VON NEUMANN ALGEBRAS AND LATTICES IN HIGHER-RANK GROUPS

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Abstract. These are my personal lecture notes from a course given by **Jesse Peterson** at the conference *Young Mathematicians in C*-algebras* at the University of Münster, Germany, between August 2nd and August 6th, 2021.

Warning: little proofreading has been done.

Contents

1. Semifinite von Neumann algebras

Definition 1.1. A von Neumann algebra is a unital C*-subalgebra $M \subseteq \mathcal{B}(\mathcal{H})$ satisfying the following equivalent conditions:

- M is closed in the WOT topology ($T_j \to T$ in the WOT topology if $\langle T_j\xi, \eta\rangle \to \langle T\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$).
- M is closed in the SOT topology ($T_j \to T$ in the SOT topology if $\|(T_j - T)\xi\| \to 0$ for all $\xi \in \mathcal{H}$).
- $M = M''$.

The equivalence of the above listed properties is a result of von Neumann. The equivalence of the first two is easy, since the WOT and the SOT topology on the convex set M have the same functionals, and thus by Hahn-Banach the respective closures coincide. To see the equivalence of the first two conditions with the third one, one shows first that $\overline{M}^{\mathrm{SOT}} \subseteq M''$; this is an easy exercise. The converse inclusion $M'' \subseteq \overline{M}^{\mathrm{SOT}}$ is proved using a clever argument of von Neumann: let $T \in M''$ and $\xi \in \mathcal{H}$. One shows that the projection P onto $\overline{M\xi}$ belongs to M'. Since $[T, P] = 0$, it follows that $T(\xi) = \lim_i S_i(\xi)$ for some S_i in M, since $T(\xi)$ belongs to the range of P. This shows that $\|(T - S_i)\xi\| \to 0$, but this is only for the specific $\xi \in \mathcal{H}$ that was chosen at the beginning. To prove it for a finite subset of $\mathcal{H}$, one uses a standard matrix trick: given $\xi_1, \ldots, \xi_n \in \mathcal{H}$, set $\mathcal{H}^{(n)} = \bigoplus_{j=1}^n \mathcal{H}$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^{(n)}$. One then repeats the argument for $T^{(n)} = \mathrm{diag}(T, \ldots, T)$ and ξ.

Examples 1.2. Some standard von Neumann algebras:

(1) It is immediate that $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.
(2) For a measure space (X, μ), consider $L^\infty(X, \mu)$. If μ is σ-finite, then there is an inclusion $L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$ via multiplication operators. One can check that $L^\infty(X, \mu)' = L^\infty(X, \mu)$, and thus $L^\infty(X, \mu) = L^\infty(X, \mu)''$ is a von Neumann algebra.
(3) If Γ is a discrete group, there is a left regular representation $\lambda\colon \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ given by $\lambda_s(\delta_t) = \delta_{st}$ for $s, t \in \Gamma$. The *group von Neumann algebra* $L(\Gamma)$ of Γ is defined to be the double commutant of $\lambda(\Gamma)$ in $\mathcal{B}(\ell^2(\Gamma))$.

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(4) Let Γ be a discrete group, let (X,μ) be a σ-finite measure space, and let $\Gamma \curvearrowright (X,\mu)$ be a measure preserving action[1]. Set $A = L^\infty(X,\mu)$ and let α denote the action of Γ on A induced by $\Gamma \curvearrowright (X,\mu)$. Set

$$A\Gamma = \Big\{ \sum_{t\in\Gamma} a_t u_t \colon a_t \in A, a_t = 0 \text{ for all but finitely many } t \in \Gamma \Big\},$$

and define operations via

$$(au_t)(bu_s) = a\alpha_t(b)u_{ts} \quad \text{and} \quad (au_t)^* = \alpha_{t^{-1}}(a^*)u_{t^{-1}}$$

for $a,b \in A$ and $s,t \in \Gamma$. We represent the $*$-algebra $A\Gamma$ on $L^2(X,\mu) \otimes \ell^2(\Gamma) \cong \ell^2(\Gamma, L^2(X,\mu))$ by setting

$$(au_t)(\delta_s\xi) = \delta_{ts} a\alpha_t(\xi)$$

for all $a \in A = L^\infty(X,\mu)$, all $s,t \in \Gamma$ and all $\xi \in L^2(X,\mu)$. The double commutant of the image of this representation is the *crossed product* $L^\infty(X,\mu) \rtimes \Gamma$ of $\Gamma \curvearrowright (X,\mu)$.

Remark 1.3. Let Γ be a discrete group. Then $\mathcal{Z}(L(\Gamma)) = \mathbb{C}$ if and only if Γ is an ICC group[2]. If $\Gamma \curvearrowright (X,\mu)$ is free[3], then $\mathcal{Z}(L^\infty(X,\mu) \rtimes \Gamma) = \mathbb{C}$ if and only if $\Gamma \curvearrowright (X,\mu)$ is ergodic[4].

