MODEL THEORY FOR C^*-ALGEBRAS

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ABSTRACT. These are lecture notes of a course given by **Martino Lupini** (plus an introductory lecture by Greg Oman) at the conference *Applications of model theory to operator algebras* at the University of Houston, USA, July 31 to August 4, 2017.

Warning: little proofreading has been done.

CONTENTS

1. INTRODUCTION

Logic for metric structures is a generalization of classical (or discrete) logic, suitable for applications to metric objects such as C^*-algebras. The most apparent contribution of first order logic is to provide a syntactic counterpart to the semantic construction of ultraproducts: the notion of formulas. Indeed, formulas allow one to express the fundamental properties of ultrapowers of C^*-algebras (saturation) and diagonal embeddings into them (elementarity). These features have appeared in the literature under various names (Kirchberg's ε-test, reindexation tricks, etc). Isolating such principles allows one to distinguish properties that are consequences of "general nonsense" from those that are special to C^*-algebras. The abstract model-theoretic framework also makes it easier to transfer ideas and argument to other contexts, such as the equivariant one.

2. FIRST ORDER LOGIC FOR DISCRETE STRUCTURES

This was an introductory lecture delivered by Greg Oman. Warning: the notation in this section differs from that in the other sections.

2.1. **First order languages.** A *first order language (with equality)* consists of a set $\mathcal{L}$ whose members are arranged as follows:

(1) Logical symbols: parentheses; logical operators ($\neg$, $\vee$, $\wedge$, $\to$ and $\leftrightarrow$); a variable x_n for every $n \in \mathbb{N}$; and the equality symbol.
(2) Parameters: quantifier symbols ($\forall$, $\exists$); predicate symbols, each of which has an arity (a natural number indicating how many inputs it takes); constant symbols; function symbols, each of which has an arity.

Example 2.1. The language of set theory consists of a single binary predicate symbol $\in$, no constant symbols, and no function symbols.

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Example 2.2. The language of unital ring theory consists of no predicate symbols, constant symbols 0 and 1, a binary function symbol $+$, a binary symbol $\cdot$, and a unary function ι (to be interpreted as $\iota(x) = -x$).

2.2. **Terms, formulas and sentences.** Let $n \in \mathbb{N}$, let S be a set, and let $f\colon S^n \to S$ be a function. A subset $X \subseteq S$ is said to be *closed* under f if $f(X^n) \subseteq X$. If $\mathcal{F}$ is a family of functions on S (with potentially different arity), we say that X is closed under $\mathcal{F}$ if it is closed under every member of $\mathcal{F}$. If $B \subseteq S$ is a subset and $\mathcal{F}$ is a family of functions on S, we denote by $\overline{B}$ the intersection of all $\mathcal{F}$-closed subsets of S containing B.

Definition 2.3. Given a language $\mathcal{L}$, the set of $\mathcal{L}$-expressions $\mathrm{seq}(\mathcal{L})$ is the set of all finite sequences of elements in $\mathcal{L}$. There is a semigroup operation, given by concatenation.

Example 2.4. In the language of ring theory, $(\cdot, \cdot, +, \iota, \exists, \forall, 1, \to)$ is an expression.

Definition 2.5. Let f be an n-ary function symbol, and define an operation $\varphi_f\colon \mathrm{seq}(\mathcal{L})^n \to \mathrm{seq}(\mathcal{L})$ by $\varphi_f(\varepsilon_1, \ldots, \varepsilon_n) = f\varepsilon_1 \cdots \varepsilon_n$. Set $\mathcal{F} = \{\varphi_f\colon f \text{ is a function symbol}\}$. Then the set of *terms* of $\mathcal{L}$ is the $\mathcal{F}$-closure of the set of constant symbols and variables in $\mathcal{L}$.

Example 2.6. In the language of ring theory, 0 is a term because it is a constant. Also $+00$ (thought of as $0+0$) is a term, and similarly with $++000$ (thought of as $0 + (0+0)$).

Definition 2.7. An *atomic formula* is an expression of the form $Pt_1 \cdots t_n$, where P is an n-ary predicate and $t_1, \ldots, t_n$ are terms.

Given a language $\mathcal{L}$, define the following operations on $\mathrm{seq}(\mathcal{L})$:

(a) $\varphi_\neg(\varepsilon) = \neq \varepsilon$;
(b) $\varphi_*(\alpha, \beta) = \alpha * \beta$ for all $* \in \{\vee, \wedge, \to, \leftrightarrow\}$;
(c) $\varphi_{\forall n}(\varepsilon) = \forall x_n \varepsilon$, for $n \in \mathbb{N}$; and
(d) $\varphi_{\exists n}(\varepsilon) = \exists x_n \varepsilon$, for $n \in \mathbb{N}$.

Definition 2.8. Denote by $\mathcal{F}$ the collection of all the functions in (a)–(d) above. The collection of all *$\mathcal{L}$-formulas* is the $\mathcal{F}$-closure of the atomic formulas. A formula is called a *sentence* if it has no free variables.

Example 2.9. Consider the language consisting of a single predicate symbol $<$, and let x and y be variables. Then $\forall x \exists y < xy$ is a formula (which we regard as saying *for all x there exists y such that $x < y$*.

2.3. **Satisfiability and models.** It does not make sense to ask whether a specific formula is true or not in a language. This will depend on the intended interpretation of a formula in a given structure.

Definition 2.10. An $\mathcal{L}$-structure is a function $\mathcal{U}$ defined on a subset of $\mathcal{L}$ as follows:

(1) $\mathcal{U}$ assigns to $\forall$ some nonempty set $|\mathcal{U}|$, called the *universe* of $\mathcal{U}$.
(2) $\mathcal{U}$ assigns to equality in $\mathcal{L}$ the equality relation on $|\mathcal{U}|$.
(3) $\mathcal{U}$ assigns to each n-ary predicate symbol p an n-ary relation $p^{\mathcal{U}}$ on $\mathcal{U}$.
(4) $\mathcal{U}$ assigns to each constant symbol c an element $c^{\mathcal{U}} \in |\mathcal{U}|$.
(5) $\mathcal{U}$ assigns to each n-ary function symbol f an n-ary function $f^{\mathcal{U}}\colon |\mathcal{U}|^n \to |\mathcal{U}|$.

A *variable assignment* (relative to $\mathcal{U}$) is a function $s\colon \{x_n \colon n \in \mathbb{N}\} \to |\mathcal{U}|$ If s is a variable assignment and $c \in |\mathcal{U}|$, we denote by $s(x|c)$ the variable assignment which agrees with s everywhere, except that it maps x to c.

We are now ready to define what it means for a formula to be *satisfied* in a certain structure and relative to a given interpretation of the variables.

Definition 2.11. If φ is a formula, we say that φ is true in $\mathcal{U}$ relative to the variable assignment s, denoted $\models_{\mathcal{U}} \varphi[s]$ and read $\mathcal{U}$ *satisfies φ with s*, as follows.

We first extend s to an assignment $\overline{s}$ on all terms by recursion: for a variable x, we set $\overline{s}(x) = s(x)$, while $\overline{s}(c) = c^{\mathcal{U}}$ for every constant c. Now, if $t_1, \ldots, t_n$ are terms, and f is an n-ary function symbol, we set $\overline{s}(ft_1 \cdots t_n) = f^{\mathcal{U}}(\overline{s}(t_1), \ldots, \overline{s}(t_1))$.

Now, $\models_{\mathcal{U}} \varphi[s]$ is also defined by recursion on the complexity of φ:

- $\models_{\mathcal{U}} pt_1 \cdots t_n[s]$ if and only if $(\overline{s}(t_1), \ldots, \overline{s}(t_n)) \in p^{\mathcal{U}}$ for an n-ary predicate p.

- $\models_{\mathcal{U}} (\alpha \wedge \beta)[s]$ if and only if $\models_{\mathcal{U}} \alpha[s]$ and $\models_{\mathcal{U}} \beta[s]$.
- $\models_{\mathcal{U}} (\alpha \vee \beta)[s]$ if and only if $\models_{\mathcal{U}} \alpha[s]$ or $\models_{\mathcal{U}} \beta[s]$.
- $\models_{\mathcal{U}} (\alpha \to \beta)[s]$ if and only if either $\not\models_{\mathcal{U}} \alpha[s]$ or $\models_{\mathcal{U}} \beta[s]$.
- $\models_{\mathcal{U}} (\alpha \leftrightarrow \beta)[s]$ if and only if $\models_{\mathcal{U}} \alpha[s]$ and $\models_{\mathcal{U}} \beta[s]$ are both simultaneously true or false.
- $\models \exists x\alpha[s]$ if and only if there is $c \in |\mathcal{U}|$ such that $\models_{\mathcal{U}} \alpha[s(x|c)]$.
- $\models \forall x\alpha[s]$ if and only if $\models_{\mathcal{U}} \alpha[s(x|c)]$ for all $c \in |\mathcal{U}|$.

