UNIVERSAL COEFFICIENT THEOREMS IN TRIANGULATED CATEGORIES

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Warning: little proofreading has been done.

Contents

1. Preliminaries: KK-theory

Let B be a C^*-algebra.

Proposition 1.1. Isomorphism classes of Hilbert B-modules $\mathcal{E}$ for which $\mathrm{id}_{\mathcal{E}}$ is B-compact are in bijection with the Murray-von Neumann classes of projections in $M_\infty(B)$.

Proof. Given a projection $p \in M_n(B)$, set $\mathcal{E} = pB^n$. Then $\mathrm{id}_{\mathcal{E}}$ can be written as the composition

$$\mathcal{E} \hookrightarrow B^n \to B^n \to \mathcal{E}$$

and hence has finite B-rank.

Conversely, let $\mathcal{E}$ be a Hilbert B-module such that $\mathrm{id}_{\mathcal{E}}$ is B-compact. Choose $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ in $\mathcal{E}$ such that

$$\left\| \mathrm{id}_{\mathcal{E}} - \sum_{j=1}^{n} \theta_{\xi_j,\eta_j} \right\| < 1.$$

It follows that $\sum_{j=1}^{n} \theta_{\xi_j,\eta_j}$ is invertible in $\mathcal{B}(\mathcal{E})$. Consider the finite B-rank maps

$$\eta\colon \mathcal{E} \to B^n, \zeta \mapsto (\langle \eta_j, \zeta\rangle)_{j=1}^n \quad \text{and} \quad \xi\colon B^n \to \mathcal{E}, (b_j)_{j=1}^n \mapsto \sum_{j=1}^{n} \xi_j b_j.$$

Then $\xi \circ \eta = \sum \theta_{\xi_j,\eta_j}$ is invertible and $\eta \circ \xi\colon B^n \to B^n$ corresponds to a matrix $e \in M_n(B)$. Using polar decomposition for $\xi \circ \eta$, one gets a projection in $M_n(B)$. □

It follows that the Grothendieck group of isomorphism classes of Hilbert B-modules $\mathcal{E}$ for which $\mathrm{id}_{\mathcal{E}}$ is B-compact is $K_0(B)$ when B is unital. Notice that the Grothendieck group of isomorphism classes of *all* Hilbert B-modules is trivial, since one always has $0 \oplus \mathcal{H}_B \cong \mathcal{E} \oplus \mathcal{H}_B$, where $\mathcal{H}_B = \ell^2 \otimes B$ is the universal separable Hilbert B-module.

Definition 1.2. Let

$$0 \longrightarrow I \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

be a unital ring extension. Then the *relative K-group*, written $K_0(A, B)$, is the Grothendieck group of the monoid of triple (P_0, P_1, φ), where P_0 and P_1 are finitely generated projective A-modules, and $\varphi\colon f_*(P_0) \to f_*(P_1)$ is a morphism.

We have the following natural result.

Theorem 1.3. (Excision) Let $0 \to I \to A \to B \to 0$ be a unital ring extension. Then $K_0(A, B) \cong K_0(I)$.

In particular,

$$K_0(B) \cong K_0\left(M(B \otimes \mathcal{K}), M(B \otimes \mathcal{K})/(B \otimes \mathcal{K})\right).$$

The group $K_0\left(M(B \otimes \mathcal{K}), M(B \otimes \mathcal{K})/(B \otimes \mathcal{K})\right)$ is the Grothendieck group of triples $(\mathcal{E}_0, \mathcal{E}_1, \varphi)$, where $\mathcal{E}_0$ and $\mathcal{E}_1$ are countably generated Hilbert B-modules, and $\varphi\colon \mathcal{E}_0 \to \mathcal{E}_1$ is an adjointable map with $1 - \varphi\varphi^*$ and $1 - \varphi^*\varphi$ being B-compact. The isomorphism $K_0(B) \to K_0\left(M(B \otimes \mathcal{K}), M(B \otimes \mathcal{K})/(B \otimes \mathcal{K})\right)$ is given by the index map. It turns out that

$KK_0(\mathbb{C}, B)$ is naturally isomorphic to $K_0\left(M(B \otimes \mathcal{K})/(B \otimes \mathcal{K})\right)$.

We now turn to the definition of KK-theory.

Definition 1.4. Let A and B be C^*-algebras. Let $\mathrm{Fred}(A, B)$ be the class of all triples $(\mathcal{E}_0, \mathcal{E}_1, F)$, where $\mathcal{E}_0$ and $\mathcal{E}_1$ are correspondences from A to B (this is, Hilbert B-modules with a non-degenerate left A-action), and $F \in \mathcal{B}(\mathcal{E}_0, \mathcal{E}_1)$ is an adjointable operator (usually called Fredholm operator) such that $1 - FF^*$, $1 - F^*F$ and $[F, a]$ are proper for all $a \in A$. (Recall that an operator $t\colon \mathcal{E}_0 \to \mathcal{E}_1$ between correspondences from A to B is said to be *proper* if $t \cdot a$ and $a \cdot t$ are compact for all $a \in A$. This is the same as being compact if A is unital, but it is weaker in general.)

A triple $(\mathcal{E}_0, \mathcal{E}_1, F)$ as above is called a *KK-cocycle*.

Lemma 1.5. $[F, a_1a_2] = a_1[F, a_2] + [F, a_1]a_2$ for all $a_1, a_2 \in A$.

In particular, if $[F, a]$ is proper for all $a \in A$, it follow that is is compact for all $a \in A$.

A correspondence $\mathcal{E}$ from A to B is said to be *proper* if $A \subseteq \mathcal{K}(\mathcal{E})$. One then gets a msp

$$K_*(A) \to K_*(\mathcal{K}(\mathcal{E})) \to K_*(B)$$

using Morita equivalence $\mathcal{K}(\mathcal{E}) \sim_M \langle \mathcal{E}, \mathcal{E} \rangle \lhd B$. In particular, proper correspondences should give KK-cycles directly. In this sense, Fredholm operator means unitary up to proper perturbations.

Remark 1.6. Any operator between proper correspondences is automatically proper. Hence

$$\{\text{Hilbert } B - \text{modules with } \mathrm{id}_{\mathcal{E}} \in \mathcal{K}(\mathcal{E})\} \cong \{\text{proper correspondences } \mathbb{C} \to B\}.$$

We say that two KK-cocycles $(\mathcal{E}_0, \mathcal{E}_1, F)$ and $(\mathcal{E}_0', \mathcal{E}_1', F')$ are *equivalent* if there exist unitaries $u_0\colon \mathcal{E}_0 \to \mathcal{E}_0'$ and $u_1\colon \mathcal{E}_1 \to \mathcal{E}_1'$ such that

- $au_j = u_ja$ for all $a \in A$ and $j = 0, 1$.
- $u_1F - F'u_0$ is proper.

We can now give a picture of KK-theory. (This is not the original definition.)

Definition 1.7. (Cuntz-Skandalis) Let A and B be separable C^*-algebras. Then

$$KK_0(A, B) = \mathrm{G}(\mathrm{Fred}(A, B)).$$

That this definition is the same as the original one provided by Kasparov follows from the following theorem.

Theorem 1.8. (Cuntz-Skandalis) Let $F_1, F_2 \in \mathrm{Fred}(A, B)$. Then $F_1 \sim_h F_2$ if and only if there exists $F \in \mathrm{Fred}(A, B)$ such that $F_1 \oplus F$ is equivalent to $F_2 \oplus F$.