1.1. **Traces on von Neumann algebras.**

Definition 1.4. For a von Neumann algebra M, a trace on it is a function $\mathrm{Tr}\colon M_+ \to [0,\infty]$ which is $\mathbb{R}_+$-linear and satisfies $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$ for all $x \in M$. We say that Tr is:

(1) *normal*, if whenever $(x_j)_j$ is an increasing net in M_+ which converges in the SOT-topology to $x \in M_+$, then $\mathrm{Tr}(x_j) \to \mathrm{Tr}(x)$.
(2) *faithful*, if $\mathrm{Tr}(x) = 0$ implies $x = 0$ for $x \in M_+$.
(3) *semifinite*, if for all $x \in M_+ \setminus \{0\}$ there is $0 \leq y \leq x$ such that $0 < \mathrm{Tr}(y) < \infty$.

We set

$$n_{\mathrm{Tr}} = \{x \in M \colon \mathrm{Tr}(x^*x) < \infty\} \quad \text{and} \quad m_{\mathrm{Tr}} = \Big\{ \sum_{j=1}^n x_j^* y_j \colon x_j, y_j \in n_{\mathrm{Tr}} \Big\}.$$

The following is standard but not at all trivial; for example, it is already not immediate to show that n_{Tr} and m_{Tr} are vector spaces.

Lemma 1.5. Both n_{Tr} and m_{Tr} are (not necessarily closed) ideals in M. In particular, if $\mathrm{Tr}(1) < \infty$, then $n_{\mathrm{Tr}} = m_{\mathrm{Tr}} = M$.

Examples 1.6. (1) If $M = L^\infty(X,\mu)$ for a σ-finite, fully supported measure μ, then $\mathrm{Tr}(f) = \int f\, d\mu$ is a normal, faithful, seminifinite trace. In this case,

$$n_{\mathrm{Tr}} = L^\infty(X,\mu) \cap L^2(X,\mu) \quad \text{and} \quad m_{\mathrm{Tr}} = L^\infty(X,\mu) \cap L^1(X,\mu).$$

(2) If $M = \mathcal{B}(\mathcal{H})$ and $(\xi_j)_j$ is an orthonormal basis for $\mathcal{H}$, then $\mathrm{Tr}(T) = \sum_j \langle T(\xi_j), \xi_j\rangle$ is a normal, faithful, semifinite trace. In this case,

$$n_{\mathrm{Tr}} = \{\text{Hilbert-Schmidt operators}\} \quad \text{and} \quad m_{\mathrm{Tr}} = \{\text{Trace class operators}\}.$$

Lemma 1.7. Let Tr be a normal, faithful, semifinite trace on a von Neumann algebra M. Then Tr extends uniquely to a finite trace $\mathrm{Tr}\colon m_{\mathrm{Tr}} \to \mathbb{C}$.

The above lemma gives an inner product on n_{Tr} given by $\langle x,y\rangle_{\mathrm{Tr}} = \mathrm{Tr}(y^*x)$ for $x,y \in n_{\mathrm{Tr}}$. We let $L^2(M,\mathrm{Tr})$ be the Hilbert space obtained by completing n_{Tr} with respect to this inner product. For $x \in n_{\mathrm{Tr}}$, we write $[x]_{\mathrm{Tr}} \in L^2(M,\mathrm{Tr})$ for the corresponding class. There is a canonical injective representation $\varphi_\lambda\colon M \to \mathcal{B}(L^2(M,\mathrm{Tr}))$ given by left multiplication: $\varphi_\lambda(a)([x]_{\mathrm{Tr}}) = [ax]_{\mathrm{Tr}}$ for $a \in M$ and $x \in n_{\mathrm{Tr}}$. There is also a canonical injective representation $\varphi_\rho\colon M^{\mathrm{opp}} \to \mathcal{B}(L^2(M,\mathrm{Tr}))$ by right multiplication: $\varphi_\rho(a)([x]_{\mathrm{Tr}}) = [xa]_{\mathrm{Tr}}$ for $a \in M$ and $x \in n_{\mathrm{Tr}}$. These representations commute with each other. Moreover:

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[1] This construction can also be carried out if Γ only preserves the measure *class* of μ, but we will restrict to the measure-preserving case in these lectures.

[2] A discrete group Γ has *infinite conjugacy classes*, or is *ICC*, if for all $s \in \Gamma \setminus \{1\}$, the set $\{tst^{-1} \colon t \in \Gamma\}$ is infinite.

[3] An action $\Gamma \curvearrowright (X,\mu)$ is *free*, if stabilizer groups $\Gamma_x = \{s \in \Gamma \colon s \cdot x = x\}$ are trivial for μ-almost every $x \in X$.