Remark 2.12. Satisfiability of a sentence is independent of the variable assignment, since it has no free variables.

If φ is a sentence which is satisfied in $\mathcal{U}$ (with respect to some, and hence all, variable assignments), then we say that $\mathcal{U}$ is a *model* of φ, and write $\models_{\mathcal{U}} \varphi$. If Σ is a collection of sentences, we say that $\mathcal{U}$ is a model of Σ if it is a model of every sentence in Σ.

Example 2.13. Consider the language consisting of a constant symbol e, a binary symbol $\cdot$, and a unary function symbol ι. Then an $\mathcal{L}$-structure is a group if and only if the following sentences are satisfied in it:

$$\forall x \exists y((x \cdot y = e) \wedge (y \cdot x = e)) \quad \text{and} \quad \forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z)).$$

2.4. **Fundamental theorems in first-order logic.** .

Theorem 2.14 (Compactness theorem)**.** . Let Σ be a collection of sentences in a language $\mathcal{L}$. If every finite subset of Σ has a model, then Σ has a model.

As an application: if G is a graph with the property that every *finite* subgraph of G can be k-colored, then G can be k-colored.

Theorem 2.15 (Lowenheim-Skolem theorem)**.** . Let $\mathcal{L}$ be a language of cardinality κ, and let Σ be a collection of sentences. If Σ has an infinite model, then it has a model of every cardinality $\alpha \geq \kappa$.

As an application: for every infinite cardinal κ, there exists a field of cardinality κ. (This is not difficult directly: take $\mathbb{Q}[x_j \colon j \in \kappa]$.)

Definition 2.16. Let $\mathcal{U}$ and $\mathcal{V}$ be $\mathcal{L}$-structures. We say that $\mathcal{U}$ and $\mathcal{V}$ are *elementarily equivalent*, written $\mathcal{U} \equiv \mathcal{V}$, if they satisfy the same sentences.

Theorem 2.17. Let $\mathcal{U}$ be an $\mathcal{L}$-structure. If $A \subseteq |\mathcal{U}|$ is an infinite subset, then there exists a substructure $\mathcal{V}$ of $\mathcal{U}$ such that:

(1) $|\mathcal{V}|$ contains A;
(2) the cardinality of $|\mathcal{V}|$ is the same as the cardinality of A; and
(3) $\mathcal{V} \equiv \mathcal{U}$.

The principle above has also been repeatedly discovered in the context of C^*-algebras.

3. First order logic for metric structures

In first order logic, the focus shifts from a single structure to a class of them. The notion of language has the purpose of formalizing the assertion that a certain class of objects "are of the same kind".

Definition 3.1. A *language (for metric structures)* is a set $\mathcal{L}$ of symbols, which come in two kinds:

(1) *function symbols*;
(2) *relation/predicate symbols.*

Each function symbol f has an *ariety* $n_f \in \mathbb{N} \cup \{0\}$ attached. When $n_f = 0$, we regard f as a constant symbol. Similarly, a relation symbol R also has an ariety $n_R \in \mathbb{N}$. If ℓ is either a function or a relation symbol in $\mathcal{L}$, then ℓ has an associated uniform continuity modulus $\omega_\ell \colon [0,\infty)^{n_\ell} \to [0,\infty)$ which is continuous at 0 and satisfies $\omega_\ell(0) = 0$. For each relation symbol R, there is an associated *bound* J_R, which is a compact interval in $\mathbb{R}$.

Finally, there is a distinguished binary relation symbol d with $\omega_d(s,t) = s + t$ for all $s, t \in [0,\infty)$ and $J_d = [0,1]$.

Example 3.2. The smallest language is $\mathcal{L}_0 = \{d\}$, consisting only of the metric d. One can also consider $\mathcal{L}_1 = \{d, \cdot\}$, where $\cdot$ is a binary relation symbol with $\omega_\cdot(s,t) = s + t$ for all $s,t \in [0,\infty)$.

Definition 3.3. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-structure* is a complete metric space (M, d^M) endowed with interpretations of the symbols in $\mathcal{L}$:

(1) for a function symbol f, its interpretation $f^M : M^{n_f} \to M$ is a uniformly continuous function with modulus ω_f, that is, satisfying

$$d^M(f^M(\overline{a}), f^M(\overline{b})) \leq \omega_f(d^M(\overline{a}, \overline{b})$$

for all $\overline{a}, \overline{b} \in M^{n_f}$.

(2) for a relation symbol R, its interpretation $R^M : M^{n_R} \to [0,\infty)$ is a uniformly continuous function with modulus ω_R on M^{n_R}, and moreover $R^M(M^{n_R}) \subseteq J_M$.

(3) Additionally, the interpretation of d is assumed to be d^M.

Example 3.4. Adopting the notation from Example 3.2, an $\mathcal{L}_0$ structure is precisely a complete metric space with diameter at most 1, while an example of an $\mathcal{L}_1$ structure is a complete bi-invariant metric group with diameter at most 1.

It is somewhat inconvenient to regard C^*-algebras as structures in the sense of Definition 3.3. One option is to restrict one's attention to the unit ball of a C^*-algebra, which is however not invariant under sum or scalar multiplication. There are ways around this issue, and the most satisfactory one consists in introducing a more general framework where languages have *domains of quantification*: these are to be interpreted as closed subsets of the structure, and boundedness conditions are only required to hold relative to a given tuple of domains.

Definition 3.5. A *language with quantification domains* is a set $\mathcal{L}$ of symbols, which come in three kinds:

(1) *function symbols*;

(2) *relation/predicate symbols.*

(3) an upward directed set $\mathcal{D}$ of *quantification domains.*

Each function symbol f has an *ariety* $n_f \in \mathbb{N} \cup \{0\}$ attached. When $n_f = 0$, we regard f as a constant symbol. Similarly, a relation symbol R also has an ariety $n_R \in \mathbb{N}$.

Let ℓ be either a function or a relation symbol in $\mathcal{L}$, and let $D_1, \dots, D_{n_\ell} \in \mathcal{D}$ be any choice of input domains. Then there exist

(a.1) when $\ell = f$ is a function symbol, a (relative) output domain $D_f^{D_1,\dots,D_{n_f}}$;

(a.2) when $\ell = R$ is a relation symbol, a (relative) bound $J_R^{D_1,\dots,D_{n_R}}$, which is a compact interval in $\mathbb{R}$; and

(b) a (relative) continuity modulus $\omega_\ell^{D_1,\dots,D_{n_\ell}} : [0,\infty)^{n_\ell} \to [0,\infty)$.

Finally, there is a distinguished binary relation symbol d with $\omega_d(s,t) = s + t$ for all $s,t \in [0,\infty)$ and $J_d = [0,1]$.

Any language in the usual sense can be regarded as a language with domains of quantification by setting $\mathcal{D} = \{D\}$.

Definition 3.6. Let $\mathcal{L}$ be a language with domains of quantification. An *$\mathcal{L}$-structure* is a complete metric space (M, d^M) endowed with interpretations of the symbols in $\mathcal{L}$:

(1) for $D \in \mathcal{D}$, its interpretation D^M is a closed subset of M, with $D \mapsto D^M$ order-preserving, and $\bigcup_{D\in\mathcal{D}} D^M$ is dense in M;

(2) for a function symbol f, its interpretation $f^M : M^{n_f} \to M$ is a function, and for every $D_1, \dots, D_{n_f} \in \mathcal{D}$, the restriction of f^M to $D_1 \times \cdots \times D_{n_f}$ is uniformly continuous with modulus $\omega_f^{D_1,\dots,D_{n_f}}$, and $f^M(D_1 \times \cdots \times D_{n_f}) \subseteq D^f_{D_1,\dots,D_{n_f}}$;

(3) for a relation symbol R, its interpretation $R^M : M^{n_R} \to [0,\infty)$ is a function, and for every collection $D_1, \dots, D_{n_R} \in \mathcal{D}$, the restriction of R^M to $D_1 \times \cdots \times D_{n_R}$ is uniformly continuous with modulus $\omega_R^{D_1,\dots,D_{n_R}}$, and $R^M(D_1 \times \cdots \times D_{n_R}) \subseteq J^R_{D_1,\dots,D_{n_R}}$;

(4) Additionally, the interpretation of d is assumed to be d^M.

We are now ready to regard C^*-algebras as structure in a language which we proceed to define.

