $d(\nu) = n$ and $d(\mu) = m$.

We can reformulate the definition of $C^*(E)$ using E^* as follows.

Definition 2.6. The C^*-algebra $C^*(E^*)$ is the universal C^*-algebra generated by a family $\{t_\lambda \colon \lambda \in E^*\}$ of partial isometries satisfying

(1) $\{t_v \colon v \in E^0\}$ is a family of orthogonal projections;
(2) for all $\lambda \in E^*$, we have $t_\lambda^* t_\lambda = t_{s(\lambda)}$;
(3) if $s(\lambda) = r(\mu)$, then $t_{\lambda\mu} = t_\lambda t_\mu$;
(4) For all $v \in E^0$ and for all $n \in \mathbb{N}$, we have

$$t_v = \sum_{r(\lambda) = v, d(\lambda) = n} t_\lambda t_\lambda^*.$$

3. Higher rank graphs

Throughout this section, we fix $k \in \mathbb{N}$.

Definition 3.1. Let Λ be a small category, and let $d \colon \Lambda \to \mathbb{N}^k$ be a functor. We say that (Λ, d) is a *k-graph* if it satisfies the unique factorization property: for all $\lambda \in \Lambda$ and for all $m, n \in \mathbb{N}^k$ satisfying $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ with $\lambda = \mu\nu$, such that $d(\mu) = m$ and $d(\nu) = n$.

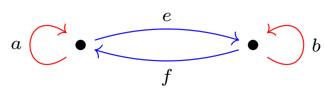
For $n \in \mathbb{N}^k$, set $\Lambda^n = d^{-1}(\{n\})$, and identify $\Lambda^0 = \mathrm{Ob}(\Lambda)$ with the set of vertices. For $j = 1, \ldots, k$, an element $\lambda \in \Lambda^{e_j}$ is called an *edge*. For $v, w \in \Lambda^0$ and $X \subseteq \Lambda$, we set $vX = r^{-1}(v) \cap X$ and $Xv = s^{-1}(v) \cap X$. We assume that Λ is row finite ($v\Lambda^n$ is finite and nonempty for all $v \in \Lambda^0$).

A *morphism* between k-graphs is a functor between the respective categories that intertwines the degree functors.

Remark 3.2. If $k = 0$, then d is trivial and Λ is just a set. If $k = 1$, then $\Lambda = E^*$ for some directed graph E. If $k \geq 2$, we think of Λ as generated by k directed graphs of different colors that share the same set of vertices.

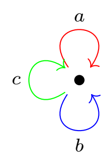
In sketching k-graphs, we include only vertices and edges. One has to specify "commuting squares", unless there is no choice.

Example 3.3. Consider the following k-graph:



Here, we must have $be = ea$ and $ae = eb$, because there are no other possible decompositions.

Example 3.4. The simplest k-graph T_k has one vertex and one edge for each color. That is, $T_k = \mathbb{N}^k$ and d is the identity map. For example, the following is T_3:



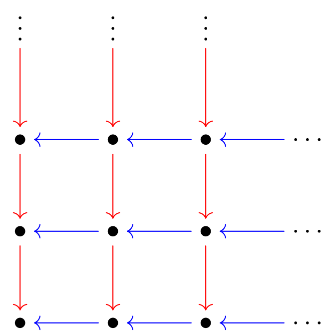
(Observe that one does not need to specify commuting squares.)

This is the k-graph analog of a torus; see Example 3.14.

Example 3.5. Denote by Ω_k the k-graph given by $\Omega_k^0 = \mathbb{N}^k$ and $\Omega_k^1 = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \colon m \leq n\}$, with structure maps

$$s(m, n) = n \quad r(m, n) = m \quad (m, n)(n, \ell) = (m, \ell) \quad d(m, n) = n - m.$$

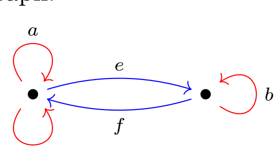
For $k = 2$, the k-graph is



Definition 3.6. If Λ is a k-graph and Σ is an ℓ-graph, then $\Lambda \times \Sigma$, with coordinate wise operations, is a $(k + \ell)$-graph.

In a cartesian product, we always have $ea = ae$ for edges a and e, so the commuting squares do not need to be specified. In some cases, commuting squares must be made explicit:

Example 3.7. Consider the following graph:



One has to determine whether $ae = ea$ and $be = eb$, or $ae = eb$ and $be = ea$. In the first case, one gets $\mathcal{O}_2 \otimes C(\mathbb{T})$, and in the second one, one gets $\mathcal{O}_2 \rtimes \mathbb{Z}$, where the automorphism is determined by exchanging the canonical generators.