[4] An action $\Gamma \curvearrowright (X,\mu)$ is *ergodic* if whenever $E \subseteq X$ is measurable and Γ-invariant, then either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Proposition 1.8. Let Tr be a normal, faithful, semifinite trace on a von Neumann algebra M. Then $\varphi_\lambda(M)' = \varphi_\rho(M^{\mathrm{opp}})$ in $\mathcal{B}(L^2(M, \mathrm{Tr}))$.

Examples 1.9. Let Γ be a discrete group. Then the map $\tau\colon L(\Gamma) \to \mathbb{C}$ given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ for $x \in L(\Gamma)$ is a normal, faithful, finite trace. More generally, if $\Gamma \curvearrowright (X, \mu)$ is a measure-preserving action, then the map $\tau\colon L^\infty(X, \mu) \rtimes \Gamma \to \mathbb{C}$ given by

$$\tau(x) = \langle x(1 \otimes \delta_e), 1 \otimes \delta_e\rangle,$$

for $x \in L^\infty(X, \mu) \rtimes \Gamma$, is also a normal, faithful, finite trace.

If a factor admits a normal, (semi)finite trace (which is then automatically faithful), then this normal (semi)finite trace is unique up to multiplicative scalars.

Problem 1.10. How much does $L(\Gamma)$ remember about Γ? How much does $L^\infty(X, \mu) \rtimes \Gamma$ remember about $\Gamma \curvearrowright (X, \mu)$?

There has been a lot of work done on both of these questions. We will use the concepts developed in this talk to show that $L(\Gamma)$ remembers whether Γ is an (ICC) amenable group.

We write $\mathtt{Lt}$ for the action of left translation of Γ on $\ell^\infty(\Gamma)$.

Definition 1.11. A discrete group Γ is said to be *amenable* if there is an $\mathtt{Lt}$-invariant state on $\ell^\infty(\Gamma)$.

We will need the following observation.

Remark 1.12. Let (M, τ_M) and (N, τ_N) be tracial von Neumann algebras, and let $\theta\colon (M, \tau_M) \to (N, \tau_N)$ be a trace-preserving isomorphism. Then θ extends to a unitary $u_\theta\colon L^2(M, \tau_M) \to L^2(N, \tau_N)$, and viewing M and N in their standard representations it follows that θ is implemented by u_θ, in the sense that $\theta(x) = u_\theta x u_\theta^*$ for all $x \in M$.

Theorem 1.13. Let Γ and Λ be discrete groups such that $L(\Gamma)$ is isomorphic to $L(\Lambda)$. Then Γ is ICC and amenable if and only if Λ is ICC and amenable.

Proof. Assume that Γ is ICC and amenable. Then Λ is ICC because in this case $L(\Lambda)$ is a factor. We show that it is also amenable. Denote by $E\colon \mathcal{B}(\ell^2(\Gamma)) \to \ell^\infty(\Gamma)$ the conditional expectation given by $E(T)(s) = \langle T(\delta_s), \delta_s\rangle$ for all $T \in \mathcal{B}(\ell^2(\Gamma))$ and all $s \in \Gamma$. Let $\phi\colon \ell^\infty(\Gamma) \to \mathbb{C}$ be a left invariant state, and define $\widetilde{\phi}\colon \mathcal{B}(\ell^2(\Gamma)) \to \mathbb{C}$ by $\widetilde{\phi} = \phi \circ E$. Note that $\widetilde{\phi}$ is positive and unital. In particular, it is a state on $\mathcal{B}(\ell^2(\Gamma))$. On the other hand, for $x \in L(\Gamma)$ and $s \in \Gamma$ we have

$$E(x)(s) = \langle x\delta_s, \delta_s\rangle = \langle x\lambda_s\delta_e, \lambda_s\delta_e\rangle = \tau(\lambda_s^* x\lambda_s) = \tau(x).$$

In particular, $\widetilde{\phi}|_{L(\Gamma)} = \tau$. Moreover, for $T \in \mathcal{B}(\ell^2(\Gamma))$ and $s, t \in \Gamma$, we get

$$E(\lambda_t T\lambda_{t^{-1}}) = \langle \lambda_t T\lambda_{t^{-1}}\delta_s, \delta_s\rangle = E(T)(t^{-1}s) = \mathtt{Lt}_t(E(T))(s).$$