Definition 3.7. We will denote by $\mathcal{L}^{C^*}$ the *language of (unital) C^*-algebras*, defined as follows:

(1) functions symbols $+, \cdot, \lambda$ (for $\lambda \in \mathbb{C}$), $*, 0, 1$;
(2) the metric binary relation symbol d, and n^2-ary relation symbols $\|\cdot\|_{M_n}$, for $n \in \mathbb{N}$ (to be interpreted as matrix norms);
(3) quantification domains D_n, for $n \in \mathbb{N}$, satisfying $D_n < D_{n+1}$ for all $n \in \mathbb{N}$ (to be interpreted as the balls of radius n).

In order to regard a C^*-algebra as an $\mathcal{L}^{C^*}$-structure, one needs to specify continuity moduli, output domains, and bounds. This is mostly straightforward. For example, for the multiplication symbol $\cdot$ and input domains D_n and D_m, the output domain is D_{nm} and the continuity modulus is the function $\omega_{\cdot}^{D_n,D_m}(s,t) = ns + mt$ for all $s, t \in [0, \infty)$.

3.1. **Terms, formulas, and sentences.** Regarding a class of objects as structures in continuous logic allows one to speak about first-order properties, which are the properties that can be expressed through *formulas*. Intuitively, an $\mathcal{L}$-formula is an expression that described a property of an $\mathcal{L}$-structure, or of a tuple of elements in an $\mathcal{L}$-structure, by only referring to the given $\mathcal{L}$-structure, its elements, and its operations given by the interpretation of the function and relation symbols of $\mathcal{L}$.

We first consider the notion of terms. Assume that we have a collection of variables $x_1, x_2, \ldots$, and that each variable x has a uniquely attached domain of quantification D_x.

Definition 3.8. An *$\mathcal{L}$-term* is an expression that can be formed by starting from variables and constant symbols, and applying function symbols from $\mathcal{L}$. More precisely, one declares that:

- variables are $\mathcal{L}$-terms;
- constant symbols are $\mathcal{L}$-terms;
- if $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, and f is an n-ary function symbol in $\mathcal{L}$, then $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.

When the language $\mathcal{L}$ is clear from the context, we just speak about terms.

Given a term t, one can define what it means for a variable to appear in t. One then writes $t(x_1, \ldots, x_n)$ to denote the fact that the variables that appear in t are within $x_1, \ldots, x_n$.

Definition 3.9. An *atomic formula* is an expression φ of the form $R(t_1, \ldots, t_n)$ for some n-ary relation R and terms $t_1, \ldots, t_n$. If $t_1, \ldots, t_n$ have variables within $x_1, \ldots, x_m$, one says that φ has free variables within $x_1, \ldots, x_m$.

We define arbitrary formulas starting from atomic ones as follows:

- atomic formulas are formulas;
- if $\varphi_1, \ldots, \varphi_n$ are formulas and $q \colon \mathbb{R}^n \to \mathbb{R}$ is continuous, then $q(\varphi_1, \ldots, \varphi_n)$ is a formula;
- if φ is a formula and x is a variable with domain D, then $\inf_{x \in D} \varphi$ and $\sup_{x \in D} \varphi$ are formulas.

A variable appearing in a formula can be either bound or free, depending on whether it is in the scope of a quantifier or not. If φ has free variables within $x_1, \ldots, x_m$, one writes $\varphi(x_1, \ldots, x_m)$.

Definition 3.10. A *sentence* is a formula without free variables.

Sentences should be thought of as expressions describing how close a given structure is to satisfying a certain property.

Interpretation of terms and formulas is defined in the obvious way, by induction on their complexity. The interpretation of a sentence is a real number.

Example 3.11. In the language $\mathcal{L} = \{d, \cdot\}$, a term in the free variables $x_1, x_2, \ldots$ is simply a parenthesized word in those variables. For example, $(x_1 \cdot (x_2 \cdot x_3))$ or $((x_1 \cdot x_2) \cdot x_3)$. These two terms have the same interpretation in any metric group (or whenever the operation $\cdot$ is associative), although they are formally different terms.

Example 3.12. We specialize to the language $\mathcal{L}^{C^*}$ of C^*-algebras.

- A term is a complex $*$-polynomial on a finite number of free variables. (Formally this is a bit inaccurate, since the terms $((x_1+x_2)^*)$ and $(x_1^*+x_2^*)$ are formally different, although their interpretation is the same in every C^*-algebra. We will ignore this distinction.)
- An atomic formula is an expression of the form $\|p(x_1,\ldots,x_n)\|$ for some $*$-polynomial p.
- A generic formula is an expression of the form $\sup_{x_1\in D_1}\cdots\sup_{x_k\in D_n}\|p(x_1,\ldots,x_n)\|$ for some $*$-polynomial p and some $0\leq k\leq n$.
- A sentence is an expression of the form $\sup_{x_1\in D_1}\cdots\sup_{x_n\in D_n}\|p(x_1,\ldots,x_n)\|$ for some $*$-polynomial p.

We close this section with a discussion of *multi-sorted languages*, which will be needed later. In this setting, a language prescribed a collection $\mathcal{S}$ of *sorts*. Each sort S comes with a corresponding collection $\mathcal{D}_S$ of domains of quantification for S, and each n-ary function or relation symbol in $\mathcal{L}$ has a prescribed n-tuple of *input* sorts and, for function symbols, also an *output* sort. A structure M then consists of a family $(M^S)_{S\in\mathcal{S}}$ of complete metric spaces, subject to the usual requirements.

4. Axiomatizability and definability

Throughout, we fix a language $\mathcal{L}$. We will be primarily interested in $\mathcal{L}=\mathcal{L}^{C^*}$, but most of what we do here holds in general.

4.1. **Axiomatizable classes.** The notion of language allows one to define precisely what it means for a class of objects to be of *the same kind*. In this subsection, we will focus of making sense of what it means for a class of objects to satisfy *the same property* (axiomatizable property).

Recall that an $\mathcal{L}$-condition is an expression of the form $\varphi\leq r$, where φ is an $\mathcal{L}$-sentence and $r\in\mathbb{R}$. We say that an $\mathcal{L}$-structure M satisfies the condition $\varphi\leq r$ if $\varphi^M\leq r$.

Definition 4.1. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{C}$ is *axiomatizable* if there exists a family of $\mathcal{L}$-conditions $\{\varphi_j\leq r_j\colon j\in J\}$, such that an $\mathcal{L}$-structure M belongs to $\mathcal{C}$ if and only if $\varphi_j^M\leq r_j$ for all $j\in J$. Moreover, we say that a property P for $\mathcal{L}$-structures is *axiomatizable* if the class of structures that satisfy it is an axiomatizable class.

Remark 4.2. In the definition of axiomatizable class, and upon replacing a sentence φ with $\max\{\varphi-r,0\}$, one can assume that all sentences φ_j appearing in the definition attain only non-negative values, and that all the r_j are zero.

Proposition 4.3. The class $\mathcal{C}^*$ of unital C^*-algebras is axiomatizable in $\mathcal{L}^{C^*}$.

Proof. It is straightforward to write down sentences that describe that an $\mathcal{L}^{C^*}$-structure satisfies the axiom of a unital C^*-algebra. For example, the C^*-identity is captured by the conditions

$$\sup_{x\in D_n}\left|\|x^*x\|-\|x\|^2\right|\leq 0$$

for all $n\in\mathbb{N}$. The only subtle point is to insist that the domain D_n is interpreted as the ball of radius n centered at zero. This is enforced by the following conditions

$$\sup_{x\in D_m}\inf_{y\in D_n}\left(\|x-y\|-\max\{\|x\|-n,0\}\right)\leq 0,$$

for $n\leq m$. The verification that an $\mathcal{L}^{C^*}$-structure is a unital C^*-algebra if and only if it satisfies all these axioms is easy but tedious. $\square$

It is possible to show that a number of relevant classes of C^*-algebras are axiomatizable:

- Abelian C^*-algebras are captured by the formula $\sup_{x,y\in D_1}\|xy-yx\|\leq 0$.
- A C^*-algebra is not abelian if and only if it contains a nilpotent contraction. Hence, nonabelian C^*-algebras are captured by the formula $\inf_{x\in D_1}\|x^2\|-\|x\|+1\leq 0$.

In order to see that other classes of C^*-algebras are axiomatizable, we want to be able to use more general formulas, for example involving continuous functional calculus. We call these general formulas *definable predicates*.