Notice in particular that the inverse of the class of $F\colon \mathcal{E}_0 \to \mathcal{E}_1$ is the class of $F^*\colon \mathcal{E}_1 \to \mathcal{E}_0$. For the Kasparov product, one should define a way of composing

$$A \xrightarrow{(\mathcal{E}_0, \mathcal{E}_1, F)} B \xrightarrow{(\mathcal{G}_0, \mathcal{G}_1, H)} C\,.$$

Interpreting $(\mathcal{E}_0, \mathcal{E}_1, F)$ as $\mathcal{E}_0 - \mathcal{E}_1$ and multiplying by $\mathcal{G}_0 - \mathcal{G}_1$, one sees that the natural choice for the correspondence from A to C is

$$(\mathcal{E}_0 \otimes_B \mathcal{G}_0 \oplus \mathcal{E}_1 \otimes_B \mathcal{G}_1, \mathcal{E}_1 \otimes_B \mathcal{G}_0 \oplus \mathcal{E}_0 \otimes_B \mathcal{G}_1, F\sharp H).$$

Defining the Fredholm operator $F\sharp H$ is difficult. Instead of exhibiting its construction, we present its universal property. We need some preparation first.

Let $(\mathcal{E}_0, \mathcal{E}_1, F)$ be a $KK(A, B)$-cocycle. Denote by $\varphi_j\colon A \to \mathcal{B}(\mathcal{E}_j)$ the left A-action, and set

$$D = \mathcal{K}(\mathcal{E}_0 \oplus \mathcal{E}_1) + \varphi_0(A) + \varphi_1(A) + F\varphi_0(A) + F^*\varphi_1(A).$$

Using that $\varphi_0(a)F^* = F^*\varphi_1(a)$ and $F\varphi_0(a) = \varphi_1(a)F$ for all $a \in A$, one shows that D is a C^*-algebra. Moreover, the triple $(\mathcal{E}_0, \mathcal{E}_1, F)$ determines a partially split extension of C^*-algebras

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}(\mathcal{E}_0 \oplus \mathcal{E}_1) \arrow[r] & D \arrow[r, "\pi"] & M_2(A) \arrow[r] & 0\,. \\ & & & A \oplus A \arrow[ul] \arrow[u] & \end{tikzcd}$$

Conversely, such a partially split extension gives $(\mathcal{E}_0, \mathcal{E}_1, F)$ back: the composition $A \oplus A \to D \to M(\mathcal{K}(\mathcal{E}))$ extends to $M(A \oplus A)$, and by taking the images of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, one obtains the B-modules $\mathcal{E}_0$ and $\mathcal{E}_1$, and one gets a decomposition $\mathcal{E} \cong \mathcal{E}_0 \oplus \mathcal{E}_1$. The operator F is recovered up to proper perturbations.

Theorem 1.9. (Universal property of the functor KK) Let $\mathcal{A}$ be an additive category and let $H\colon C^* \to \mathcal{A}$ be a split exact $\mathcal{K}$-stable functor. Then H factors uniquely through kk, meaning that there is a functor $\widehat{H}\colon \text{kk} \to \mathcal{A}$ such that

$$\begin{tikzcd} C^* \arrow[r, "H"] \arrow[d, "KK"'] & \mathcal{A} \\ \text{kk} \arrow[ur, "\widehat{H}"'] & \end{tikzcd} .$$

Proof. If $\mathcal{E}$ is a Hilbert B-module, then $H(\mathcal{K}(\mathcal{E})) \cong H(\langle \mathcal{E}, \mathcal{E}\rangle) \to H(B)$, and moreover the embedding $j\colon A \to M_2(A)$ given by $j(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ induces an isomorphism $j_*\colon H(A) \to H(M_2(A))$. Split exactness gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H(\mathcal{K}(\mathcal{E}) \arrow[r] & H(\pi^{-1}(j(A))) \arrow[r] \arrow[d, "\cong"] & H(A) \arrow[r] \arrow[d, "\cong"] & 0 , \\ 0 \arrow[r] & H(\mathcal{K}(\mathcal{E}) \arrow[r] & H(D) \arrow[r] & H(M_2(A)) \arrow[r] & 0 \\ & & H(A) \arrow[u, "H(j_0)"] & & \end{tikzcd}$$

where j_0 and j_0 are the canonical embeddings of A into $M_2(A)$ as corners. Hence $H(j_0)-H(j_1)$ maps $H(A)$ into $H(\mathcal{K}(\mathcal{E}) \subseteq H(D)$ because both give the same map on $H(M_2(A))$. The difference $H(j_0) - H(j_1)$ composed with $H(\mathcal{K}((E))) \to H(B)$ gives a map

$$KK(A, B) \to \mathrm{Hom}(H(A), H(B)).$$

This is actually a functor and it is the unique extension of H to all homomorphisms. Also, kk is an additive category and $C^* \to$ kk is split exact and stable. The bijection becomes now clear. $\square$

2. $KK-$ and $E-$theory for C^*-algebras over topological spaces

Even in the case of a finite primitive ideal space, there are many non-homeomorphic T_0 spaces with the same cardinality. We are therefore led to the following problem. Fix a topological space X, and classify all pairs (A, ψ), where A is a C^*-algebra, and $\psi\colon \mathrm{Prim}(A) \to X$ is a homeomorphism. In this context, we have the following result that remains unpublished.

Theorem 2.1. (Kirchberg, unpublished) Let A and B be nuclear, separable, $\mathcal{O}_\infty \otimes \mathcal{K}$-absorbing with $\mathrm{Prim}(A) \cong X \cong \mathrm{Prim}(B)$ with given homeomorphisms. Then A and B are isomorphic if and only if there is an invertible element in $KK_0(X; A, B)$.

The proof is difficult C^*-analysis. A natural question is then

Question 2.2. How to detect when there is an invertible element in $KK_0(X; A, B)$ using simpler invariants?

Answering this question will involve homological algebra in KK-theory. Nuclearity and $\mathcal{O}_\infty$-absorption are no longer relevant in this topic, and UCT considerations become crucial. Hence, it is important to understand how to generalize the UCT from KK to $KK(X)$.

Recall that there is a canonical bijective correspondence

$$\mathcal{O}(\mathrm{Prim}(A)) \cong \mathrm{Ideals}(A)$$

as complete lattices (this is, least upper bounds and greatest lower bounds exist). Moreover, $\mathrm{Prim}(A)$ can be recovered from $\mathcal{O}(\mathrm{Prim}(A))$: its points are the irreducible closed subsets. A space X is called *sober* is the only closed irreducible sets are the closure of the points in X. Kirchberg showed that $\mathrm{Prim}(A)$ is sober for every C^*-algebra A. We may therefore restrict ourselves to sober spaces X. Also, primitive ideal spaces are locally (quasi-)compact, and they are second countable if A is separable. Any finite T_0 space is the primitive ideal space of some nuclear, separable, $\mathcal{O}_\infty \otimes \mathcal{K}$-absorbing C^*-algebra. (Recall that X is T_0 if $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$, or also for every $x, y \in X$, there is an open set $U \subseteq X$ containing either x or y and not the other point. Any T_0 space is automatically sober.)

Remark 2.3. If X is finite and T_1, then it is discrete. Hence, if one is working with finite spaces, there is no interest in considering T_1 spaces.

If X is a finite T_0 space, define a partial order on X by $x \leq y$ if $\overline{\{x\}} \subseteq \overline{\{y\}}$.

Lemma 2.4. Let X be a finite T_0 space, and let A be a subset of X.

(1) A is closed if and only if whenever $x \in A$ and $y \leq x$, then $y \in A$.
(2) A is open if and only if whenever $x \in A$ and $x \leq y$, then $y \in A$.

Conversely, a partial order on a set X defines a topology in this way, called the *Alexandrov topology*.