Since ϕ is $\mathtt{Lt}$-invariant, it follows that $\widetilde{\phi}(\lambda_t T\lambda_{t^{-1}}) = \widetilde{\phi}(T)$. Equivalently, $\widetilde{\phi}(\lambda_t T) = \widetilde{\phi}(T\lambda_t)$. By linearity, we get $\widetilde{\phi}(xT) = \widetilde{\phi}(Tx)$ for all $x \in \mathbb{C}[\Gamma]$. Now, given $x \in \mathbb{C}[\Gamma]$ and $T \in \mathcal{B}(\ell^2(\Gamma))$, we use Cauchy-Schwarz at the first step, and the fact that $\widetilde{\phi}$ restricts to the canonical trace on $\mathbb{C}([\Gamma]$ at the second, to get

$$|\widetilde{\phi}(xT)| \leq \widetilde{\phi}(xx^*)^{1/2}\widetilde{\phi}(T^*T)^{1/2} = \|x\|_2\widetilde{\phi}(T^*T)^{1/2}.$$

It follows that the identity $\widetilde{\phi}(xT) = \widetilde{\phi}(Tx)$ holds for all x in the $\|\cdot\|_2$-closure of $\mathbb{C}[\Gamma]$, which is $L(\Gamma)$.

It is an easy exercise to check that $L^2(L(\Gamma), \tau_\Gamma)$ can be canonically identified with $\ell^2(\Gamma)$. Since any isomorphism between the factors $L(\Gamma)$ and $L(\Lambda)$ must identify τ_Γ and τ_Λ (since these are the unique traces on them), an application of Remark 1.12 gives a unitary $u\colon \ell^2(\Gamma) \to \ell^2(\Lambda)$ such that $\mathrm{Ad}(u)$ restricts to an isomorphism $L(\Gamma) \cong L(\Lambda)$.

Define $\psi\colon \ell^\infty(\Lambda) \to \mathbb{C}$ by $\psi(f) = \widetilde{\phi}(M_f)$. For $t \in \Lambda$, a computation similar to the one we performed before shows that

$$\psi(\mathtt{Lt}_t(f)) = \widetilde{\phi}(\lambda_t M_f \lambda_t^*) = \psi(f).$$

It follows that ψ is a left-invariant state on $\ell^\infty(\Lambda)$, so Λ is amenable. □

Remark 1.14. The proof above also works in the non-ICC case, if one instead of Remark 1.12 uses the non-trivial fact that given two faithful normal traces on a von Neumann algebra, the corresponding standard representations are unitarily equivalent. (This uses the Radon-Nikodym theorem in the abelian case, and the general case follows similarly but uses unbounded operators.)

2. Some approximation properties for groups: inner amenability and biexactness

If Γ is amenable, we have seen in the proof of Theorem 1.13 that there is a state $\widetilde{\phi}$ on $\mathcal{B}(\ell^2(\Gamma))$ extending τ_Γ such that $\widetilde{\phi}(xT) = \widetilde{\phi}(Tx)$ for all $x \in L(\Gamma)$ and all $T \in \mathcal{B}(\ell^2(\Gamma))$. We call such a state a *hypertrace* on $\mathcal{B}(\ell^2(\Gamma))$.

Definition 2.1. A tracial von Neumann algebra (M, τ) is said to be *injective* if there exists a conditional expectation $\mathcal{B}(L^2(M, \tau)) \to M$.

Lemma 2.2. Let (M, τ) be a tracial von Neumann algebra. Then there is a hypertrace on $\mathcal{B}(L^2(M, \tau))$ if and only if M is injective.

Proof. The "if" implication follows by taking $\phi = \tau \circ E$; this is then a hypertrace on $\mathcal{B}(L^2(M, \tau))$.

Conversely, suppose that there is a hypertrace ϕ on $\mathcal{B}(L^2(M, \tau))$...

$\phi(T) = \langle \pi(T)\xi, \xi\rangle$. Then the assignment $M \to \mathcal{H}$, given by mapping $x \in M$ to $\pi(x)\xi$, extends to an isometry $L^2(M, \tau) \to \mathcal{H}$.

Define $\varphi\colon \mathcal{B}(L^2(M, \tau)) \to \mathcal{B}(L^2(M, \tau))$ by $\varphi(T) = e\pi(T)e$. Note that $\varphi|_M = \mathrm{id}_M$. Let $x, y, z \in M$ and $T \in M$. Then

$$\langle \varphi(T)z^{\mathrm{opp}}[x], [y]\rangle = \langle e\pi(y^*Txz)e\xi, \xi\rangle = \phi(y^*Txz) = \phi(zy^*Tx) = \langle z^{\mathrm{opp}}\varphi(T)[x], [y]\rangle.$$

So $\varphi(T)$ belongs to the commutant of M^{opp}, which is M by Proposition 1.8. □

It is a famous (award-winning) result of Connes from 1976 that there is a unique separable, injective II_1-factor. It is usually denoted by $\mathcal{R}$, and it can be identified as the weak closure of $\bigotimes_{n=1}^\infty M_2$ in its GNS representation.

Property Γ was introduced in order to show that the group algebra of certain ICC groups is not injective.