The rigourous definition is as follows. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. Fix a tuple $\overline{x} = (x_1, \ldots, x_n)$ of variables with corresponding domains $\overline{D} = (D_1, \ldots, D_n)$, and let $\mathcal{F}(\overline{x})$ be the collection of $\mathcal{L}$-formulas with free variables from $\overline{x}$. Then $\mathcal{F}(\overline{x})$ admits a natural real Banach algebra structure (with pointwise operations), induced from the algebra structure on $\mathbb{R}$, and one can define a seminorm on $\mathcal{F}(\overline{x})$ by setting

$$\|\varphi\| = \sup_{M \in \mathcal{C}} \sup_{a \in D_1^M \times \cdots \times D_n^M} \left|\varphi^M(\overline{a})\right|.$$

The Hausdorff completion $\mathcal{M}(\overline{x})$ of $\mathcal{F}(\overline{x})$ with respect to this seminorm is thus a real Banach algebra, whose elements are called the *definable predicates* on the tuple $\overline{x}$.

Definable predicates are uniform limits of formulas, where the uniformity condition is over all elements in the class $\mathcal{C}$. Given a definable predicate φ and an $\mathcal{L}$-structure M, its interpretation $\varphi^M \colon D_1^M \times \cdots \times D_n^M \to \mathbb{R}$ can be defined as the limit of the interpretations of approximating formulas. In practice, it is easier to work with definable predicates. In fact, the definition of axiomatizable class does not change if one uses definable predicates instead of sentences.

4.2. **Definability.** In discrete first-order logic, a subset of a structure is called definable whenever it can be written as the set of elements that satisfy a certain formula. The direct analog of this notion in the metric setting turns out to be too generous, and the right generalization involves the notion of *stability* for formulas (or predicates) and relations.

Definition 4.4. Let $\varphi(x_1, \ldots, x_n)$ be a definable predicate, and let M be an $\mathcal{L}$-structure. Then the *zero* set of φ^M is

$$Z^M(\varphi) = \{\overline{a} = (a_1, \ldots, a_n) \in D_{x_1}^M \times \cdots \times D_{x_n}^M \colon \varphi(\overline{a}) = 0\}.$$

Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. We say that $\varphi(x_1, \ldots, x_n)$ is *stable* (with respect to $\mathcal{C}$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $M \in \mathcal{C}$ and $\overline{a} \in D_{x_1}^M \times \cdots \times D_{x_n}^M$ satisfies $|\varphi(\overline{a})| < \delta$, then there exists $\overline{b} \in\in D_{x_1}^M \times \cdots \times D_{x_n}^M$ with $\varphi(\overline{b}) = 0$ and $d(\overline{a}, \overline{b}) = 0$.

From now on, we fix a class $\mathcal{C}$ of $\mathcal{L}$-structures.

Definition 4.5. A *definable set (or property) S* if an assignment $M \mapsto S(M) \subseteq M^n$ from elements of $\mathcal{C}$ to closed subsets of M^n, such that there is a stable definable predicate whose M-zero set is precisely $S(M)$.

For example, the set of all projections in a C^*-algebras is a definable set; see the examples below.

The upshot of definable sets, is that one is allowed to quantify over them, even though they are not quantification domains:

Proposition 4.6. Let $\psi(\overline{x}, \overline{y})$ be a definable predicate and let S be a definable set. Then there exists a definable predicate $\varphi(\overline{x})$ such that

$$\varphi^M(\overline{a}) = \inf_{\overline{b} \in S(M)} \psi(\overline{a}, \overline{b})$$

for all $M \in \mathcal{C}$ and for all $\overline{a} \in M^n$.

In the context of the proposition above, we write $\varphi(\overline{x}) = \inf_{\overline{y} \in S} \psi(\overline{x}, \overline{y})$. The definable predicate φ is not unique, but its interpretation in elements of $\mathcal{C}$ does not depend on the choice.

Examples 4.7. Some examples of definable sets:

(1) The formula $\varphi(x) = \max\{\|x^*x - 1\|, \|xx^* - 1\|\}$ is stable, and its zero set is the definable set of unitaries.
(2) The formula $\varphi(x) = |x^*x - x\|$ is stable, and its zero set is the definable set of projections.
(3) The formula $\varphi(x) = \inf_{y \in D_1} \|x - y^*y\|$ is stable, and its zero set is the definable set of positive contractions.
(4) The formula $\varphi(x) = \max\{\|(x^*x)^2 - x^*x\|, \|(xx^*)^2 - (xx^*)\|\}$ is stable, and its zero set is the definable set of partial isometries.
(5) The definable predicate $\varphi(x, y) = \inf_{s \text{ partialisometry}} \max\{\|x - s^*s\|, \|y - ss^*\|\}$ is stable, and its zero set is the definable set of pairs Murray-von Neumann equivalent projections. (Observe that here we are quantifying over the definable set of partial isometries, using Proposition 4.6.)

(6) Let $\varphi(x, y)$ be the definable predicate from the item above, and consider the definable predicate

$$\psi(x) = \inf_{y \in D_1} \max\{\varphi(x, y), \|yx - y|, \|xy - x\|, |1 - \|x - y\||\}$$

is stable, and its zero set is the definable set of infinite projections.

Using definable sets, one can show that further classes of C^*-algebras are axiomatizable.

Example 4.8. A unital C^*-algebra has *real rank zero* if for every positive contractions $a, b \in A$ and for every $\varepsilon > 0$, there exists a projection $p \in A$ such that

$$\|pa\|^2 < \|ab\| + \varepsilon \quad \text{and} \quad \|(1-p)b\|^2 \leq \|ab\| + \varepsilon.$$

Since projections and positive contractions are definable sets, we can axiomatize the class of real rank zero C^*-algebras. More explicitly, a C^*-algebra has real rank zero if and only if it satisfies the condition

$$\sup_{x,y \in D_1 \text{ positive}} \inf_{z \in D_1 \text{ projection}} \max\{\|zx\|, \|(1-z)y\|\} - \|xy\|^{1/2} \leq 0.$$

Similarly, the class of purely infinite simple C^*-algebras is axiomatizable.

5. Ultraproducts and ultrapowers

Throughout this section, we fix a set I of indices. An *ultrafilter* over I is a nonempty collection $\mathcal{U}$ of subsets of I not containing the empty set, such that $A \cap B \in \mathcal{U}$ whenever $A, B \in \mathcal{U}$, and such that for every $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

From now on, we fix an ultrafilter $\mathcal{U}$ over I. We think of $\mathcal{U}$ as giving us a notion of "largeness": a subset of I belongs to $\mathcal{U}$ whenever it is large.

Proposition 5.1. If $(a_i)_{i \in I}$ is a bounded sequence in $\mathbb{R}$, then there exists a unique element $a \in \mathbb{R}$ such that for every $\varepsilon > 0$, the set $\{i \in I : |a - a_i| < \varepsilon\}$ belongs to $\mathcal{U}$.

Proof. Consider the collection $\mathcal{F}$ of nonempty compact subsets of $\mathbb{R}$ of the form $\overline{\{a_i : i \in A\}}$ for every $A \in \mathcal{U}$. Then $\mathcal{F}$ satisfies the finite intersection property, and by compactness there must be an element x in its intersection. Let U be an open neighborhood of x in $\mathbb{R}$. We claim that $\{i \in I : a_i \in U\}$ belongs to $\mathcal{U}$. If this is not the case, then the set $\{i \in I : a_i \notin U\}$ belongs to $\mathcal{U}$, so there exists $A \in \mathcal{U}$ such that $\overline{\{a_i : i \in A\}}$ is contained in $\mathbb{R} \setminus U$. Since $\overline{\{a_i : i \in A\}}$ belongs to $\mathcal{F}$, this is a contradiction.

Finally, since $\mathbb{R}$ is Hausdorff, it can be at most one element in this intersection, finishing the proof. ☐

In the context of the proposition above, we say that a is the *limit along* $\mathcal{U}$ of $(a_i)_{i \in I}$.

We now fix a language $\mathcal{L}$ and a family $(M_i)_{i \in I}$ of $\mathcal{L}$-structures.

Definition 5.2. The *ultraproduct* of M_i, for $i \in I$, is the $\mathcal{L}$-structure $M = \prod_{\mathcal{U}} M_i$ defined as follows.

(1) For every domain D in $\mathcal{L}$, we let D^M be the Hausdorff completion of the ultraproduct $D_0^M = \prod_{\mathcal{U}} D^{M_i}$ with respect to the pseudometric

$$d^M(a, b) = \lim_{i \to \mathcal{U}} d^{M_i}(a_i, b_i)$$

for all $a, b \in D_0^M$. It is easy to check that the family $\{D^M : D \in \mathcal{D}\}$ is directed. Its union is a metric space, and we let M be its completion $M = \overline{\bigcup_{D \in \mathcal{D}} D^M}$.