We now proceed to define the C^*-algebra associated to a k-graph.

Definition 3.8. Let Λ be a k-graph. We define $C^*(\Lambda)$ to be the universal C^*-algebra generated by a family $\{t_\lambda\colon \lambda \in \Lambda\}$ of partial isometries satisfying:

(1) $\{t_v\colon v \in \Lambda^0\}$ is a family of orthogonal projections;
(2) for all $\lambda \in \Lambda$, we have $t_\lambda^* t_\lambda = t_{s(\lambda)}$;
(3) if $s(\lambda) = r(\mu)$, then $t_{\lambda\mu} = t_\lambda t_\mu$;
(4) For all $v \in \Lambda^0$ and for all $n \in \mathbb{N}^k$, we have

$$t_v = \sum_{v\Lambda^n} t_\lambda t_\lambda^*.$$

For $v \in \Lambda^0$, we usually denote t_v by p_v.

Remark 3.9. One can check that $C^*(\Lambda)$ is unital if and only if Λ^0 is finite.

Remark 3.10. The set $\mathrm{span}\{t_\lambda t_\mu\colon s(\mu) = s(\lambda)\}$ is dense in $C^*(\Lambda)$.

By universality, there is a strongly continuous action $\gamma\colon \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$ which, for $z = (z_1, \dots, z_k) \in \mathbb{T}^k$, is given by

$$\gamma_z(t_\lambda) = z_1^{d(\lambda)_1} \cdots z_k^{d(\lambda)_k} t_\lambda$$

for all $\lambda \in \Lambda$. This is called the *gauge action*.

The following result is useful when determining whether a given representation is faithful.

Theorem 3.11. Let $\pi\colon C^*(\Lambda) \to B$ be a homomorphism, and suppose there exists an action $\beta\colon \mathbb{T}^k \to \mathrm{Aut}(B)$ such that $\pi \circ \gamma_z = \beta_z \circ \pi$ for all $z \in \mathbb{T}^k$. Then π is injective if and only if $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.

Proof. We give a sketch of the argument. It is known that $C^*(\Lambda)^\gamma$ is an AF-algebra, and that every projection in it is Murray-von Neumann equivalent to a vertex projection. Since π does not vanish on the vertex projections, it follows that the restriction of π to $C^*(\Lambda)^\gamma$ is injective. Now, there is a faithful conditional expectation $E\colon C^*(\Lambda) \to C^*(\Lambda)^\gamma$, given by integration, and using the commutative diagram

$$\begin{tikzcd} C^*(\Lambda) \arrow[r, "\pi"] \arrow[d, "E"'] & B \arrow[d, "E"] \\ C^*(\Lambda)^\gamma \arrow[r, "\pi|_{C^*(\Lambda)^\gamma}"'] & B^\beta, \end{tikzcd}$$

we deduce that π is also faithful. $\square$

Remark 3.12. The existence of a conditional expectation from $C^*(\Lambda)$ to a nuclear C^*-algebra (in this case, the AF-algebra $C^*(\Lambda)^\gamma$) implies that $C^*(\Lambda)$ is nuclear.

A simple application of the above result gives the following:

Corollary 3.13. If Λ is a k-graph and Σ is an ℓ-graph, then there is a canonical isomorphism

$$C^*(\Lambda \times \Sigma) \cong C^*(\Lambda) \otimes C^*(\Sigma).$$

Example 3.14. Recall the k-graph T_k from Example 3.4. Since $T_k = T_1 \times \cdots T_1$, we immediately see that there are natural isomorphisms

$$C^*(T_k) \cong \bigotimes_{j=1}^k C^*(T_1) \cong \bigotimes_{j=1}^k C(\mathbb{T}) \cong C(\mathbb{T}^k).$$

Example 3.15. Recall the k-graph Ω_k from Example 3.5. Since $\Omega_k = \Omega_1 \times \cdots \Omega_1$, we immediately see, using Example 2.3 at the second step, that there are natural isomorphisms

$$C^*(\Omega_k) \cong \bigotimes_{j=1}^k C^*(\Omega_1) \cong \bigotimes_{j=1}^k \mathcal{K}(\ell^2(\mathbb{N})) \cong \mathcal{K}(\ell^2(\mathbb{N}^k)).$$

Example 3.16. Let $m, n \in \mathbb{N}$, and let C_m and C_n be as in Example 2.2. Then there is a natural isomorphism $C^*(C_m \times C_n) \cong M_{mn}(C(\mathbb{T}^2))$.