Definition 2.3. A tracial von Neumann algebra (M, τ) is said to have *property* Γ if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in M such that $\tau(u_n) \to 0$ and $\|u_n x - x u_n\|_2 \to 0$ for all $x \in M$. (One can equivalently ask that $(u_n)_{n\in\mathbb{N}}$ is asymptotically central in the SOT, since on bounded sets the SOT and the $\|\cdot\|_2$-topology agree.)

Example 2.4. The hyperfinite II_1-factor $\mathcal{R}$ has property Γ, as can be seen by taking $u_n = 1\otimes\cdots\otimes 1\otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. More generally, $M\overline{\otimes}\mathcal{R}$ has property Γ for every tracial von Neumann algebra M.

Definition 2.5. A discrete group Γ is said to be *inner amenable* if there is a conjugation-invariant state ϕ on $\ell^\infty(\Gamma)$ which vanishes on $c_0(\Gamma)$.

Theorem 2.6. (Effros) If $L(\Gamma)$ has property Γ, then Γ is inner amenable.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of unitaries in $L(\Gamma)$ as in the definition of property Γ. Define a state $\phi\colon \mathcal{B}(\ell^2(\Gamma)) \to \mathbb{C}$ by taking any weak-$*$ limit point of $T \mapsto \langle T[u_n], [u_n]\rangle$. Since $\ell^\infty(\Gamma) \hookrightarrow \mathcal{B}(\ell^2(\Gamma))$, we can restrict ϕ to a state on $\ell^\infty(\Gamma)$. Since $u_n \to 0$ in the WOT, we have $\phi|_{\mathcal{K}(\ell^2(\Gamma))} = 0$, so this restriction vanishes in particular on $c_0(\Gamma)$. Moreover,

$$\phi(\lambda_t \rho_t T \rho_{t^{-1}} \lambda_{t^{-1}}) = \phi(T)$$

for $t \in \Gamma$ and $T \in \mathcal{B}(\ell^2(\Gamma))$. If T is the multiplication operator by $f \in \ell^\infty(\Gamma)$, we get $\lambda_t \rho_t T \rho_{t^{-1}} \lambda_{t^{-1}} = f \circ \mathrm{Ad}(t)$. It follows that $\phi(f) = \phi(f \circ \mathrm{Ad}(t))$, so $\phi|_{\ell^\infty(\Gamma)}$ is conjugation-invariant, as desired. □

Example 2.7. The free group $\mathbb{F}_2 = \langle a, b\rangle$ is not inner amenable. To see this, for $c \in \mathbb{F}_2$, denote by $W(c)$ all reduced words in $\mathbb{F}_2$ which start with c, and let $\chi_c \in \ell^\infty(\mathbb{F}_2)$ denote the characteristic function of $W(c)$. Then

$$a^{-1}W(a)a = W(b) \sqcup W(b^{-1}) \sqcup W(a^{-1}) \quad \text{and} \quad b^{-1}W(b)b = W(a) \sqcup W(a^{-1}) \sqcup W(b^{-1}).$$

If $\phi \in \ell^\infty(\mathbb{F}_2) \to \mathbb{C}$ is a conjugation invariant state, the above imply that $\phi(\chi_a) = \phi(\chi_{a^{-1}}) + \phi(\chi_b) + \phi(\chi_{b^{-1}})$ and $\phi(\chi_b) = \phi(\chi_{b^{-1}}) + \phi(\chi_a) + \phi(\chi_{a^{-1}})$. Using these, we get

$$\begin{aligned} 1 = \phi(1) &= \phi(\delta_e) + \phi(\chi_a) + \phi(\chi_{a^{-1}}) + \phi(\chi_b) + \phi(\chi_{b^{-1}}) \\ &= \phi(\delta_e) + 2(\phi(\chi_{a^{-1}}) + \phi(\chi_b) + \phi(\chi_{b^{-1}})) \\ &= \phi(\delta_e) + 2(\phi(\chi_{b^{-1}}) + \phi(\chi_a) + \phi(\chi_{a^{-1}})). \end{aligned}$$

It follows that $\phi(\chi_a) = \phi(\chi_b)$. Using the first displayed equality above, we deduce that $\phi(\chi_{a^{-1}}) = \phi(\chi_{b^{-1}}) = 0$. Since one likewise shows that

$$aW(a^{-1})a^{-1} = W(b) \sqcup W(b^{-1}) \sqcup W(a),$$

it follows that $\phi(\chi_a) = \phi(\chi_b) = 0$, and thus $\phi(\delta_e) = 1$. In particular, this implies that ϕ does not vanish on $c_0(\mathbb{F}_2)$, and hence $\mathbb{F}_2$ is not inner amenable.