(2) If f is an n-ary symbol in $\mathcal{L}$ and $D_1, \dots, D_n \in \mathcal{D}$, let $D_f^{D_1, \dots, D_n}$ be the output domain, which we abbreviate to just D. We define $f^M : D_1^M \times \cdots \times D_n^M \to D^M$ by setting

$$f^M(a_1, \dots, a_n) = \left[(f^{M_i}(a_{1,i}, \dots, a_{n,i}))_{i \in I}\right]$$

for all $a_1, \dots, a_n \in M^n$. It is easy to check that f^M is uniformly continuous and that its modulus of continuity is precisely the prescribed modulus $\omega_f^{D_1, \dots, D_n}$. Finally, by letting the input domains vary and take their union and completion, we can extend f^M to a function $M^n \to M$ with the same modulus of continuity.

(3) If R is an n-ary symbol, we argue similarly as before, with the only difference that for $D_1, \dots, D_n \in \mathcal{D}$ and with $J = J_R^{D_1,\dots,D_n}$, the assignment $R^M\colon D_1^M \times \cdots \times D_n^M \to J^M$ is defined by

$$R^M(a_1, \dots, a_n) = \lim_{i \to \mathcal{U}} R^{M_i}(a_{1,i}, \dots, a_{n,i})$$

for all $a_1, \dots, a_n \in M^n$. The rest is analogous.

Applied to C^*-algebras, this notion recovers the usual (categorical) notion of ultraproduct of C^*-algebras. Explicitly, for C^*-algebras A_i, for $i \in I$, their ultraproduct can be identified as the quotient $\prod_{\mathcal{U}} A_i = \prod_{i \in I} A_i / \bigoplus_{i \in I} A_i$. One gets bounded sequences in the model-theoretic construction because one constructs the ultrapower $\prod_{\mathcal{U}} A_i$ essentially from its bounded subsets.

5.1. **Saturation.** Let $\mathcal{L}$ be a language and let A be any set. One can then consider the (extended) language $\mathcal{L}(A)$ to be the union of $\mathcal{L}$ with the set of constant symbols $\{c_a \colon a \in A\}$. If M is an $\mathcal{L}$-structure containing A, then we can canonically regard M as an $\mathcal{L}(A)$-structure in which the interpretation of c_a is a. We refer to the formulas in $\mathcal{L}(A)$ as $\mathcal{L}$-formulas *with parameters from* A.

Definition 5.3. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{L}$ is *separable for* $\mathcal{C}$ if all the Banach algebras of definable predicates (for any finite set of variables and for any family of input domains) is separable.

Let $\varphi(\overline{x})$ be a definable predicate and let $r \in \mathbb{R}$. If M is an $\mathcal{L}$-structure and $\overline{a} \in D_1^M \times \cdots \times D_n^M$ satisfies $\varphi^M(\overline{a}) \leq r$, we say that $\overline{a}$ *realizes* the condition $\varphi(\overline{x}) \leq r$. Recall that a type is a (possibly infinite) collection of conditions of the form $\varphi(\overline{x}) \leq r$. Given a type t, we denote by t^+ the type consisting of all conditions of the form $\varphi(\overline{x}) \leq r + \varepsilon$, for $\varepsilon > 0$, where $\varphi(\overline{x}) \leq r$ is a condition in t.

Definition 5.4. Let $t(\overline{x})$ be a type, and let M be an $\mathcal{L}$-structure.

- We say that $t(\overline{x})$ is *realized* in M if there exist $a_n \in D_n^M$, for $n \in \mathbb{N}$, such that $(a_n)_{n \in \mathbb{N}}$ realizes every condition in $t(\overline{x})$.
- We say that $t(\overline{x})$ is *approximately realized* in M if every finite set of conditions in $t^+(\overline{x})$ is realized in M.

Definition 5.5. Let M be an $\mathcal{L}$-structure. We say that M is *countable saturated* if for every separable subset $A \subseteq M$ and every $\mathcal{L}(A)$-type $t(\overline{x})$, if $t(\overline{x})$ is approximately realized in M, then it is realized in M.

A fundamental feature of ultraproducts over countably incomplete ultrafilters is their being countably saturated. The proof is a diagonalization argument, which we omit.

Theorem 5.6. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures for which $\mathcal{L}$ is separable. If $\mathcal{U}$ is a countably incomplete ultrafilter over a directed set I, and $(M_i)_{i \in I}$ is a family in $\mathcal{C}$, then the ultraproduct $\prod_{\mathcal{U}} M_i$ is countably saturated.

5.2. **Łos' theorem.** The following result, known as Łos' theorem, is a fundamental result in model theory that relates the construction of ultraproducts with the notion of formula.

Theorem 5.7. Let $\mathcal{L}$ be a language, let I be an index set, let $(M_i)_{i \in I}$ be a family of $\mathcal{L}$-structures, and let $\mathcal{U}$ be an ultrafilter over I. Abbreviate their ultraproduct as $M_{\mathcal{U}}$. Let $\varphi(x_1, \dots, x_n)$ be a definable predicate with free variables within domains $D_1, \dots, D_n$. Then

$$\varphi^{M_{\mathcal{U}}}(a_1, \dots, a_n) = \lim_{i \to \mathcal{U}} \varphi^{M_i}(a_{1,i}, \dots, a_{n,i})$$

for all $(a_1, \dots, a_n) \in M_{\mathcal{U}}^n$. In particular, if φ is an $\mathcal{L}$-sentence, then

$$\varphi^{M_{\mathcal{U}}} = \lim_{i \to \mathcal{U}} \varphi^{M_i}.$$

Proof. Let $t(x_1, \dots, x_n)$ be a term. One can easily see, by induction on the complexity of t, that one can define an output domain $D = D^t_{D_1,\dots,D_n}$ and a continuity modulus $\omega = \omega^t_{D_1,\dots,D_n}$ in terms of the output domains and continuity moduli of the function symbols in $\mathcal{L}$, such that for any $\mathcal{L}$-structure N, the interpretation t^N is a function $D_1^N \times \cdots \times D_n^N \to D^N$ with continuity modulus ω. In particular, this guarantees that the function $t^M\colon D_1^{M_{\mathcal{U}}} \times \cdots \times D_n^{M_{\mathcal{U}}} \to D^{M_{\mathcal{U}}}$ given by $t(a_1, \dots, a_n) = \big[(t^{M_i}(a_{1,i}, \dots, a_{n,i}))_{i \in I}\big]$, for all $a_1, \dots, a_n \in M_{\mathcal{U}}^n$, is well defined and has continuity modulus ω. Furthermore, it is also shown by induction on the complexity of

t, and using the definition of the interpretation of function symbols in $M_{\mathcal{U}}$, that this function coincides with the interpretation of t in $M_{\mathcal{U}}$ (thus justifying the notation t^M).

Let now φ be the formula in the statement. Again, using induction on its complexity one can define a bound $J = J_\varphi^{D_1,\dots,D_n}$ and a continuity modulus $\omega = \omega_\varphi^{D_1,\dots,D_n}$ such that for any $\mathcal{L}$-structure N, the interpretation φ^N is a function $D_1^N \times \cdots \times D_n^N \to J^N$ with continuity modulus ω. In particular, this guarantees that the function $\varphi^M : D_1^{M_{\mathcal{U}}} \times \cdots \times D_n^{M_{\mathcal{U}}} \to J^{M_{\mathcal{U}}}$ given by $\varphi(a_1,\dots,a_n) = \lim_{i\to\mathcal{U}} \varphi^{M_i}(a_{1,i},\dots,a_{n,i})$, for all $a_1,\dots,a_n \in M_{\mathcal{U}}^n$, is well defined and has continuity modulus ω. Furthermore, it is also shown by induction on the complexity of φ, and using the definition of the interpretation of function symbols in $M_{\mathcal{U}}$, that this function coincides with the interpretation of φ in $M_{\mathcal{U}}$, as desired.

The proof for definable predicates is analogous, and we omit it. $\square$

Corollary 5.8. The axiomatizable class of C^*-algebras is closed under (model-theoretic) ultraproducts.

We can now give a semantic characterization of stable predicates. Recall that an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ is called *nonprincipal* if all of its elements are infinite subsets of $\mathbb{N}$.