Example 2.5. If $X = [0,1]$, then the Alexandrov topology on it induced by the usual order of $\mathbb{R}$ is the topology whose open sets are

$$\{(a,1]\colon a \in [0,1]\} \cup \{[a,1]\colon a \in [0,1]\}.$$

If $\mathrm{Prim}(A) = X$, then any open subset U of X has associated to it an ideal $A(U)$ of A, and if C is a closed subset of X, then

$$A(C) = A/A(X \setminus C).$$

In particular, if $U \subseteq V \subseteq X$ are open subsets, then

$$A(U \setminus V) = A(U)/A(V),$$

and $A(U \setminus V)$ only depends on $U \setminus V$.

Definition 2.6. Let A be a C^*-algebra. The *filtrated K-theory* of A is

$$FK(A) = (K_*(A(U \setminus V))_{V,U}.$$

It is not clear yet in what category this object lives.

Suppose that $\psi\colon \mathrm{Prim}(A) \to X$ is a continuous function. Then there is a correspondence $\psi^{-1}\colon \mathcal{O}(X) \to \mathcal{O}(\mathrm{Prim}(A)) = \mathrm{Ideals}(A)$.

Lemma 2.7. A map $f\colon \mathcal{O}(X) \to \mathrm{Ideals}(A)$ is ψ^{-1} for some continuous map $\psi\colon \mathrm{Prim}(A) \to X$ if and only if f commutes with arbitrary suprema and finite infima ($f(\emptyset) = 0, f(X) = A$ and f is monotone).

A morphism $f\colon (A,\psi) \to (B,\varphi)$ between C^*-algebras over a space X is a $*$-homomorphism $f\colon A \to B$ such that $f(A(U)) \subseteq B(U)$ for all $U \subseteq X$ open.

In the case in which $\mathrm{Prim}(A)$ is Hausdorff, the notion of C^*-algebras over X agrees with a previously known and well-studied notion: that of $C_0(X)$-C^*-algebras.

Theorem 2.8. Let X be a locally compact Hausdorff space. Then there is an isomoprhism of categories

$$C^*\text{-algebras over } X \cong C_0(X) - C^*\text{-algebras}.$$

We now turn to the definitions of $KK(X)$- and $E(X)$-theory.

Definition 2.9. Let E be a graded Hilbert B-module with a left action, and let $F \in \mathcal{B}(E)^{odd}$ be a self-adjoint operator such that

(1) $(1-F^2)a \in \mathcal{K}(B)$ for all $a \in A$,
(2) $[F,a] \in \mathcal{K}(B)$ for all $a \in A$,
(3) E is X-equivariant: $A(U) \cdot E = E \cdot B(U)$ for all $U \subseteq X$ open.

In the locally compact Hausdorff case, instead of (4) one requires that

$$(f \cdot a) \cdot \xi = a \cdot (\xi \cdot f)$$

for all $f \in C_0(X)$, for all $x \in A$, and for all $\xi \in E$.

Kasparov's theory goes through in the X-equivariant case: KK-product works as usual, no conditions on F means that there is no need to modify the difficult bits in Kasparov's theory. We therefore get a category $\mathrm{kk}(X)$ with separable C^*-algebras over X as objects and

$$KK_0(X;A,B) = \text{ homotopy classes of } X-\text{equivariant } KK\text{-cycles } A \to B$$

as morphisms, with Kasparov's product as composition. If A is a C^*-algebra over X and B is any C^*-algebra, then $A \otimes B$, for any choice of the tensor product, is a C^*-algebra over X. This descends to a functor $\mathrm{kk}(X) \times \mathrm{kk} \to \mathrm{kk}(X)$. This implies for example that

$$A \otimes C_0(\mathbb{R}^2) \sim_{\mathrm{kk}(X)} A$$

for any C^*-algebra A over X.

Just as in Kasparov's theory, $\mathrm{kk}(X)$ is universal. The functor

$$\mathrm{kk}(X) \leftarrow \; C^*\text{-algebras over } X$$

is the universal split-exact, $\mathcal{K}$-stable (homotopy invariance follows from these two) functor. For a C^*-algebra A over X, the functor $B \mapsto H(A \otimes B)$ is split exact and stable on the category of all C^*-algebras if H is exact and stable on all C^*-algebras over X.

Long exact sequences in $KK(X)$-theory work similarly to long exact sequences in KK-theory. An extension of C^*-algebras over X is a diagram

$$0 \to I \to E \to Q \to 0$$

in the category of C^*-algebras over X, such that for every open set U of X, the induced diagram

$$0 \to I(U) \to E(U) \to Q(U) \to 0$$

is an extension in the category of C^*-algebras. Not every extension induces a long exact sequence in KK-theory. An extra condition must be added, and this is *semi-splitness*. An extension $I \to E \to Q$ as above is called semi-split if there is a completely positive (contractive follows) X-equivariant section $s\colon Q \to E$. For a semi-split extension, there are long exact sequences in $KK(X)$-theory in both variables (for one of them, the corresponding functor is contravariant).

Example 2.10. Consider the extension

$$C_0((0,1]) \to C([0,1]) \to \mathbb{C},$$

where the second map is given by evaluation at 0. This extension is not semi-split.

For infinite, second countable spaces X, it is better to consider the X-equivariant analog of E-theory instead.

Definition 2.11. $E(X)$ is the universal exact $\mathcal{K}$-stable homotopy functor on separable C^*-algebras over X.

There is also a concrete description using X-equivariant asymptotic morphisms, where X-equivariancy for a path $(\varphi_t)_{t\in[0,\infty)}$ of functions $\varphi_t\colon A \to B$ means that if $U \subseteq X$ is open and $a \in A(U)$, then

$$\lim_{t\to\infty} \|\varphi_t(a)\|_{B(X\setminus U)} = 0.$$

Theorem 2.12. The obvious map $KK(X;A,B) \to E(X;A,B)$ is an isomorphism whenever X is finite and A is nuclear.

When X is infinite, $E(X)$ can potentially be computed using finite spaces and projective limits as follows. Choose a countable basis $(U_n)_{n\in\mathbb{N}}$ for the topology of X (which was assumed to be second countable). Denote by X_n the T_0-quotient of X with topology given by $U_1,\dots,U_n$. There are maps $X_n \to X_{n-1}$, and X is homeomorphic to the projective limit

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1.$$

If A and B are C^*-algebras over X, we may view them as C^*-algebras over X_n, and hence $E_*(X;A,B)$ can potentially be computed from $E_*(X_n;A,B)$ using the short exact sequence

$$0 \leftarrow \varprojlim E_*(X_n;A,B) \leftarrow E_*(X;A,B) \leftarrow \varprojlim{}^1 E_{*+1}(X_n;A,B) \leftarrow 0.$$

3. Homological algebra in triangulated categories

We begin by recalling some facts about the UCT for KK-theory. For any C^*-algebras A and B, one has

$$KK_*(\mathbb{C},B) = K_*(B) \quad KK_1(A,B) = KK_0(C_0(\mathbb{R},A),B) \quad KK_0(C_(\mathbb{R}),B) = K_1(B).$$

The product on KK-theory gives a pair of maps

$$KK_0(A,B) \to \mathrm{Hom}(K_*(A),K_*(B)) \quad KK_1(A,B) \to \mathrm{Hom}(K_*(A),K_{*+1}(B)).$$

Moreover, $J_K = \ker(\gamma) \leq KK(A,B)$ is an ideal in the category kk, meaning that if $\alpha \in J_K$, then $\mathrm{id}_{C_0(\mathbb{R})} \otimes \alpha \in J_K$. In other words, J_K is a stable homological ideal in kk.