The following definition is due to Claire Anantharaman-Delaroche:

Definition 2.8. A topological action $\Gamma \curvearrowright X$ of a discrete group Γ on a compact Hausdorff space X is said to be *amenable* if there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of continuous functions $\mu_n \colon X \to \mathrm{Prob}(\Gamma) \subseteq \ell^1(\Gamma)$ such that

$$\sup_{x\in X} \|\mu_n(t\cdot x) - t_* \cdot \mu_n(x)\|_1$$

for all $t \in \Gamma$.

Example 2.9. If Γ is amenable, then every action of it is amenable. (For $\Gamma \curvearrowright \{*\}$, the converse is also true.) More generally, if $\Gamma \curvearrowright X$ has a Γ-invariant measure, then $\Gamma \curvearrowright X$ is amenable.

Example 2.10. $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$ is amenable. This can be seen, for example, by letting $\mu_n \colon \partial\mathbb{F}_2 \to \mathrm{Prob}(\mathbb{F}_2)$ be given by $\mu_n(w) = \frac{1}{n}\sum_{k=1}^n \delta_{w_k}$ for all $w \in \partial\mathbb{F}_2$, where w_k denotes the k-th element in the infinite word w. In this case, one shows that

$$\|\mu_n(sw) - s_*\mu_n(w)\| \leq \frac{1}{n} d(s,e)$$

which converges uniformly to zero as $n \to \infty$.

In general, one can show that $\Gamma \curvearrowright X$ is amenable if and only if $C(X) \rtimes_{\mathrm{r}} \Gamma$ is nuclear, and if and only if $(C(X) \rtimes_{\mathrm{r}} \Gamma)^{**}$ is injective.

Definition 2.11. A discrete group Γ is said to be *exact* if there exist a compact Hausdorff space X and an amenable action $\Gamma \curvearrowright X$. Equivalently, the left translation action of Γ on its Stone-Chec compactification (the C*-spectrum of $\ell^\infty(\Gamma)$) is amenable.

Definition 2.12. Let Γ be a discrete group. Set

$$A(\Gamma) = \{f \in \ell^\infty(\Gamma) \colon f - \mathtt{Rt}_t(f) \in c_0(\Gamma) \ \text{ for all } t \in \Gamma\}.$$

Then $A(\Gamma)$ is a unital, commutative, $\mathtt{Lt}$-invariant C*-subalgebra of $\ell^\infty(\Gamma)$ containing $c_0(\Gamma)$ as an essential ideal. Denote by $\overline{\Gamma}$ its spectrum, which is a compact Hausdorff space containing Γ as an open dense, Γ-invariant subset. Set $\Delta\Gamma = \overline{\Gamma} \setminus \Gamma$, and note that $C(\Delta\Gamma) = A(\Gamma)/c_0(\Gamma)$.

We say that Γ is *biexact* if $\Gamma \curvearrowright \Delta\Gamma$ is amenable.

The reason for the terminology is the following. A group Γ is exact if and only if $\mathtt{Lt} \colon \Gamma \curvearrowright \ell^\infty(\Gamma)$ is amenable (can use $\mathtt{Rt}$ instead), while Γ is biexact if $\mathtt{Lt} \times \mathtt{Rt} \colon \Gamma \times \Gamma \curvearrowright \ell^\infty(\Gamma)/c_0(\Gamma)$ is amenable.

Example 2.13. $\mathbb{F}_2$ is biexact. To see this, we first observe that $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$ is a factor of $\mathbb{F}_2 \curvearrowright \Delta\mathbb{F}_2$. To prove this, let $w \in \partial\mathbb{F}_2$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{F}_2$ converging to w. Then $st_nr \to sw$ for all $s, r \in \mathbb{F}_2$. Thus, if $f \in C(\partial\mathbb{F}_2 \cup \mathbb{F}_2) \subseteq \ell^\infty(\mathbb{F}_2)$, then $f - \mathtt{Rt}_r(f) \in c_0(\Gamma)$, and hence $C(\partial\mathbb{F}_2 \cup \mathbb{F}_2) \subseteq A(\mathbb{F}_2)$. It follows that $C(\partial\mathbb{F}_2) \hookrightarrow C(\Delta\mathbb{F}_2)$ equivariantly, and thus there exists an equivariant surjective map $\pi \colon \Delta\mathbb{F}_2 \to \partial\mathbb{F}_2$. Once we have this, we compose π with the probability-valued functions $\mu_n \colon \partial\mathbb{F}_2 \to \mathrm{Prob}(\mathbb{F}_2)$ from Example 2.10 to get functions witnessing the fact that $\mathbb{F}_2 \curvearrowright \Delta\mathbb{F}_2$ is amenable. Thus $\mathbb{F}_2$ is biexact.

Remark 2.14. Amenability implies biexactness, but inner amenability does not.

Theorem 2.15. (Ozawa). If Γ is biexact, then Γ is amenable if and only if $L(\Gamma)$ has property Γ.

A major obstruction to biexactness is the following result of Ozawa.