Proposition 5.9. Let $\mathcal{C}$ be an elementary class of $\mathcal{L}$-structures, and let $P(\overline{x})$ be definable predicate. Then the following are equivalent:

(1) $P(\overline{x})$ is stable.
(2) For any sequence $(M_n)_{n\in\mathbb{N}}$ of structures in $\mathcal{C}$, for any nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, and for any tuple $\overline{a}$ in $\prod_{\mathcal{U}} M_n$ satisfying $P(\overline{a}) = 0$, there exists a representative sequence $(\overline{a}_n)_{n\in\mathbb{N}}$ of tuples in M_n such that
$$\{n \in \mathbb{N} \colon P^{M_n}(\overline{a}_n) = 0\} \in \mathcal{U}.$$

Łos' theorem naturally leads to the notion of elementary inclusion. A morphism $M \to N$ of $\mathcal{L}$-structures is a function $\Phi\colon M \to N$ satisfying $\Phi(D^M) \subseteq D^N$ for every $D \in \mathcal{D}$, and $\varphi(\Phi(\overline{a})) \leq \varphi(\overline{a})$ for every *atomic* formula $\varphi(\overline{x})$. We moreover say that Φ is an embedding if it is injective as a function $M \to N$ and moreover $\varphi(\Phi(\overline{a})) = \varphi(\overline{a})$ for every atomic formula $\varphi(\overline{x})$. In this case, we also say that M is a substructure of N (since we can always assume that an embedding is simply an inclusion).

Definition 5.10. Let $M \subseteq N$ be a substructure. We say that M is an *elementary substructure* of N if for every formula $\varphi(\overline{x})$ with tuple domain $\overline{D}$, we have $\varphi^M(\overline{a}) = \varphi^N(\overline{a})$ for every $\overline{a} \in \overline{D}^M$.

Example 5.11. By Łos' theorem, the diagonal embedding $M \to M_{\mathcal{U}}$ is an elementary embedding.

A useful criterion to verify that an inclusion $M \subseteq N$ is elementary is the following:

Theorem 5.12 (Tarski-Vaught test)**.** . Let $M \subseteq N$ be a substructure of an $\mathcal{L}$-structure N. Then $M \subseteq N$ is elementary if and only if for every $\mathcal{L}$-formula $\varphi(\overline{x}, y)$, where $\overline{x}$ have domains $\overline{D}$ and y has domain D, one has
$$\inf\{\varphi(\overline{a}, b) \colon b \in D^N\} = \inf\{\varphi(\overline{a}, b) \colon b \in D^M\}$$
for all $\overline{a} \in \overline{D}^M$.

5.3. **Theory and elementary/existential equivalence.** The notion of elementary embedding is closely related to that of elementary equivalence.

Definition 5.13. Let M be an $\mathcal{L}$-structure. We define its *theory* $\mathrm{Th}(M)$ to be the multiplicative functional $\mathrm{Th}(M)\colon \mathcal{S} \to \mathbb{R}$ given by $\mathrm{Th}(M)(\varphi) = \varphi^M$ for every $\mathcal{L}$-sentence $\varphi \in \mathcal{S}$.

Two $\mathcal{L}$-structures M and N are said to be *elementary equivalent*, written $M \equiv N$, if $\mathrm{Th}(M) =\mathrm{Th}(N)$.

It follows from Łos' theorem that $\mathrm{Th}(M) =\mathrm{Th}(M_{\mathcal{U}})$ for any countably incomplete ultrafilter $\mathcal{U}$.

Proposition 5.14. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. Then $\mathcal{C}$ is axiomatizable if and only if for every uncountably incomplete ultrafilter $\mathcal{U}$ over an index set I, and for every is a family $(M_i)_{i\in I}$ in $\mathcal{C}$, the ultraproduct $\prod_{\mathcal{U}} M_i$ belongs to $\mathcal{C}$.

We now define several classes of formulas.

Definition 5.15. Let $\mathcal{L}$ be a language and let φ be an $\mathcal{L}$-formula.

- φ is said to be *quantifier-free* if no quantifiers appear in it. Equivalently, it has the form $q(\varphi_1, \ldots, \varphi_n)$, where $\varphi_1, \ldots, \varphi_n$ are atomic formulas and $q\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function.
- φ is said to be *positive quantifier-free* if it is quantifier free and the function q as before satisfies $q(r_1, \ldots, r_n) \leq q(s_1, \ldots, s_n)$ whenever $r_j \leq s_j$ for all $j = 1, \ldots, n$.
- φ is said to be *existential* if it has the form $\inf_{x \in D} \psi$, for a quantifier-free formula ψ. (Infima are to be regarded as the continuous analog of the discrete quantifier $\exists$.)
- φ is said to be *positive existential* if it has the form $\inf_{x \in D} \psi$, for a positive quantifier-free formula ψ.

We define the *(positive) existential theory* $\mathrm{Th}_{\mathrm{e}}(M)$ (respectively $\mathrm{Th}_{\mathrm{pe}}(M)$) of an $\mathcal{L}$-structure M as the restriction of the theory of M to the (positive) existential formulas.

Proposition 5.16. Let M and N be separable $\mathcal{L}$-structures, and let $\mathcal{U}$ be a countably incomplete ultrafilter.

(1) $\mathrm{Th}(M) = \mathrm{Th}(N)$ if and only if $M_{\mathcal{U}}$ and $N_{\mathcal{U}}$ are isomorphic.
(2) $\mathrm{Th}_{\mathrm{e}}(M) = \mathrm{Th}_{\mathrm{e}}(N)$ if and only if there are embeddings $N \hookrightarrow M_{\mathcal{U}}$ and $M \hookrightarrow N_{\mathcal{U}}$.
(3) $\mathrm{Th}_{\mathrm{pe}}(M) = \mathrm{Th}_{\mathrm{pe}}(N)$ if and only if there are morphisms $N \to M_{\mathcal{U}}$ and $M \to N_{\mathcal{U}}$.

Proof. Parts (2) and (3) follow from Łos' theorem. The first part is the continuous analog of a result of Keisler and Shelah. □

Similarly, we have the following:

Proposition 5.17. Let $\mathcal{U}$ be a countable incomplete ultrafilter over an index set I, and let $\Phi\colon M \hookrightarrow N$ be an embedding of $\mathcal{L}$-structures. Then the following are equivalent:

(1) Φ is existential (positive existential);
(2) there exists an embedding (morphism) $\Psi\colon N \to M_{\mathcal{U}}$ such that $\Psi \circ \Phi$ is equal to the diagonal embedding $\Delta_M\colon M \to M_{\mathcal{U}}$.

For C^*-algebras, morphisms and embeddings have the usual meanings. In this context, an inclusion $A \subseteq B$ is positive existential if and only if for every tuple $\overline{a} \in A^n$, for all polynomials $p_1(\overline{x}, \overline{y}), \ldots, p_k(\overline{x}, \overline{y})$, and for all reals $r_1, \ldots, r_k \in \mathbb{R}$, whenever there exists a tuple $\overline{b} \in B^n$ such that $\|p_j(\overline{a}, \overline{b})\| \leq r_j$ for all $j = 1, \ldots, k$, then for every $\varepsilon > 0$ there exists a tuple $\overline{c}_\varepsilon \in A^n$ such that $\|p_j(\overline{a}, \overline{c}_\varepsilon)\| \leq r_j + \varepsilon$ for all $j = 1, \ldots, k$.

In this context, we can give the following model-theoretic description of relative commutants. Recall that an ultrafilter $\mathcal{U}$ is countably incomplete if there exists a sequence $(I_n)_{n \in \mathbb{N}}$ in $\mathcal{U}$ such that $\bigcap\limits_{n \in \mathbb{N}} I_n = \emptyset$. (For ultrafilters over $\mathbb{N}$, this is equivalent to being nonprincipal, which means that all of its elements are infinite sets.)

Proposition 5.18. Let A and C be separable unital C^*-algebras and let $\mathcal{U}$ be a countably incomplete ultrafilter over $\mathbb{N}$. Then the following are equivalent:

(1) the factor embedding $A \to C \otimes_{\max} A$ is existential (respectively, positive existential);
(2) there is an embedding (respectively, morphism) $C \to A_{\mathcal{U}} \cap A'$.

5.4. **A model-theoretic approach to the Rokhlin property.** In this subsection, we focus on compact (quantum) group actions on C^*-algebras. We fix a compact (quantum) group G. We want to define a language $\mathcal{L}_G^{C^*}$ in which G-actions on a unital C^*-algebra can be regarded as structures. When G is finite, we may just add to $\mathcal{L}^{C^*}$ one symbol for every group element. In general, we need to take the topology of G into account.

Given an irreducible representation $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$, we fix an orthonormal basis $\{\xi_k\colon k = 1, \ldots, d_\pi\}$ of $\mathcal{H}_\pi$, and define

$$C(G)_\pi = \mathrm{span}\{\pi_{i,j} \in C(G)\colon \pi_{i,j}(g) = \langle \pi(g)\xi_i, \xi_j \rangle \text{ for all } g \in G, 1 \leq i, j \leq d_\pi\},$$

which is a finite dimensional subspace of $C(G)$.