Theorem 3.1. (Brown) There is a natural map

$$\kappa\colon \ker(\gamma(A,B)) \to \mathrm{Ext}(K_*(A),K_{*+1}(B)).$$

Any element in $KK_1(A,B)$ is given by a semi-split C^*-algebra extension

$$0 \to B\otimes\mathcal{K} \to E \to A \to 0.$$

With the picture above, given $\alpha \in KK_1(A,B)$, consider the 6-term exact sequence on K-theory

$$\begin{tikzcd}
K_0(B) \arrow[r] & K_0(E) \arrow[r] & K_0(A) \arrow[d, "\pm\gamma(\alpha)"] \\
K_1(A) \arrow[u, "\pm\gamma(\alpha)"] & K_1(E) \arrow[l] & K_1(B) \arrow[l]
\end{tikzcd}$$

If $\gamma(\alpha) = 0$, this gives us two extensions of abelian groups, giving $K(\alpha) \in \mathrm{Ext}(K_*(A),K_{*+1}(B))$.

Definition 3.2. The UCT holds for A and B if the map

$$\gamma\colon KK_*(A,B) \to \mathrm{Hom}_*(K_*(A),K_*(B))$$

is an isomorphism. A C^*-algebra A is sait to be in the *bootstrap class*, also called UCT class, if the UCT holds for A and B for every C^*-algebra B.

The bootstrap class contains $\mathbb{C}$ and it is closed under:

1. Countable direct sums
2. Suspensions
3. Two out of three in any extension
4. Semi-split extensions
5. KK-equivalece

It turns out that the bootstrap class is the smallest subclass of kk with these properties.

Example 3.3. If A is contractible, then A satisfies the UCT, because all the K-groups vanish.

Theorem 3.4. (Skandalis) A C^*-algebra A is in the bootstrap class if and only if A is KK-equivalent to an abelian (type I) C^*-algebra.

The category kk is a triangulated category. This framework formalizes some general techniques for working with long exact sequences. We present the definition below.

Definition 3.5. A *triangulated category* is an additive category τ with a suspension automorphism $\Sigma\colon \tau \to \tau$ and a class of *exact triangles*:

$$A \to B \to C \to \Sigma A.$$

A functor F from τ to an abelian category is said to be *homological* if $F(A) \to F(B) \to F(C)$ is exact for all exact triangles $A \to B \to C \to \Sigma A$. The asioms of a triangulated category imply that F has long exact sequences

$$\cdots \to F(\Sigma^{-1}C) \to F(A) \to F(B) \to F(C) \to F(\Sigma A) \to F(\Sigma B) \to \cdots .$$

A diagram $A \to B \to C \to \Sigma A$ in kk is *exact* if and only if there are KK-equivalences α, β and γ and an extension triangle

$$\begin{tikzcd}
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A' \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & \Sigma A. \arrow[u, "\Sigma\alpha"]
\end{tikzcd}$$

Extension triangle means that $0 \to A' \to B' \to C' \to 0$ is semi-split extension, and that $C' \to \Sigma A'$ is the boundary map.

These are the axioms of triangulated categories:

(1) $A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \Sigma A$ is exact for every A.
(2) For every morphism $f\colon A \to B$, there exists an exact triangle $A \to B \to C \to \Sigma A$ containing the morphism f. (This is false for kk for $\mathbb{Z}_2$-graded C^*-algebras.)
(3) A triangle $A \to B \to C \to \Sigma A$ is exact if and only if the induced triangle $B \to C \to \Sigma A \to \Sigma B$ is exact.

Proposition 3.6. For every object D of τ, the assignments $A \mapsto \tau(D, A)$ and $A \mapsto \tau(A, D)$ are (co)-homological.

The axiom ensures that ensures this is the following. If

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, dashed, "\gamma"] & \Sigma A' \arrow[d, "\Sigma\alpha"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A',
\end{tikzcd}$$

then there exists $\gamma\colon C \to C'$ making the resulting diagram exact and commutative. Notice that γ is not assumed to be unique; in fact it will not in general be unique.

Corollary 3.7. If α and β are isomorphisms in τ, then so is γ.

Proof. For any object D in τ, the map $\tau(D, \gamma)\colon \tau(D, C) \to \tau(D, C')$ is an isomorphism by the 5 Lemma. This implies that γ is an isomorphism itself. ☐

Corollary 3.8. Two exact triangles $A \to B \to C \to \Sigma A$ with the same map $f\colon A \to B$ are isomorphic.

This object C is called the *cone* of f. It is unique up to isomorphism by the Corollary, but the isomorphism is not natural. The octahedral axiom gives, among other things, an exact triangle relating C_f, $C_{f\circ g}$ and C_g in a triangle

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[dr, "f\circ g"'] & B \arrow[d, "f"] \\
 & D.
\end{tikzcd}$$

These axioms all hold for KK-theory. We want to explore the UCT in the context of triangulated categories. In the usual context, we started with a stable (commutes with suspension) homological functor $K_*\colon \mathrm{kk} \to Ab_c^{\mathbb{Z}_2}$ (countable $\mathbb{Z}_2$-graded abelian groups). We could instead take nay triangulated category τ and a stable homological functor $F\colon \tau \to \mathcal{A}$.

Example 3.9. Let $\tau = \mathrm{kk}(X)$ for some finite T_0 space X, and set

$$F(A) = (K_*(A(U_x)))_{x \in X}$$

where U_x is the minimal open set in X containing x. This would not give a nice UCT, though. One can instead take

$$F(A) = \left(K_*(A(\overline{\{x\}}))\right)_{x\in X} \qquad \text{or} \qquad FK(A) = (K_*(A(C)))_{C \text{ locally closed}} \cdot$$

It is not clear however whether there is a nice UCT using these invariants.

Given two objects A and B, the map $\gamma\colon \tau(A,B) \to \mathrm{Hom}_{\mathcal{A}}(F(A),F(B))$ should be composition with the functor F. There is an obvious problem: in general, there is no reason why γ should be surjective, unless we restrict the category $\mathcal{A}$. Nevertheless, the kernel of this map is still interesting, and it does not depend on the choice of the target category $\mathcal{A}$. We therefore turn to a more systematic approach to proving the UCT involving projective resolutions.

In the case of KK-theory, we have $F = K_*$.

Theorem 3.10. For every C^*-algebra A, there exists an abelian C^*-algebra $\widehat{A}$ satisfying the UCT and with isomorphic K-groups. Moreover, A satisfies the UCT if and only if it is KK-equivalent to $\widehat{A}$.

Proof. Let A be an object in kk and find a projective resolution

$$0 \longrightarrow P_1 \xrightarrow{d} P_0 \xrightarrow{\pi} K_*(A) \longrightarrow 0$$

of $K_*(A)$ in $Ab_c^{\mathbb{Z}_2}$. (Recall that in the category of abelian groups, projective is the same as free.) The trick to prove the UCT is to lift the diagram to KK-theory. Write

$$P_j = \bigoplus_{i\in I_j^+} \mathbb{Z}[0] \oplus \bigoplus_{i\in I_j^-} \mathbb{Z}[1]$$

for $j = 0,1$, where the first summand is the degree zero component. To lift the diagram, set

$$\widehat{P}_j = \bigoplus_{i\in I_j^+} \mathbb{C} \oplus \bigoplus_{i\in I_j^-} C_0(\mathbb{R})$$

and note that $K_*(\widehat{P}_j) = P_j$ for $j = 0,1$. Since KK-theory takes direct sums to direct products, we get isomorphisms

$$KK(\widehat{P}_j, B) = \prod_{i\in I_j^+} KK(\mathbb{C}, B) \times \prod_{i\in I_j^-} KK(C_0(\mathbb{R}), B) \cong \mathrm{Hom}_{Ab_c^{\mathbb{Z}_2}}(P_j, K_*(B)).$$