Theorem 2.16. (Ozawa). Let Γ be a biexact group and let $B \subseteq L(\Gamma)$ be a von Neumann subalgebra without minimal projections (in other words, B is diffuse). Then $B' \cap L(\Gamma)$ is injective.

Proof. (Boutonet-Carderi). Since B is diffuse, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in B such that $u_n \to 0$ in the WOT. Note that

$$A(\Gamma) \rtimes \Gamma \subseteq \ell^\infty(\Gamma) \rtimes \Gamma \cong \mathcal{B}(\ell^2(\Gamma)).$$

Let ϕ be a state on $\mathcal{B}(\ell^2(\Gamma))$ which is a limit point of the sequence of states given by $T \mapsto \langle T[u_n], [u_n]\rangle$. Then $\phi|_{L(\Gamma)} = \tau_\Gamma$ and $\phi|_{L(\Gamma)^{\mathrm{opp}}} = \tau_\Gamma$. Also, $\phi|_{\mathcal{K}(\ell^2(\Gamma))} = 0$ and in particular ϕ vanishes on $c_0(\Gamma)$. For $f \in C(\overline{\Gamma})$ and $t \in \Gamma$, since $f - \mathtt{Rt}_t(f)$ belongs to $c_0(\Gamma)$, we have $\phi(f - \rho_t f \rho_{t^{-1}}) = 0 = \phi(f\rho_t - \rho_t f)$. In fact, we can multiply f on the right by λ_s, for $s \in \Gamma$, since conjugation by λ_s leavs $c_0(\Gamma)$ invariant. Thus we get $\phi(x\rho_t - \rho_t x) = 0$ for all x in the C*-algebra generated by $C(\overline{\Gamma})$ and the unitaries λ_s, so for all $x \in C(\overline{\Gamma}) \rtimes \Gamma$. That is,

$$\phi(xy - yx) = 0 \tag{1}$$

whenever $x \in C(\overline{\Gamma}) \rtimes \Gamma$ and $y \in C^*_\rho(\Gamma)$. Note that

$$|\phi(xy) - \phi(xy_0)| \leq \|x\|\|y - y_0\|_2$$

for all $x \in C(\overline{\Gamma}) \rtimes \Gamma$ and $y \in C^*_\rho(\Gamma)$, since ϕ is a state which restricts to the canonical trace on $C^*_\rho(\Gamma)$. This implies that (1) holds also in the SOT-closure of $C^*_\rho(\Gamma)$, which is $R(\Gamma) = C^*_\rho(\Gamma)''$.

Let $a \in B' \cap L(\Gamma)$, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $C^*_\lambda(\Gamma)$ such that $\|a_n - a\|_2 \to 0$. For $x \in C(\overline{\Gamma}) \rtimes \Gamma$, we use at the first step that $\phi|_{L(\Gamma)} = \tau$ to get

$$\phi(ax) = \lim_n \phi(a_n x) = \lim_n \lim_k \langle a_n x[u_k], [u_k]\rangle = \lim_n \lim_k \langle x[u_k], [u_k a_n^*]\rangle = \lim_n \phi(a_n^* x) \overset{(1)}{=} \lim_n \phi(x a_n^*) = \phi(xa),$$

where at the third step we used the fact that a and x commute in $\mathcal{B}(\ell^2(\Gamma))$.

Denote by $\pi\colon C(\overline{\Gamma}) \rtimes \Gamma \to \mathcal{B}(\mathcal{H})$ and $\xi_0 \in \mathcal{H}$ the GNS construction associated to ϕ, so that $\phi(x) = \langle \pi(x)\xi_0, \xi_0\rangle$ for all $x \in C(\overline{\Gamma}) \rtimes \Gamma$. Set $M = \overline{C(\overline{\Gamma}) \rtimes \Gamma}^{**}$, and extend π to a representation $\pi\colon M \to \mathcal{B}(\mathcal{H})$. Since ϕ restricts to the trace on $C^*_\lambda(\Gamma)$, there is a projection $p \in \pi(C^*_\lambda(\Gamma))''$ which "supports" λ, in the sense that $L(\Gamma) \subseteq pMp$. Thus ϕ gives a state φ on pMp satisfying $\varphi(ax) = \varphi(xa)$ for all $a \in B' \cap L(\Gamma)$ and all $x \in pMp$. This implies that there exists a conditional expectation $E\colon pMp \to B' \cap L(\Gamma)$. Since Γ is biexact, $\Gamma \curvearrowright \overline{\Gamma}$ is amenable and thus M and pMp are injective. Composing E with a conditional expectation $\mathcal{B}(\mathcal{H}) \to pMp$, we get conditional expectations $\mathcal{B}(\mathcal{H}) \to B' \cap L(\Gamma)$, which shows that $B' \cap L(\Gamma)$ is injective. ☐

In the setting above, one can show that $L(\Gamma)$ does not have property Γ as soon as it is not injective: one can take $a \in L(\Gamma)$ in the above argument (and not just in $B' \cap L(\Gamma)$), since a asymptotically commutes with the u_n's. The contradiction is then easier, because we only need that there is no central state on $L(\Gamma)$.