Example 3.17. Let $m, n \in \mathbb{N}$ with $m, n \geq 2$, and let O_m and O_n be as in Example 2.1. Then there is a natural isomorphism $C^*(O_m \times O_n) \cong \mathcal{O}_m \otimes \mathcal{O}_n$.

4. Crossed products of higher rank graphs

4.1. **Higher rank graph automorphisms.** Suppose that α is an automorphism of Λ. Then there is a (unique) automorphism $\widetilde{\alpha}$ of $C^*(\Lambda)$ given by $\widetilde{\alpha}(t_\lambda) = t_{\alpha(\lambda)}$ for all $\lambda \in \Lambda$. We can construct a $(k+1)$-graph $\Lambda \rtimes_\alpha \mathbb{Z}$ from α as follows. As a set, we have $\Lambda \rtimes_\alpha \mathbb{Z} = \Lambda \times \mathbb{N}$, and the structure maps are given by

$$s(\lambda, n) = (s(\alpha^{-n}(\lambda)), 0) \quad r(\lambda, n) = (r(\lambda), 0) \quad (\lambda, \ell)(\mu, m) = (\lambda \alpha^\ell(\mu), \ell + m) \quad d(\lambda, n) = (d(\lambda), n).$$

Theorem 4.1. Let $\pi \colon C^*(\Lambda) \to M(C^*(\Lambda) \rtimes_{\widetilde{\alpha}} \mathbb{Z})$ be the canonical embedding. Let $u \in M(C^*(\Lambda) \rtimes_{\widetilde{\alpha}} \mathbb{Z})$ be the canonical unitary implementing $\widetilde{\alpha}$ in the crossed product. Then there is a canonical isomorphism

$$\varphi \colon C^*(\Lambda \rtimes_\alpha \mathbb{Z}) \to C^*(\Lambda) \rtimes_{\widetilde{\alpha}} \mathbb{Z}$$

given by $\varphi(t_{(\lambda,n)}) = \pi(t_\lambda) u^n$ for all $(\lambda, n) \in \Lambda \rtimes_\alpha \mathbb{Z} = \Lambda \times \mathbb{N}$.

4.2. **1-cocycles and group actions.** Let G be a locally compact abelian group. A map $c \colon \Lambda \to G$ is called a *G-valued 1-cocycle on* Λ if $c(\lambda\mu) = c(\lambda) + c(\mu)$ for all $\lambda, \mu \in \Lambda$. The collection of all G-valued 1-cocycles on Λ is denoted by $Z^1(G, \Lambda)$.

Given $c \in Z^1(G, \Lambda)$, there is an action $\alpha^c \colon \widehat{G} \to \operatorname{Aut}(C^*(\Lambda))$ given by

$$\alpha^c_\chi(t_\lambda) = \chi(c(\lambda)) t_\lambda$$

for all $\chi \in \widehat{G}$ and for all $\lambda \in \Lambda$.

Example 4.2. The degree map $d \colon \Lambda \to \mathbb{N}^k \subseteq \mathbb{Z}^k$ can be regarded as a $\mathbb{Z}^k$-valued 1-cocycle on Λ. The induced action $\alpha^d \colon \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ is precisely the gauge action.

Suppose that G is discrete. Then the skew product $\Lambda \times_c G$ is a k-graph (same rank as Λ) with structure maps given by

$$s(\lambda, g) = (s(\lambda), g + c(\lambda)) \quad r(\lambda, g) = (r(\lambda), g) \quad d(\lambda, g) = d(\lambda)$$

and with composition given by $(\lambda, g)(\mu, g + c(\lambda)) = (\lambda\mu, g)$ if $s(\lambda) = r(\mu)$.

Proposition 4.3. There is a canonical isomorphism $C^*(\Lambda \times_c G) \cong C^*(\Lambda) \rtimes_{\alpha^c} \widehat{G}$. In particular, and with $\gamma \colon \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ denoting the gauge action, we have

$$C^*(\Lambda) \rtimes_\gamma \mathbb{T}^k \cong C^*(\Lambda \rtimes_d \mathbb{Z}^k).$$

Suppose that a discrete group G acts on Λ by k-graph automorphisms. Then the quotient Λ/G inherits a natural k-graph structure from Λ. The following result allows us to identify, up to Morita equivalence, the crossed product by a free action with the C^*-algebra of the orbit space k-graph.