Hence π lifts to $\widehat{\pi}\colon \widehat{P}_0 \to A$ and d lifts to $\widehat{d}\colon \widehat{P}_1 \to \widehat{P}_0$, so we get

$$0 \longrightarrow \widehat{P}_1 \xrightarrow{\widehat{d}} \widehat{P}_0 \xrightarrow{\widehat{\pi}} A \longrightarrow 0\ ,$$

and $\widehat{\pi} \circ \widehat{d} = \widehat{\pi \circ d} = 0$ since lifts are unique. Since KK is a triangulated category, we can find an exact triangle in KK containing $\widehat{d}$:

$$\widehat{P}_1 \xrightarrow{\widehat{d}} \widehat{P}_0 \longrightarrow \widehat{A} \longrightarrow \Sigma\widehat{P}_1\ .$$

Then $\widehat{\pi} \in KK(\widehat{P}_0, A)$ is in the kernel of $KK(\widehat{d}, A)$. Since $KK(-,A)$ is homological, we get an exact sequence

$$KK(\widehat{A}, A) \longrightarrow KK(\widehat{P}_0, A) \longrightarrow KK(\widehat{P}_1, A)$$

where the last map sends $\widehat{\pi}$ to 0. Thus there exists an element $\varphi \in KK(\widehat{A}, A)$ that is mapped to $\widehat{\pi}$. We claim that φ induces an isomorphism on K-theory. Ultimately, the goal is to show that A is in the Bootstrap class if and only if φ is invertible. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_*(\widehat{P}_1) \arrow[r] \arrow[d, "\mathrm{id}"] & K_*(\widehat{P}_0) \arrow[r] \arrow[d, "\mathrm{id}"] & K_*(A) \arrow[r] \arrow[d, "K_*(\varphi)"] & 0 \\
K_{*+1}(\widehat{A}) \arrow[r] & K_*(\widehat{P}_1) \arrow[r] & K_*(\widehat{P}_0) \arrow[r] & K_*(\widehat{A}) \arrow[r] & K_{*-1}(\widehat{P}_1).
\end{tikzcd}$$

A diagram chase shows that $K_{*+1}(\widehat{A}) \to K_*(\widehat{P}_1)$ and $K_*(\widehat{A}) \to K_{*-1}(\widehat{P}_1)$ are zero, and hence $K_*(\varphi)$ is an isomorphism. Now take B in kk and write down a long exact sequence for $KK(-,B)$ and $\widehat{P}_1 \to \widehat{P}_0 \to \widehat{A} \to \Sigma\widehat{P}_1$:

$$\begin{tikzcd}
KK_0(\widehat{P}_1, B) \arrow[d] & KK_0(\widehat{P}_0, B) \arrow[l] & KK_0(\widehat{A}, B) \arrow[l] \\
KK_1(\widehat{A}, B) \arrow[r] & KK_1(\widehat{P}_0, B) \arrow[r] & KK_1(\widehat{P}_1, B) \arrow[u]
\end{tikzcd}$$

Recall that $KK_0(\widehat{P}_j, B) = \mathrm{Hom}_*(P_j, K_*(B))$. Now,

$$\begin{aligned}
\mathrm{Hom}(K_*(A), K_*(B)) &= \ker\left(\mathrm{Hom}(P_0, K_*(B)) \to \mathrm{Hom}(P_1, K_*(B))\right)\\
\mathrm{Ext}(K_*(A), K_*(B)) &= \mathrm{coker}\left(\mathrm{Hom}(P_0, K_*(B)) \to \mathrm{Hom}(P_1, K_*(B))\right),
\end{aligned}$$

and thus we can extract the short exact sequence

$$0 \to \mathrm{Ext}(K_*(A), K_*(B)) \to KK_*(\widehat{A}, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

and since $K_*(A) \cong K_*(\widehat{A})$, we get the UCT for $\widehat{A}$.

Starting with a C^*-algebra A, we constructed an abelian C^*-algebra $\widehat{A}$ with isomorphic K-theory and that satisfies the UCT. We will now show that A satisfies the UCT if and only if φ is a KK-equivalence. MISSING BIT. $\square$

We wish to generalize this construction to arbitrary triangulated categories. A major question is how to go back and forth between a triangulated category τ and a stable abelian category $\mathcal{A}$. For example, it could be $\tau = \mathrm{kk}$ and $\mathcal{A} = Ab_c^{\mathbb{Z}_2}$, or $\tau = \mathrm{kk}(X)$ and $\mathcal{A} =$? Suppose we start with a stable homological functor $F\colon \tau \to \mathcal{A}$ such that for a projective object $P \in \mathcal{A}$, there is a canonical $\widehat{P} \in \tau$ with $F(\widehat{P}) = P$ and

$$\mathrm{Hom}_\tau(\widehat{P}, B) = \mathrm{Hom}_{\mathcal{A}}(P, F(B))$$

for all $B \in \tau$. It follows that $\widehat{P}$ is unique and hence $P \mapsto \widehat{P}$ is a partially defined left-adjoint functor of F. Indeed, if P_1 and P_2 are projective objects in $\mathcal{A}$, then

$$\mathrm{Hom}_\tau(\widehat{P}_1, \widehat{P}_2) = \mathrm{Hom}_{\mathcal{A}}(P_1, F(\widehat{P}_2)) = \mathrm{Hom}_{\mathcal{A}}(P_1, P_2).$$

We can also lift projective resolutions. Given $B \in \tau$, choose a projective resolution

$$\cdots \to P_1 \to P_0 \to F(B) \to 0$$

in $\mathcal{A}$ for $F(B)$. Then the isomorphism $\mathrm{Hom}_{\mathcal{A}}(P_0, F(B)) \cong \mathrm{Hom}_\tau(\widehat{P}_0, B)$ lifts the map $P_0 \to F(B)$ uniquely to a map $\widehat{P}_0 \to B$. The maps $P_n \to P_{n-1}$ for $n \geq 1$ also lift uniquely to maps $\widehat{P}_n \to \widehat{P}_{n-1}$, giving

$$\cdots \to \widehat{P}_1 \to \widehat{P}_0 \to B \to 0.$$

Since lifts are unique, the compositions are again zero, and we therefore get a chain complex. This is a $\ker(F)$-projective resolution of G, and any such is of this form.

To finish the proof of the UCT, we will assume that the resolution has length one, so that we can choose $P_n = 0$ for $n \geq 2$. We can embed $\widehat{P}_1 \to \widehat{P}_0$ into an exact triangle

$$\begin{tikzcd} \widehat{P}_1 \arrow[r] \arrow[dr, "0"'] & \widehat{P}_0 \arrow[r] \arrow[d] & \widetilde{B} \arrow[r] \arrow[dl, dashed, "\exists\varphi"] & \Sigma\widehat{P}_1 . \\ & B & & \end{tikzcd}$$

Since $\tau(-, B)$ is cohomological, there is a morphism $\varphi\colon \widetilde{B} \to B$. Applying F to this triangle, we get a commutative diagram

$$\begin{tikzcd} F(\widehat{P}_1) \arrow[r] \arrow[d, "\cong"] & F(\widehat{P}_0) \arrow[r] \arrow[d, "\cong"] & F(\widetilde{B}) \arrow[r] \arrow[d, "F(\varphi)"] & F(\Sigma\widehat{P}_1) \arrow[r] \arrow[d, "\cong"] & F(\Sigma\widehat{P}_0) \arrow[d, "\cong"] \\ P_1 \arrow[r] & P_0 \arrow[r] & F(B) \arrow[r] & \Sigma P_1 \arrow[r] & \Sigma P_0. \end{tikzcd}$$

Since $P_1 \to P_0$ is a monomorphism, so is its suspension $\Sigma P_1 \to \Sigma P_0$, and hence $F(B) \to \Sigma P_1$ is an epimorphism, and so is $P_0 \to F(B)$. Thus $\varphi\colon \widetilde{B} \to B$ induces an isomorphism $F(\varphi)\colon F(\widetilde{B}) \to F(B)$. Recall that $\widetilde{B}$ is constructed out if $\ker(F)$-projective objects.