We regard an action $\alpha\colon G \to \mathrm{Aut}(A)$ as a unital homomorphism $A \to C(G) \otimes A$ satisfying certain conditions. We set

$$A_\pi = \{a \in A\colon \alpha(a) \in C(G)_\pi \otimes A\}.$$

One can check that $\alpha(A_\pi) \subseteq C(G)_\pi \otimes A_\pi$. (In fact, for $a \in A_\pi$, there is a relatively easy formula for $\alpha(a)$ in terms of the matrix coefficients of π.) We call A_π the *spectral subspace* of α associated to π.

One can check that $A_\pi \cap A_\sigma = \{0\}$ whenever $\pi \neq \sigma$, and that $A = \overline{\oplus_{\pi \in \widehat{G}} A_\pi}$. Finally, α is completely determined by this decomposition of A. This motivates the following definition of the multi-sorted language $\mathcal{L}_G^{C^*}$:

Definition 5.19. Let G be a compact (quantum) group. We define the multi-sorted language $\mathcal{L}_G^{C^*}$???

Similarly to Proposition 4.3, we have the following:

Proposition 5.20. Let G be a compact (quantum) group. Then G-C^*-algebras form an axiomatizable class in $\mathcal{L}_G^{C^*}$.

Proof. One has to write down formulas that guarantee that the quantification domains D_π, for $\pi \in \widehat{G}$, are interpreted as A_π and $C(G)_\pi \otimes A_\pi$. We omit the details. $\square$

Recall that $\alpha\colon G \to \mathrm{Aut}(A)$ has the *Rokhlin property* if there exists a unital equivariant homomorphism $C(G) \to A_{\mathcal{U}} \cap A'$ for some (any) ultrafilter $\mathcal{U}$. We can give a model-theoretic reformulation as follows:

Theorem 5.21. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then α has the Rokhlin property if and only if the G-equivariant homomorphism $\alpha\colon (A, \alpha) \to (C(G) \otimes A, \mathtt{Lt} \otimes \mathrm{id}_A)$ is positive existential.

The following summarizes one of the most important features of the Rokhlin property.

Corollary 5.22. If $\alpha\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property, then the natural embeddings $A^\alpha \hookrightarrow A$ and $A \rtimes_\alpha G \hookrightarrow A \otimes \mathcal{K}(L^2(G))$ are positive existential. In particular, if A and $A \otimes \mathcal{K}(L^2(G))$ belong to a positive existential axiomatizable class, then so do A^α and $A \rtimes_\alpha G$.

6. The effect of the Continuum Hypothesis

The continuum is, by definition, the cardinality of the reals. The Continuum Hypothesis (CH) asserts that the continuum equals the least uncountable cardinal $\aleph_1$. It is known that CH is independent from the usual axioms in set theory ZFC, in the sense that it cannot be proved or disproved using ZFC.

The value of the continuum, or more generally additional set-theoretic axioms, can have a deep influence on the structure and properties of "massive C^*-algebras". One paradigmatic instance of this phenomenon is the question of whether all automorphisms of the Calkin algebra $\mathcal{Q}$ are inner. Originally raised by Brown-Douglas-Fillmore, it was shown by Phillips-Weaver that CH implies the existence of outer automorphisms of $\mathcal{Q}$, while Farah showed that the negation of CH (he actually used a stronger axiom) implies that all automorphisms of $\mathcal{Q}$ are inner.

In this section, we focus on a different problem which is also sensitive to the value of the continuum, namely the number of ultrapowers and relative commutants of a given infinite dimensional, separable C^*-algebra with respect to nonprincipal ultrafilters over $\mathbb{N}$.

Fix an infinite dimensional, separable C^*-algebra A. If $\mathcal{U}$ and $\mathcal{V}$ are two nonprincipal ultrafilters over $\mathbb{N}$, then it follows from Łos' theorem that $A_{\mathcal{U}}$ and $A_{\mathcal{V}}$ are elementary equivalent. It was open for some time whether these ultrapowers are necessarily isomorphic. It turns out that the answer to this question depends (and is equivalent to) CH:

Theorem 6.1. [Farah-Hart-Sherman, Farah-Shelah]. Let A be an infinite dimensional, separable C^*-algebra.

(1) If CH holds, then $A_{\mathcal{U}} \cong A_{\mathcal{V}}$ and $A_{\mathcal{U}} \cap A' \cong A_{\mathcal{V}} \cap A'$ for any nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over $\mathbb{N}$.
(2) If CH fails, then there exist two nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over $\mathbb{N}$ (in fact, $2^{|\mathbb{R}|}$ many) such that $A_{\mathcal{U}} \ncong A_{\mathcal{V}}$ and $A_{\mathcal{U}} \cap A' \ncong A_{\mathcal{V}} \cap A'$.

Part (1) of the result above follows from the following general result, which does not need to assume CH.

Theorem 6.2. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structure for which $\mathcal{L}$ is separable. Let $M, N \in \mathcal{C}$ be elementary equivalent, countably saturated structures, both with density cardinality $\aleph_1$. Then M and N are isomorphic. Moreover, if $M_0 \subseteq M$ is a separable elementary substructure and $\Phi_0\colon M_0 \to N$ is an elementary embedding, then Φ_0 extends to an isomorphism $\Phi\colon M \to N$.

The result above is used to prove Part (1) of Theorem 6.1 in the following way. It can be shown that if $\mathcal{U}$ is a nonprincipal ultrafilter, and A is separable and infinite dimensional, then the density characters of $A_{\mathcal{U}}$ and $A_{\mathcal{U}} \cap A'$ are precisely the continuum. (These density characters are a priori *at most* the continuum, but one can prove, with some work, that this upper bound is attained. One uses that the continuum is also the cardinality of the collection of infinite subsets of $\mathbb{N}$.) Now, under CH, the density characters of these objects is then $\aleph_1$, so Theorem 6.1 applies and gives the result.

7. Strongly self-absorbing C*-algebras

The class of strongly-self absorbing C^*-algebras, defined by Toms and Winter, has played in recent years a pivotal role in the study of the structure and classification of simple nuclear C^*-algebras. In this section, we present some model-theoretic results related to these algebras, their ultrapowers, and relative commutants.

Definition 7.1. Let $\mathbb{D}$ be a separable, unital C^*-algebra. We say that $\mathbb{D}$ is *strongly self-absorbing* if $\mathbb{D} \not\cong \mathbb{C}$ and there is an isomorphism $D \to D \otimes_{\min} D$ which is approximately unitarily equivalent to the factor embedding $D \to D \otimes_{\min} D$.

The choice of the tensor product is irrelevant, since strongly self-absorbing C^*-algebras are automatically nuclear (in addition to simple and at most monotracial).

Theorem 7.2. Let $\mathbb{D}$ be a (separable) strongly self-absorbing C^*-algebra, let A be a separable C^*-algebra, and let $\mathcal{U}$ be a countably incomplete ultrafilter. Then the following are equivalent:

(1) A is $\mathbb{D}$-absorbing;
(2) the factor embedding $A \to A \otimes \mathbb{D}$ is positive existential;
(3) the factor embedding $A \to A \otimes \mathbb{D}$ is approximately unitarily equivalent to an isomorphism;
(4) there is a unital embedding $\mathbb{D} \to A_{\mathcal{U}} \cap A'$;
(5) If $t(\overline{x})$ is a positive quantifier-free type which is approximately realized in $\mathbb{D}$, theb the type $t(\overline{x}) \cup \{\|x_j a - a x_j\| \leq 0 \colon a \in A\}$ is approximately realized in A.

Proof. To show that (1) $\Rightarrow$ (2), one shows, using the definition of strongly self-absorbing, that the factor embedding $\mathbb{D} \otimes A \to \mathbb{D} \otimes \mathbb{D} \otimes A$ given by $d \otimes a \mapsto 1_D \otimes d \otimes a$, is positive existential. That (2) $\Rightarrow$ (3) follows by using an intertwining argument, while (3) $\Rightarrow$ (1) is immediate. The equivalence (2) $\Leftrightarrow$ (4) is a consequence of Proposition 5.18 (since $\mathbb{D}$ is simple), and finally the equivalence (4) $\Leftrightarrow$ (5) is a consequence of Łos' theorem and countable saturation of ultrapowers. $\square$

Corollary 7.3. Being $\mathbb{D}$-absorbing is axiomatizable, as witnessed by the conditions

$$\sup_{x_1,\ldots,x_n} \inf_{y_1,\ldots,y_n} \max\{\varphi(x_1,\ldots,x_n), \|x_j y_k - y_k x_j\| \colon j,k = 1,\ldots,n\} \leq 0,$$

where φ varies among all the positive quantifier-free formulas for which the condition $\varphi(\overline{x}) \leq 0$ is realized in $\mathbb{D}$.