Theorem 4.4. Suppose that a discrete group G acts freely on Λ. (This condition can be checked on the vertices.) Denote by $\beta \colon G \to \operatorname{Aut}(C^*(\Lambda))$ the induced action. Then there is a natural isomorphism

$$C^*(\Lambda) \rtimes_\beta G \cong C^*(\Lambda/G) \otimes \mathcal{K}(\ell^2(G)).$$

Let G be an abelian discrete group and let $c \in Z^1(G, \Lambda)$. Define a k-graph action $\beta^c \colon G \to \operatorname{Aut}(\Lambda \times_c G)$ by

$$\beta^c_g(\lambda, h) = (\lambda, g + h)$$

for all $g, h \in G$ and for all $\lambda \in \Lambda$. Then β^c is free. Moreover, every free action is of this form.

Corollary 4.5. Let G be an abelian discrete group and let $c \in Z^1(G, \Lambda)$. Then there is a natural isomorphism

$$C^*(\Lambda \times_c G) \rtimes_{\beta^c} G \cong C^*(\Lambda) \otimes \mathcal{K}(\ell^2(G)).$$

In the corollary above, and under the identification $C^*(\Lambda \times_c G) \rtimes_{\beta^c} \cong C^*(\Lambda) \rtimes_{\alpha^c} \widehat{G}$, the action β^c is identified with $\widehat{\alpha^c}$. In this sense, the above result can be deduced from Takai duality.

Proposition 4.6. Let Σ be a k-graph. Suppose that its degree map is a coboundary, that is, that there exists a function $b \colon \Sigma^0 \to \mathbb{Z}^k$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Sigma$. Then $C^*(\Sigma)$ is AF.

We will use the above proposition to prove the following result.

Theorem 4.7. Let $\gamma\colon \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$ denote the gauge action. Then $C^*(\Lambda) \rtimes_\gamma \mathbb{T}^k$ is AF.

Proof. By Example 4.2, γ is the action associated to the 1-cocycle d. By Proposition 4.3, there is a canonical isomorphism $C^*(\Lambda) \rtimes_\gamma \mathbb{T}^k \cong C^*(\Lambda \rtimes_d \mathbb{Z}^k)$.

We claim that the degree map for $\Lambda \times_d \mathbb{Z}^k$ is a coboundary with $b\colon (\Lambda \times_d \mathbb{Z}^k)^0 \to \mathbb{Z}^k$ given by $b(\xi, n) = n$ for all $(\xi, n) \in (\Lambda \times_d \mathbb{Z}^k)^0$. Indeed, for $(\lambda, n) \in \Lambda \times_d \mathbb{Z}^k$, we have

$$b(s(\lambda, n)) - b(r(\lambda, n)) = b(s(\lambda), n + d(\lambda)) - b(r(\lambda), n) = d(\lambda) = d(\lambda, n),$$

and the claim is proved. The result now follows from Proposition 4.6. $\square$

As a consequence, we get

Theorem 4.8. There is a canonical isomorphism

$$C^*(\Lambda \times_d \mathbb{Z}^k) \rtimes_{\widetilde{\beta}^d} \mathbb{Z}^k \cong C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\mathbb{Z}^k)).$$

In particular, $C^*(\Lambda)$ is strongly Morita equivalent to the crossed product of an AF-algebra by $\mathbb{Z}^k$. We deduce that $C^*(\Lambda)$ is nuclear and satisfies the UCT.

We now turn to gauge invariant ideals.

Definition 4.9. Let H be a subset of Λ^0.

(1) We say that H is *hereditary* if for every $\lambda \in \Lambda$, whenever $r(\lambda)$ belongs to H, then $s(\lambda)$ belongs to H as well. (We think of Λ^0 with the partial order given by $v \leq w$ if there exists $\lambda \in \Lambda$ with $s(\lambda) = v$ and $r(\lambda) = w$. Then H is hereditary in the usual sense.)

(2) We say that H is *saturated* if for all $v \in \Lambda^0$, if there exists $j \in \{1, \ldots, k\}$ such that $s(v\Lambda^{e_j}) \subseteq H$, then $v \in H$. (If a vertex is such that all edges of one color ending at it have sources in H, then the vertex is in H.)