The following lemma in the case $\tau = \mathrm{kk}$ and $\mathcal{A} = Ab_c^{\mathbb{Z}_2}$ states that $KK(\widetilde{B}, D) = 0$ whenever $K_*(D) = 0$. This also follows using the UCT and recalling that $\widetilde{B}$ is in the Bootstrap class.

Lemma 3.11. Let D be an object in τ such that $F(C) = 0$. Then $\tau(\widetilde{B}, D) = 0$.

Proof. Since $\tau(\widehat{P}, D) \cong \mathrm{Hom}_{\mathcal{A}}(P, F(D)) = 0$, the result follows from considering the long exact sequence for $\tau(-, D)$ associated to the exact triangle $\widehat{P}_1 \to \widehat{P}_0 \to \widetilde{B} \to \Sigma\widehat{P}_1$. $\square$

Theorem 3.12. Assume that for every $B \in \tau$, the object $F(B)$ has a projective resolutions of length 1 in $\mathcal{A}$. Then for every object D in τ, there is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{A}}(\Sigma F(\widetilde{B}), F(D)) \to \tau(\widetilde{B}, D) \to \mathrm{Hom}_{\mathcal{A}}(F(\widetilde{B}), F(D)) \to 0.$$

Proof. Apply the cohomological functor $\tau(-, D)$ to $\widehat{P}_1 \to \widehat{P}_0 \to \widetilde{B} \to \Sigma\widehat{P}_1$ and use the identification $\tau(\widehat{P}_1, D) \cong \mathrm{Hom}_{\mathcal{A}}(P_1, F(D))$ to get

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(\Sigma P_0, F(D)) \to \mathrm{Hom}_{\mathcal{A}}(\Sigma P_1, F(D)) \to \tau(\widetilde{B}, D) \to \mathrm{Hom}_{\mathcal{A}}(P_0, F(D)) \to \mathrm{Hom}_{\mathcal{A}}(P_1, F(D)) \to \cdots$$

and the desired short exact sequence can be extracted from the above long exact sequence by noticing that the Ext and Hom groups involved in the statement are the cokernel and kernel of the corresponding maps above. $\square$

Lemma 3.13. With the notation of the above theorem and discussion, B satisfies the UCT if and only if φ is an isomorphism.

Proof. If $\varphi\colon \widetilde{B} \to B$ is an isomorphism, then the same UCT sequence holds for B. Conversely, assume that B satisfies the UCT. Embed φ into a right triangle $\widetilde{B} \to B \to C \to \Sigma\widetilde{B}$. Then $F(C) = 0$ because $F(\varphi)$ (and hence $F(\Sigma\varphi)$) is invertible and using the long exact sequence. Since B satisfies the UCT, we have $\tau(B,C) = 0$ so in particular $g = 0$. We also know that $\tau(\widetilde{B}, C) = 0$ since the UCT holds for $\widetilde{B}$ as well. Now the long exact sequence for $\tau(-,C)$ implies that $\tau(C,C) = 0$, so $\mathrm{id}_C = 0$ and hence $C = 0$. It follows that φ is an isomorphism. $\square$

Corollary 3.14. We have the following description of the Bootstrap class:

$$\begin{aligned} \text{Bootstrap class} &= \{\text{objects constructible from } \widehat{P}, \text{ with } P \text{ projective in } \mathcal{A}\} \\ &= \{\text{objects } A \in \mathcal{A}\colon \tau(A,D) = 0 \text{ whenever } F(D) = 0\}. \end{aligned}$$

In other words, it suffices to check that the UCT holds for (A,D) whenever $K_*(D) = 0$ to conclude that it holds for any pair (A,B).

How does the UCT help in classification? The UCT for B implies that any map $\alpha\colon F(B) \to F(D)$ lifts to an element in $\tau(B,D)$. If α is invertible and D also satisfies the UCT, then any lift of α must also be invertible. Indeed, lift α and α^{-1} to $\widehat{\alpha}$ and $\widehat{\alpha^{-1}}$ respectively. Then $\widehat{\alpha} \circ \widehat{\alpha^{-1}}$ and $\widehat{\alpha^{-1}} \circ \widehat{\alpha}$ lift the identity maps, so we only need to check that liftings of the identity maps of $F(B)$ and $F(D)$ are invertible. This is true because $\ker(F) \subseteq \tau(B,B)$ and $\ker(F) \subseteq \tau(D,D)$ are nilpotent.

Theorem 3.15. (Naturality of the UCT) Let $f\colon D \to D'$ be a morphism in τ and let $B \in \tau$. Then there are maps $F(f)_*, f_*$ and $F(f)_*$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}}(\Sigma F(\widetilde{B}), F(D)) \arrow[r] \arrow[d, "F(f)_*"] & \tau(\widetilde{B}, D) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}_{\mathcal{A}}(F(\widetilde{B}), F(D)) \arrow[r] \arrow[d, "F(f)_*"] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}}(\Sigma F(\widetilde{B}), F(D')) \arrow[r] & \tau(\widetilde{B}, D') \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(F(\widetilde{B}), F(D')) \arrow[r] & 0 \end{tikzcd}$$

commute.

We will now discuss the range of the invariant. We still need to assume that for every object B in τ, the invariant $F(B)$ has a length one projective resolution in $\mathcal{A}$.

Lemma 3.16. Assume that for every object B in τ, the invariant $F(B)$ has a length one projective resolution in $\mathcal{A}$. Then any X in $\mathcal{A}$ with length 1 projective resolution is of the form $F(B)$ for some $B \in \tau$.

Proof. If $0 \to P_1 \to P_1 \to X \to 0$ is a projective resolution of X in $\mathcal{A}$, lift $P_1 \to P_0$ and embed it into an exact triangle $\widehat{P}_1 \to \widehat{P}_0 \to B \to \Sigma\widehat{P}_1$. Then $F(B) \cong X$. $\square$

One remarkable case in which this happens is $K_*\colon \mathrm{kk} \to Ab_c^{\mathbb{Z}_2}$.

Example 3.17. Let $X = \{a,b\}$ with topology $\{\emptyset, \{a\}, \{a,b\}\}$. Set $\tau = \mathrm{kk}(X)$ and $F(I \lhd A)$ is the 6-term exact sequence

$$\begin{tikzcd} K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d] \\ K_1(A/I) \arrow[u] & K_1(A) \arrow[l] & K_1(I) \arrow[l] \end{tikzcd}$$

Hence we may take $\mathcal{A}$ to consist of all 6-periodic chain complexes of countable abelian groups. One should not take all 6-periodic *exact* chain complexes, because they do not form an additive category. The projective objects in $\mathcal{A}$ can be described as follows: a chain $\mathcal{C}$ is projective if and only if every group appearing in it is projective (free) and $\mathcal{C}$ is exact. One such example is

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r] & 0 \arrow[d] \\ 0 \arrow[u] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

and there are 5 more such examples obtained by rotating this one. These in fact generate all projective chains. It is easy to check that $\mathbb{C} \lhd \mathbb{C}$ is a lift for this projective object. The lift for

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[d] \\ 0 \arrow[u] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

is $0 \lhd \mathbb{C}$, and the lift for

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ 0 \arrow[u] & 0 \arrow[l] & \mathbb{Z} \arrow[l] \end{tikzcd}$$

is $C_0((0,1)) \lhd C_0((0,1])$. The remaining three generators are lifted by taking suspensions in these three examples.