In fact, the argument above shows that being $\mathbb{D}$-absorbing is sup-inf axiomatizable.

We close this section with a result on relative commutants of strongly self-absorbing C^*-algebras.

Theorem 7.4. Let C be a $\mathbb{D}$-absorbing unital C^*-algebra which is quantifier-free countably saturated, and let $\theta \colon \mathbb{D} \to C$ be any embedding.

(1) Any other embedding $\mathbb{D} \to C$ is unitarily equivalent to θ.
(2) For every separable subalgebra $A \subseteq C \cap \theta(\mathbb{D})'$ and every separable subalgebra $B \subseteq C$, there exists a unitary $u \in C \cap A'$ such that $uBu^* \subseteq C \cap \theta(\mathbb{D})'$.
(3) $C \cap \theta(\mathbb{D})$ is an elementary substructure of C.
(4) If C has density character $\aleph_1$, then the inclusion $C \cap \theta(\mathbb{D})' \hookrightarrow C$ is approximately unitarily equivalent to an isomorphism.
(5) If C is countably saturated, then $C \cap \theta(\mathbb{D})'$ is countably saturated.

Corollary 7.5. Let A be a separable, unital C^*-algebra, and let $\mathcal{U}$ be any countably incomplete ultrafilter. Then the conclusions of the theorem above hold for $C = A_{\mathcal{U}}$ and for $C = A_{\mathcal{U}} \cap A'$. In particular, if CH holds, then $A_{\mathcal{U}} \cong A_{\mathcal{U}} \cap \mathbb{D}'$. For example, $\mathbb{D}_{\mathcal{U}} \cong \mathbb{D}_{\mathcal{U}} \cap \mathbb{D}'$.

8. Model theory and the classification program

Elliott's conjecture, posed in the early 90's, predicted that all simple, separable, nuclear C^*-algebras would be classified by K-theory and traces (known as the Elliott invariant). Despite the great success seen in the first decade, Rørdam and Toms constructed nonisomorphic simple, separable, nuclear C^*-algebras with the same Elliott invariant. Their examples were distinguished using alternative invariants (real rank, radius of comparison), which are interestingly captured by the first-order theory of a C^*-algebra. This motivated the following question:

Question 8.1. Is the Elliott invariant together with the first-order theory a complete invariant for simple, separable, nuclear C^*-algebras?

Unfortunately, being simple, or separable, or nuclear, or any combination of these, is not axiomatizable. Even worse:

Theorem 8.2. If $\mathcal{C}$ is a class of exact C^*-algebras which contains a non subhomogeneous C^*-algebra, then $\mathcal{C}$ is not axiomatizable.

Proof. We claim that if a C^*-algebra A is not subhomogeneous and $\mathcal{U}$ is any nonprincipal ultrafilter over $\mathbb{N}$, then $A_{\mathcal{U}}$ is not exact. Once we prove the claim, the result will follow.

For the sake of this argument, suppose that there exists a sequence $\pi_n\colon A \to M_{k_n}$ of irreducible representations with $\lim_{n\to\infty} k_n = \infty$. (The other possibility is that A contains an infinite dimensional irreducible representation.) Set $J_n = \ker(\pi_n)$, and $J = \prod_{\mathcal{U}} J_n$, which is an ideal in $A_{\mathcal{U}}$. Find a subalgebra $B_n \subseteq A$ such that $B_n/J_n \cong M_{k_n}$. Then $\prod_{\mathcal{U}} B_n/J \cong \prod_{\mathcal{U}} M_{k_n}$. Now, if $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\mathcal{B}(\mathcal{H})$ admits a complete order embedding into $\prod_{\mathcal{U}} M_{k_n}$. Since $\mathcal{B}(\mathcal{H})$ is not exact, and exactness passes to completely ordered embedded structures, it follows that $\prod_{\mathcal{U}} M_{k_n}$ is not exact either. Since exactness passes to subquotients, we conclude that $A_{\mathcal{U}}$ is not exact. $\square$

For the sake of comparison, recall that the class of subhomogeneous C^*-algebras is axiomatizable.

Corollary 8.3. The class of exact (or nuclear) C^*-algebras is not elementarily axiomatizable. The same conclusion applies to many other interesting classes of C^*-algebras, such as UHF-algebras, AF-algebras, Kirchberg algebras, etc.

Similarly, the class of *simple* C^*-algebras is not elementary. Indeed, and even though M_n is simple, the ultraproduct $\prod_{\mathcal{U}} M_n$ over any nonprincipal ultrafilter $\mathcal{U}$ is not simple, since the trace-kernel is a nontrivial ideal.

In order to capture other interesting properties such as nuclearity and simplicity, we are led to consider the more generous notion of an *infinitary formula*. In an infinitary form, one is allowed to take countably infinite conjunctions or disjunctions, which are expressions of the form $\sup_{n\in\mathbb{N}} \varphi_n$ or $\inf_{n\in\mathbb{N}} \varphi_n$, for a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of formulas with certain restrictions. In the usual notion of formulas, one is only allowed to take sup/inf over variables. In fact, we will only consider of a special kind, which we call $\sup\vee\inf$-formulas.

Recall that if $\varphi(\overline{x})$ is a formula, then its interpretation in every structure is uniformly continuous, with continuity modulus ω^φ independent of the structure and which can be computed in terms of the uniform continuity moduli of the function and relation symbols in the language and their bounds.

Definition 8.4. An *infinitary* $\sup\vee\inf$*-formula* is an expression $\varphi(\overline{x})$ of the form

$$\varphi(\overline{x}) = \sup_{\overline{y}\in\overline{D}} \inf_{n\in\mathbb{N}} \psi_n(\overline{x},\overline{y}),$$

where $\overline{y}$ is a tuple of variables with corresponding domain $\overline{D}$, and $(\psi_n(\overline{x},\overline{y}))_{n\in\mathbb{N}}$ is a sequence of existential formulas such that the function $\omega^\varphi(\overline{r},\overline{s}) = \sup_{n\in\mathbb{N}} \min\{\omega^{\psi_n}(\overline{r},\overline{s}),1\}$ satisfies $\omega^\varphi(\overline{r},\overline{s}) \to 0$ as $\overline{r}\to 0$ and $\overline{s}\to 0$.

Interpretations and continuity moduli of infinitary formulas are defined as usual.

Remark 8.5. It is important to note that the analog of Łos' theorem for infinitary $\sup\vee\inf$-formulas does not hold.

If $\mathcal{C}$ is a class of $\mathcal{L}$-structures, we say that $\mathcal{C}$ is *infinitary* $\sup\vee\inf$-*axiomatizable* if there exists a countable collection of conditions of the form $\varphi \leq r$, where φ is a infinitary $\sup\vee\inf$-formula and $r \in \mathbb{R}$, such that an $\mathcal{L}$-structure belongs to $\mathcal{C}$ if and only if it satisfies all such conditions.

Proposition 8.6. The class of UHF-algebras admits an infinitary $\sup\vee\inf$-axiomatization.

Proof. Here we use the equivalent formulation of being UHF as those algebras which are locally matricial. Hence, a C^*-algebra A is UHF if and only if for every $\ell \in \mathbb{N}$, for every $a_1, \ldots, a_\ell$ positive contractions in A, and for every $\varepsilon > 0$, there exist $d \in \mathbb{N}$ and matrix units $e_{i,j}^{(d)}$ for M_d, and scalars $\lambda_{i,j}^{(d)}$ of modulus one such that

$$\left\| a_k - \sum_{i,j=1}^{d} \lambda_{i,j}^{(d)} e_{i,j}^{(d)} \right\|$$

for all $k = 1, \ldots, \ell$. In particular, A is UHF if and only if it satisfies the following condition:

$$\sup_{x_1,\ldots,x_n} \inf_{d\in\mathbb{N}} \inf_{e_{i,j}^{(d)} \mathrm{matrixunits}} \inf_{\lambda_{i,j}^{(d)}\in S^1} \max_{k=1,\ldots,\ell} \left\| x_k - \sum_{i,j=1}^{d} \lambda_{i,j}^{(d)} e_{i,j}^{(d)} \right\| .$$

□

Proposition 8.7. The class of nuclear C^*-algebras admits an infinitary $\sup\vee\inf$-axiomatization.

Proposition 8.8. The class of simple C^*-algebras admits an infinitary $\sup\vee\inf$-axiomatization.

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