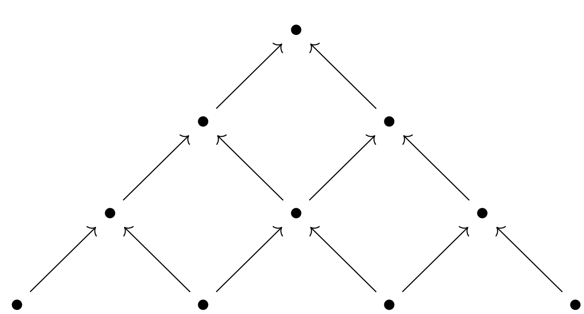
Remark 4.10. The collection of all hereditary saturated subsets of Λ^0 is a lattice with the obvious operations.

We can describe all the gauge invariant ideals. For $H \subseteq \Lambda^0$, denote by I_H the ideal in $C^*(\Lambda)$ generated by $\{p_v \colon v \in H\}$. When H is hereditary and saturated, then

$$I_H = \overline{\mathrm{span}}\{t_\lambda t_\mu^* \colon s(\lambda) = s(\mu) \in H\}.$$

Theorem 4.11. The assignment $H \mapsto I_H$ defines a one-to-one correspondence between the gauge invariant ideals in $C^*(\Lambda)$ and the collection of all hereditary saturated subsets of Λ^0.

Example 4.12. Consider the following 1-graph:



The C^*-algebra associated to this graph is the GICAR algebra: the gauge invariant CAR algebra. It is also the fixed point algebra of $\mathcal{O}_2$ under the $\mathbb{T}^2$ action given by $\alpha_{(z_1,z_2)}(s_j) = z_j s_j$ for $(z_1, z_2) \in \mathbb{T}^2$ and $j = 1, 2$.

Observe that hereditary saturated subsets of Λ^0 are in one-to-one correspondence with vertices, where a vertex has associated to it the set of its predecessors.

5. The path groupoid of a higher rank graph

Recall the definition of the k-graph Ω_k from Example 3.5.

Definition 5.1. A k-graph morphism $x\colon \Omega_k \to \Lambda$ is said to be an *infinite path* in Λ. We denote by Λ^∞ the set of all infinite paths in Λ. For $x \in \Lambda^\infty$, we set $r(x) = x(0,0)$.

The cylinder set determined by $\lambda \in \Lambda$ is

$$Z(\lambda) = \{x \in \Lambda^\infty \colon \lambda = x(0, d(\lambda))\}.$$

This set is never empty since $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$. Note that $x \in Z(r(x))$ for all $x \in \Lambda^\infty$.

The collection $\{Z(\lambda) \colon \lambda \in \Lambda\}$ forms a basis for a topology on Λ^∞.

Definition 5.2. For $p \in \mathbb{N}^k$, define the shift $\sigma^p \colon \Lambda^\infty \to \Lambda^\infty$ by

$$\sigma^p(x)(m,n) = x(m+p, n+p)$$

for all $x \in \Lambda^\infty$ and for all $(m,n) \in \mathbb{N}^k \times \mathbb{N}^k$.

If $p = d(\lambda)$, then σ^p induces a homeomorphism $Z(\lambda) \cong Z(s(\lambda))$. Thus, for all $x \in Z(s(\lambda))$, there exists a unique $y \in Z(\lambda)$ such that $x = \sigma^p(y)$. In this case, we write $y = \lambda x$. Observe that σ^p is a local homeomorphism for all $p \in \mathbb{N}^k$.

A key technical tool to analyze $C^*(\Lambda)$ is the path groupoid $\mathcal{G}_\Lambda$ introduced by Renault.

Definition 5.3. Denote by $\mathcal{G}_\Lambda$ the groupoid

$$\mathcal{G}_\Lambda = \{(x, m-n, y) \in \Lambda^I \times \mathbb{Z}^k \times \Lambda^I \colon m,n \in \mathbb{N}^k, \sigma^m(x) = \sigma^n(y)\}.$$

The unit space of $\mathcal{G}_\Lambda$ may be identified with Λ^0 via $x \mapsto (x,0,x)$. Under this identification, we have $r(x,\ell,y) = x$ and $s(x,\ell,y) = y$. Composition is given by

$$(x,\ell,y)(y,j,z) = (x,\ell+j,z),$$

and the inverse of (x,ℓ,y) is $(y,-\ell,x)$.