We look at projective resolutions. If $\mathcal{C}$ is a chain complex, there exists a projective object P_0 in $\mathcal{A}$ and an epimorphism $P_0 \to \mathcal{C} \to 0$. Let P_1 be the kernel of this map. We claim that P_1 is projective if $\mathcal{C}$ is exact. Since $P_1 \subseteq P_0$ and P_0 contains free groups, so does P_1. Now consider the long exact sequence in homology:

$$\cdots \to H_n(P_1) \to H_n(P_0) \to H_n(\mathcal{C}) \to H_{n+1}(P_1) \to H_{n+1}(P_0) \to \cdots .$$

It follows that P_1 is exact if and only if $\mathcal{C}$ is exact. We conclude that P_1 is projective if $\mathcal{C}$ is exact. If $\mathcal{C}$ is not exact, then it has infinite length of projective resolutions.

4. Lifting invariants with projective resolutions of length two

We will focus on the case where projective resolutions have length two. This is a very common situation.

Example 4.1. Let X be a finite unique path space (between any two points, there is at most one path connecting them). Then its quiver algebra $\mathbb{Z}[X]$ has cohomological dimension 2: every module has a length 2 projective resolution. For $\mathbb{Z}[X]$ itself, we have

$$0 \to \bigoplus_{x \to y} \mathbb{Z}[X]e_y \otimes e_x\mathbb{Z}[X] \to \bigoplus_{x \in X_0} \mathbb{Z}[X]e_x \otimes e_x\mathbb{Z}[X] \to \mathbb{Z}[X] \to 0,$$

where the first map is $a \otimes b \mapsto a(x \to y) \otimes b - a \otimes (x \to y)b$, and the second one is $z \otimes w \mapsto zw$. If M is a $\mathbb{Z}[X]$-module, then by tensoring the resolution for $\mathbb{Z}[X]$ with M, we get an exact sequence of $\mathbb{Z}[X]$-modules

$$0 \to \bigoplus_{x \to y} \mathbb{Z}[X]e_y \otimes M_x \to \bigoplus_{x \in X_0} \mathbb{Z}[X]e_x \otimes M_x \to M \to 0.$$

Then take long exact sequence and use the identification

$$\mathrm{Ext}^n_{\mathbb{Z}[X]}(\mathbb{Z}[X]e_x \otimes_{\mathbb{Z}} M_y, N) \cong \mathrm{Ext}^n_{\mathbb{Z}}(M_y, N_x)$$

to conclude that $\mathrm{Ext}^n_{\mathbb{Z}[X]} = 0$ for $n \geq 3$. On the other hand, $\mathrm{Ext}^2_{\mathbb{Z}[X]} = \mathrm{coker}(\mathrm{Ext}^1_{\mathbb{Z}} \to \mathrm{Ext}^1_{\mathbb{Z}})$ is non-zero in general.

Now let $\tau = \mathrm{kk}(X)$ and $\mathcal{A} = \mathbb{Z}[X]$-modules, with

$$F(A) = (K_*(A(U_x)))_{x \in X},$$

where U_x is the minimal open set containing x. Notice that if $x \leq y$, then $U_x \subseteq U_y$ and hence there is a map

$$K_*(A(U_x)) \to K_*(A(U_y)).$$

Check that $\mathbb{Z}[X]e_x$ lifts to $\mathbb{C}_x$, so liftings of resolutions work as usual.

Example 4.2. The category of countable abeliab $\mathbb{Z}_2$-graded $\mathbb{Z}[x,x^{-1}]$- and $\mathbb{Z}[x]$-modules have homological dimension two. (The proof is similar to the example above.) Our machinery can be applied to circle actions with $\tau = KK^{\mathbb{T}}$ and $F = K^{\mathbb{T}}_*$.

Let $F\colon \tau \to \mathcal{A}$ be as usual.

Theorem 4.3. Let B be an object of $\mathcal{A}$ with a length two projective resolution. Then B lifts to an object in τ.

Proof. Choose a length two projective resolution

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \longrightarrow 0$$

of B. Let $\Omega B = \ker(d_0) = \Im(d_1)$. Since there is a length one projective resolution $0 \to P_2 \to P_1 \to \Omega B \to 0$, it follows that ΩB lifts to $\widehat{\Omega B}$ in τ, and $\widehat{\Omega B}$ satisfies the UCT:

$$0 \to \mathrm{Ext}^1_{\mathcal{A}}(\Sigma F(\widetilde{B}), F(D)) \to \tau(\widetilde{B}, D) \to \mathrm{Hom}_{\mathcal{A}}(F(\widetilde{B}), F(D)) \to 0.$$

Hence the inclusion $\Omega B \hookrightarrow P_0$ lifts to $\varphi \in \tau(\widehat{\Omega B}, \widehat{P}_0)$. Set $\widehat{B} = \mathit{cone}(\varphi)$. We claim that $\widehat{B}$ is a lift of B. Since

$$\widehat{\Omega B} \xrightarrow{\varphi} \widehat{P}_0 \longrightarrow \widehat{B} \longrightarrow \Sigma\widehat{\Omega B}$$

is an exact triangle, it follows that

$$F(\widehat{\Omega B} \to F(\widehat{P}_0) \to F(\widehat{B}) \to F(\Sigma\widehat{\Omega B}) \to F(\Sigma\widehat{P}_0)$$

is also exact. Since the first three terms can be identified with $0 \to \Omega B \hookrightarrow P_0 \twoheadrightarrow F(\widehat{B})$, it follows that $F(\widehat{B}) = B$. ☐

Since we are mainly concerned with classification, we want to study to what extent the lifting is unique.

Theorem 4.4. Liftings of B are in bijection with the elements of $\mathrm{Ext}^2_{\mathcal{A}}(B, \Sigma^{-1}(B))$. Hence, there is a unique lift if and only if $\mathrm{Ext}^2_{\mathcal{A}}(B, \Sigma^{-1}(B)) = 0$.

A lift of B is a pair $(\widehat{B}, \theta)$ where $\widehat{B} \in \tau$ is in the UCT class and $\theta\colon F(\widehat{B}) \to B$ is an isomorphism.

Remark 4.5. The bijection with $\mathrm{Ext}^2_{\mathcal{A}}(B, \Sigma^{-1}(B))$ is not canonical: it depends on the choice of a particular lifting corresponding to the zero element in the $\mathrm{Ext}^2_{\mathcal{A}}$ group.

Here is a case in which the bijection becomes canonical. Assume that $\mathcal{A} = \mathcal{A}_0^{\mathbb{Z}_2}$ has a canonical decomposition into even and odd parts. Write $B = B_0 \oplus B_1$. Since $\mathrm{Ext}^2_{\mathcal{A}}(B_j, \Sigma^{-1}(B_j)) = 0$ for parity reasons, the even and odd parts of B have unique liftings, say $\widehat{B}_0$ and $\widehat{B}_1$. Hence $\widehat{B} = \widehat{B}_0 \oplus \widehat{B}_1$ lifts B, and in this way the lifting becomes natural.

Example 4.6. For gauge actions on Cuntz-Krieger algebras, the equivariant K-theory is the same as the K-theory of the fixed point algebra, which is AF. Hence the $\mathrm{Ext}^2_{\mathcal{A}}$ group vanishes for parity reasons again. In particular, gauge actions on Cuntz-Krieger algebras are uniquely determined up to $KK^{\mathbb{T}}$-equivalence by their equivariant K-theory.