If $\sigma^m(x) = \sigma^n(y)$, then there exist $\mu,\nu \in \Lambda$ and $z \in \Lambda^\infty$ such that

$$d(\mu) = m, \quad d(\nu) = n, \quad s(\mu) = s(\nu) = r(z), \quad x = \mu z \quad \text{and} \quad y = \nu z.$$

For $\mu,\nu \in \Lambda$ with $s(\mu) = s(\nu)$, set

$$Z(\mu,\nu) = \{(\mu z, d(\mu) - d(\nu), \nu z) \colon z \in Z(s(\mu))\}.$$

The family of all $Z(\mu,\nu)$ forms a basis for a topology on $\mathcal{G}_\Lambda$, and $Z(\mu,\nu)$ is compact in this topology. In particular, $\mathcal{G}_\Lambda$ is zero-dimensional. With this topology, $\mathcal{G}_\Lambda$ is an étale, ample, Hausdorff groupoid, and it is amenable since it is a Renault-Deaconu groupoid.

For each $\lambda \in \Lambda$, we let T_λ denote the characteristic function of $Z(\lambda, s(\lambda))$. (We think of $Z(\lambda, s(\lambda))$ as the set of all infinite paths in Λ terminating at $s(\lambda)$.)

Theorem 5.4. The map $t_\lambda \mapsto T_\lambda$, for $\lambda \in \Lambda$, induces a canonical isomorphism

$$C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda).$$

Proof. We sketch the proof. The operators T_λ satisfy the relations in the definition of $C^*(\Lambda)$, so there exists a canonical homomorphism $\psi\colon C^*(\Lambda) \to C^*(\mathcal{G}_\Lambda)$ given by $\psi(t_\lambda) = T_\lambda$ for all $\lambda \in \Lambda$.

Since $\psi(t_\lambda t_\mu^*) = T_\lambda T_\mu^*$ is the characteristic function of $Z(\mu,\lambda)$, the span of such functions is dense in $C^*(\mathcal{G}_\Lambda)$. It follows that ψ is surjective. To show that it is injective, we will use Theorem 3.11.

There is an action $\beta\colon \mathbb{T}^k \to \mathrm{Aut}(C^*(\mathcal{G}_\Lambda))$ given by

$$\beta_z(f)(x,\ell,y) = z_1^{\ell_1} \cdots z_k^{\ell_k} f(x,\ell,y)$$

for all $f \in C_c(\mathcal{G}_\Lambda)$ and for all $(x,\ell,y) \in \mathcal{G}_\Lambda$. It is easy to check that ψ is equivariant with respect to β. Since $\psi(t_v) = T_v \neq 0$ for all $v \in \Lambda^0$, it follows from Theorem 3.11 that ψ is injective, and thus an isomorphism. $\square$

6. Simplicity and pure infiniteness of $C^(\Lambda)$

We begin by studying simplicity of $C^*(\Lambda)$.

Definition 6.1. We say that Λ is *cofinal* if for all $v \in \Lambda^0$ and for all $x \in \Lambda^\infty$ there exist $\lambda \in \Lambda$ and $n \in \mathbb{N}^k$ such that $s(\lambda) = x(n,n)$ and $r(\lambda) = v$.

Remark 6.2. Λ is cofinal if and only if $\mathcal{G}_\Lambda$ is minimal, and if and only if there exist no nontrivial gauge invariant ideals.

For $v \in \Lambda^0$, the local periodicity group at v, denoted $P_\Lambda(v)$, is

$$P_\Lambda(v) = \{m - n \in \mathbb{Z}^k \colon m, n \in \mathbb{N}^k, \sigma^m(x) = \sigma^n(x) \text{ for all } x \in Z(v)\}.$$

Definition 6.3. We say that Λ is *aperiodic* if for all $v \in \Lambda^0$ there exists $x \in v\Lambda^\infty$ such that the map $\mathbb{N}^k \to \Lambda^\infty$, given by $n \mapsto \sigma^n(x)$, is injective.

Remark 6.4. Λ is aperiodic if and only if $P_\Lambda(v) = 0$ for all $v \in \Lambda^0$, and if and only if $\mathcal{G}_\Lambda$ has trivial isotropy groups.

It is not in general true that Λ is aperiodic if and only if every ideal in $C^*(\Lambda)$ is gauge invariant. Hence, the proof of the following result requires the combination of both properties rather than both of them separately.

Theorem 6.5. The k-graph Λ is cofinal and aperiodic if and only if $C^*(\Lambda)$ is simple.

We now turn to pure infiniteness in the simple case.

Definition 6.6. An element $\lambda \in \Lambda$ is said to be a *loop* if $d(\lambda) \neq 0$ and $s(\lambda) = r(\lambda)$.

We say that $\mu \in s(\lambda)\Lambda$ is an *entrance* for λ if $d(\mu) \leq d(\lambda)$ and λ cannot be written as $\mu\nu$ for any ν.

We say that a vertex $v \in \Lambda^0$ *can be reached from a loop with an entrance* if there exist a loop λ with an entrance and $\alpha \in \Lambda$ with $r(\alpha) = v$ and $s(\alpha) = s(\lambda)$.

Observe that the following result only provides a sufficient condition for pure infiniteness of $C^*(\Lambda)$.

Theorem 6.7. Suppose that Λ is cofinal and aperiodic. If every vertex in Λ can be reached from a loop with an entrance, then $C^*(\Lambda)$ is simple and purely infinite. In particular, $C^*(\Lambda)$ is a Kirchberg algebra that satisfies the UCT by Theorem 4.8, and it is therefore classified by its K-theory.

The condition in the theorem above is satisfied in a number of cases, for example for the graphs O_n from Example 2.1.

7. Twisted higher rank graph

Let G be a locally compact abelian group. For $n \geq 1$, let Λ^{*n} denote the set of composable n-tuples:

$$\Lambda^{*n} = \{(\lambda_1, \dots, \lambda_n) \in \Lambda^n \colon s(\lambda_j) = r(\lambda_{j+1}) \text{ for } j = 1, \dots, n\},$$

and set $\Lambda^{*0} = \Lambda^0$.

Definition 7.1. We say that a function $f \colon \Lambda^{*n} \to G$ is an *n-cochain* if either $n = 0$ or $f(\lambda_1, \dots, \lambda_n) = 0$ whenever $\lambda_j \in \Lambda^0$ for some $j = 1, \dots, n$.

Let $C^n(\Lambda, G)$ denote the group of n-chochains. For $f \in C^n(\Lambda, G)$ with $n \geq 1$, define $\delta^n f \colon \Lambda^{*(n+1)} \to G$ by

$$(\delta^r f)(\lambda_0, \dots, \lambda_n) = \sum_{j=0}^{n} (-1)^j f(\lambda_0, \dots, \widehat{\lambda_j}, \dots, \lambda_n)$$

for $(\lambda_0, \dots, \lambda_n) \in \Lambda^{*(n+1)}$. For $f \in C^0(\Lambda, G)$, define $\delta^0 f \colon \Lambda^{*1} \to G$ by

$$\delta^0 f(\lambda) = f(s(\lambda)) - f(r(\lambda))$$

for $\lambda \in \Lambda^0$.

One checks that these boundary maps satisfy $\delta^{n+1} \circ \delta^n = 0$ for all $n \geq 0$. We denote by $H^*(\Lambda, G)$ the cohomology of this complex. That is, with $Z^n(\Lambda, G) = \ker(\delta^n)$ and $B^n(\Lambda, G) = \Im(\delta^{n-1})$, we have

$$H^n(\Lambda, G) = Z^n(\Lambda, G)/B^n(\Lambda, G).$$

Remark 7.2. Our main focus will be $H^2(\Lambda, \mathbb{T})$, with $\mathbb{T}$ written multiplicatively.

Consider a 2-cocycle $c \in Z^2(\Lambda, \mathbb{T})$, that is, a map $c\colon \Lambda^{*2} \to G$ satisfying that for every $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^{*3}$ we have

$$c(\lambda_1, \lambda_2)c(\lambda_1\lambda_2, \lambda_3) = c(\lambda_1, \lambda_2\lambda_3)c(\lambda_2, \lambda_3).$$

Then c is a coboundary (that is, $c \in B^2(\Lambda, \mathbb{T})$) if there exists a function $b\colon \Lambda \to G$ satisfying

$$c(\lambda_1, \lambda_2) = b(\lambda_1)\overline{b(\lambda_1\lambda_2)}b(\lambda_2).$$

Remark 7.3. There exists a homomorphism $H^2(\Lambda, \mathbb{T}) \to H^2(\mathcal{G}_\Lambda, \mathbb{T})$, where the right-hand side is the groupoid cohomology defined by Renault.