Proof. (of the lifting result) Let $(\widehat{B}, \theta\colon F(\widehat{B}) \to B)$ be a lifting of B, and let

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \longrightarrow 0$$

be a projective resolution. Lift $d_0\colon P_0 \to B \cong F(\widehat{B})$ to $\widehat{d_0} \in \tau(\widehat{P}_0, \widehat{B})$, and construct an exact triangle

$$\Sigma\widehat{B} \to \widehat{\Omega B} \to \widehat{P}_0 \to \widehat{B},$$

where the last map is $\widehat{d_0}$. Applying F to this triangle we get

$$0 \to F(\widehat{\Omega B} \to P_0 \to B \to 0,$$

where the last map is d_0. Thus $F(\widehat{\Omega B}) = \ker(d_0) = \Omega B$. Thus every lifting comes from some $\varphi \in \tau(\widehat{\Omega B}, \widehat{P}_0)$ lifting the inclusion map $\Omega B \to P_0$. (Recall that ΩB has a unique lifting because it has a length one projective resolution.) The source of non-uniqueness of the lifting of B is the map φ. If $\widehat{B}_1$ and $\widehat{B}_2$ are two liftings of B, then the corresponding maps φ_1 and φ_2 satisfy $\varphi_1 - \varphi_2 \in \mathrm{Ext}^1_{\mathcal{A}}(\Omega B, P_0)$ by the UCT for $\widehat{\Omega B}$. Now, $\widehat{B}_1$ and $\widehat{B}_2$ are isomorphic lifts, there is an isomorphism $\beta \in \tau(\widehat{B}_1, \widehat{B}_2)$ lifting id_B. Consider the exact triangles

$$\begin{tikzcd}
\widehat{B}_1 \arrow[d, "\beta"] & \widehat{P}_0 \arrow[l] \arrow[d, "="] & \widehat{\Omega B} \arrow[l, "\varphi_1"'] \arrow[d, dashed, "\psi"] & \Sigma\widehat{B}_1 \arrow[l] \arrow[d, "\Sigma\beta"] \\
\widehat{B}_2 & \widehat{P}_0 \arrow[l] & \widehat{\Omega B} \arrow[l, "\varphi_2"'] & \Sigma\widehat{B}_2 \arrow[l]
\end{tikzcd}$$

The diagram commutes because $\tau(\widehat{P}_0, \widehat{B}_2) = \mathrm{Hom}_{\mathcal{A}}(P_0, B_2)$ and β lifts the identity on B. By the axioms of triangulated categories, there exists $\psi \in \tau(\widehat{\Omega B}, \widehat{\Omega B})$ making all squares commute. If we apply F to the above diagram, both rows become

$$0 \to \Omega B \to P_0 \to B \to 0$$

with vertical arrows being the respective identities. It follows that $F(\psi) = \mathrm{id}_{\Omega B}$ and thus $\psi - \mathrm{id}_{\widehat{\Omega B}} \in \mathrm{Ext}^1_{\mathcal{A}}(\Omega B, \Omega B)$. It turns out that $\varphi_1 - \varphi_2$ is the image of $\psi - \mathrm{id}_{\widehat{\Omega B}}$ under the map $k\colon \mathrm{Ext}^1_{\mathcal{A}}(\Omega B, \Omega B) \to \mathrm{Ext}^1_{\mathcal{A}}(\Omega B, P_0)$. Thus φ_1 and φ_2 give isomorphic liftings if and only if $\varphi_1 - \varphi_2 \in \mathrm{Im}(k)$.

There is a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(\Omega B, \Omega B) \xrightarrow{k} \mathrm{Ext}^1_{\mathcal{A}}(\Omega B, P_0) \xrightarrow{\ell} \mathrm{Ext}^1_{\mathcal{A}}(\Omega B, B) \longrightarrow \mathrm{Ext}^2_{\mathcal{A}}(\Omega B, \Omega B) \longrightarrow \cdots$$

and hence φ_1 and φ_2 give isomorphic lifts if and only if $\ell(\varphi_1 - \varphi_2) = 0$. We will make use of the fact that there is an isomorphism $\mathrm{Ext}^1_{\mathcal{A}}(\Omega B, B) \cong \mathrm{Ext}^2_{\mathcal{A}}(B, B)$. Note that $\mathrm{Ext}^2_{\mathcal{A}}(\Omega B, \Omega B) = 0$. The bijection can then be described as follows. Fix a lift φ_0. For any element in $\mathrm{Ext}^1_{\mathcal{A}}(\Omega B, B)$, since ℓ is surjective, one can find a lift in $\mathrm{Ext}^2_{\mathcal{A}}(\Omega B, P_0)$, which must have the form $\varphi_1 - \varphi_2$ for some other maps $\varphi_1 - \varphi_2$, Now take $\varphi_0 + (\varphi_1 - \varphi_2)$. ☐

The main purpose of developing this machinery is to apply it to the classification of circle actions. Many gauge actions have stably finite fixed point algebras, so $KK^{\mathbb{T}}$-equivalence will be far from cocycle equivalence. Also, for many natural examples, the groups $\mathrm{Ext}^2_{R(\mathbb{T})}(K^{\mathbb{T}}_*(A), K^{\mathbb{T}}_{*+1}(A)) = 0$.

4.1. **Computation of $\mathrm{Ext}^2_{R(\mathbb{T})}$.** Start with an $R(\mathbb{T})$-bimodule resolution

$$0 \longrightarrow R(\mathbb{T}) \otimes R(\mathbb{T}) \xrightarrow{j} R(\mathbb{T}) \otimes R(\mathbb{T}) \xrightarrow{mult} R(\mathbb{T}) \longrightarrow 0,$$

where $j(a \otimes b) = ax \otimes b - a \otimes xb$. Now, if M is an $R(\mathbb{T})$-module, then we get a short exact sequence

$$0 \to R \otimes_{\mathbb{Z}} M \to R \otimes_{\mathbb{Z}} M \to M \to 0.$$

For two R-modules M and N, we get the following long exact sequence by applying $\mathrm{Hom}_{R(\mathbb{T})}(-, N)$

$$0 \longrightarrow \mathrm{Hom}_{R(\mathbb{T})}(M,N) \longrightarrow \mathrm{Hom}_{R(\mathbb{T})}(R(\mathbb{T})\otimes_{\mathbb{Z}} M,N) \xrightarrow{j^*} \mathrm{Hom}_{R(\mathbb{T})}(R(\mathbb{T})\otimes_{\mathbb{Z}} M,N) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_{R(\mathbb{T})}(M,N) \longrightarrow \mathrm{Ext}^1_{R(\mathbb{T})}(R(\mathbb{T})\otimes_{\mathbb{Z}} M,N) \xrightarrow{j^*} \mathrm{Ext}^1_{R(\mathbb{T})}(R(\mathbb{T})\otimes_{\mathbb{Z}} M,N) \longrightarrow \mathrm{Ext}^2_{R(\mathbb{T})}(M,N).$$

Using that $\mathrm{Ext}^n_{R(\mathbb{T})}(R(\mathbb{T})\otimes_{\mathbb{Z}} M,N) = \mathrm{Ext}^n_{\mathbb{Z}}(M,N)$, the above sequence is identified with

$$0 \longrightarrow \mathrm{Hom}_{R(\mathbb{T})}(M,N) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M,N) \xrightarrow{j^*} \mathrm{Hom}_{\mathbb{Z}}(M,N) \longrightarrow \mathrm{Ext}^1_{R(\mathbb{T})}(M,N) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(M,N) \xrightarrow{j^*} \mathrm{Ext}^1_{\mathbb{Z}}(M,N) \longrightarrow \mathrm{Ext}^2_{R(\mathbb{T})}(M,N).$$

The map j^* is given by the difference of the actions of x on N and M.