Definition 7.4. For $c \in Z^2(\Lambda, \mathbb{T})$, let $C^*(\Lambda, c)$ denote the universal C^*-algebra generated by a family $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

(1) $\{p_v : v \in \Lambda^0\}$ is a family of orthogonal projections;
(2) for all $\lambda \in \Lambda$, we have $t_\lambda^* t_\lambda = t_{s(\lambda)}$;
(3) if $s(\lambda) = r(\mu)$, then $t_{\lambda\mu} = c(\lambda, \mu)t_\lambda t_\mu$;
(4) For all $v \in \Lambda^0$ and for all $n \in \mathbb{N}^k$, we have

$$p_v = \sum_{v\Lambda^n} t_\lambda t_\lambda^*.$$

One can show that if c and c' are cohomologous, then there is a canonical isomorphism $C^*(\Lambda, c) \cong C^*(\Lambda, c')$.

Theorem 7.5. For $c \in Z^2(\Lambda, \mathbb{T})$, there exist a 2-cocycle σ_c on $\mathcal{G}_\Lambda$ and a canonical isomorphism $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$.

8. Simplicity of twisted higher rank graph algebras

If $C^*(\Lambda, c)$ is simple, then Λ is cofinal, but not necessarily aperiodic. So suppose that Λ is cofinal. Then $P_\Lambda(v)$is a subgroup of $\mathbb{Z}^k$ which does not depend on $v \in \Lambda^0$, so we denote it simply by P_Λ. There is a short exact sequence of groupoids

$$0 \to \Lambda^\infty \times P_\Lambda \to \mathcal{G}_\Lambda \to \mathcal{H}_\Lambda \to 0,$$

where $\mathcal{H}_\Lambda$ is minimal and topologically principal. Denote by ι the injective map $\iota\colon \Lambda^\infty \times P_\Lambda \to \mathcal{G}_\Lambda$. Then the cohomology class of $\iota_x^*(\sigma_c)$ in $H^2(P_\Lambda, \mathbb{T})$ is independent of x. Moreover, there exist $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ and $\omega \in Z^2(P_\Lambda, \mathbb{T})$ such that $[\sigma] = [\sigma_c]$ and $\iota_x^*(\sigma) = \omega$ for all $x \in \Lambda^\infty$. It follows that there is a canonical isomorphism

$$C^*(\Lambda^\infty \times P_\Lambda, \iota^*(\sigma)) \cong C_0(\Lambda^\infty) \otimes C^*(P_\Lambda, \omega).$$

(Observe that $C^*(P_\Lambda, \omega)$ is a twisted group C^*-algebra, and P_Λ is some $\mathbb{Z}^\ell$.)

Theorem 8.1. There exist a Fell bundle $\mathcal{B}_\Lambda^c$ over $\mathcal{H}_\Lambda$ and canonical isomorphisms

$$\mathcal{B}_\Lambda^c|_{\Lambda^\infty} \cong \Lambda^\infty \times C^*(P_\Lambda, \omega) \quad \text{and} \quad C^*(\Lambda, c) \cong C^*(\mathcal{H}_\Lambda, \mathcal{B}_\Lambda^c).$$

Set $Z_\omega = \{q \in P_\Lambda : \omega(p, q)\overline{\omega(q, p)} = 1$. By results of Olesen, Pedersen and Takesaki, we have

$$\mathrm{Prim}(C^*(P_\Lambda, \omega)) \cong \widehat{\mathbb{Z}_\omega}.$$

By work of Ionescu and Williams, there exists an action of $\mathcal{H}_\Lambda$ on

$$\mathrm{Prim}(C_0(\Lambda^\infty) \otimes C^*(P_\Lambda, \omega)) \cong \Lambda^\infty \times \widehat{\mathbb{Z}_\omega}.$$

Moreover, the action is determined by a 1-cocycle $\widetilde{c} \in Z^1(\mathcal{H}_\Lambda, \widehat{\mathbb{Z}_\omega})$.

Theorem 8.2. Suppose Λ is cofinal. Then the following are equivalent:

(1) $C^*(\Lambda, c)$ is simple.
(2) The action of $\mathcal{H}_\Lambda$ on $\Lambda^\infty \times \widehat{\mathbb{Z}_\omega}$ is minimal.
(3) For each $x \in \Lambda^\infty$, the set $\{\widetilde{c}(h) : s(h) = x\}$ is dense in $\widehat{\mathbb{Z}_\omega}$.

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