More explicitly:

Proposition 2.17. If Γ is biexact and $L(\Gamma)$ has property Γ, then Γ is amenable.

Proof. The previous proof shows that if $L(\Gamma)$ has property Γ, then there is a state ϕ on $\mathcal{B}(\ell^2(\Gamma))$ such that $\phi(ax) = \phi(xa)$ for all $x \in C(\overline{\Gamma}) \rtimes \Gamma$ and all $a \in L(\Gamma)$. In particular, $\phi(\lambda_t f \lambda_{t^{-1}} - f) = 0$ for all $f \in C(\overline{\Gamma})$ and all $t \in \Gamma$. Restricting ϕ to $C(\overline{\Gamma})$, we get a Γ-invariant probability measure on $\overline{\Gamma}$. If Γ is moreover biexact, so that $\Gamma \curvearrowright \overline{\Gamma}$ is amenable, we deduce that Γ is amenable. ☐

Corollary 2.18. $L(\mathbb{F}_2)$ is solid (and hence prime: it cannot be written as the tensor product of two finite diffuse von Neumann algebras).

3. Proper proximality

Definition 3.1. (Boutonet-Ioana-Peterson). A discrete group Γ is said to be *properly proximal* if there does not exist a Γ-invariant probability measure on $\overline{\Gamma}$ (equivalently, on $\Delta\Gamma$).

What we did in the previous section, particularly in the proofs of Theorem 2.16 and Proposition 2.17, shows the following:

Proposition 3.2. If Γ is properly proximal, then $L(\Gamma)$ does not have property Γ.

We also immediately deduce the following:

Proposition 3.3. If Γ is non-amenable and biexact, then it is properly proximal.

Proof. If there was an invariant probability measure on $\overline{\Gamma}$, and since $\Gamma \curvearrowright \overline{\Gamma}$ is exact by biexactness, then it would follow that Γ is amenable. $\square$

The boundary $\Delta\Gamma$ is too complicated to gets your hands on. So the way to check proper proximality is to construct some action of Γ which has the corresponding proximality-type property:

Lemma 3.4. Γ is properly proximal if and only if there exists an action $\Gamma \curvearrowright X$ on a compact Hausdorff space X without invariant measures and there exists some $\mu \in \mathrm{Prob}(X)$ such that $\lim_{t\to\infty} t_* \cdot \mu - t_* s_* \cdot \mu = 0$ for all $s \in \Gamma$.

Proof. One direction is obvious: if Γ is properly proximal, we can just take $X = \Delta\Gamma$. Conversely, the property of μ in the assumption is precisely the one defininig $\Delta\Gamma$, and the existence of μ implies the existence of an equivariant surjection $\pi \colon \overline{\Gamma} \to \mathrm{Prob}(X)$ satisfying $\pi(t) = t_* \cdot \mu$ for $t \in \Gamma$. Given an invariant measure on $\overline{\Gamma}$, one can push it down to an invariant measure on $\mathrm{Prob}(X)$, and then the Chebyshev center will give an invariant measure on X. $\square$

We will illustrate how to use the above lemma for a class of groups, called convergence groups. (This is a notion coming from geometric group theory.) See Proposition 3.8.

Definition 3.5. We say that Γ is a *convergence group* if if there exists an action $\Gamma \curvearrowright X$ on a compact Hausdorff space without invariant measures and with a "north to south dynamics", that is, if $(t_n)_{n\in\mathbb{N}}$ is a sequence in Γ such that $t_n \to \infty$, then there exist a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ and $x, y \in X$ such that whenever V and W are neighborhoods of x and y respectively, then $t_{n_k}(X \setminus V) \subseteq X \setminus W$ for large k.

In simpler words, the action collapses everything outside of a small neighborhood of x to a small neighborhood of y. In particular, if x is not an atom of a measure μ, then $t_{n_k} \cdot \mu$ converges to $\delta_{\{y\}}$ weak-$*$.

Example 3.6. $SL_2(\mathbb{R})$, or a discrete subgroup of it, acting on S^1 by fractional linear transformations.

Example 3.7. For a hyperbolic group Γ, the action on its Gromov boundary has a "north to south dynamics", so hyperbolic groups are convergence groups.

We can show that convergence groups are properly proximal.

Proposition 3.8. Convergence groups are properly proximal.

Proof. Let $\Gamma \curvearrowright X$ be an action as in the definition of convergence group. We want to use it to verify the assumptions of Lemma 3.4, so we need to find the measure μ as in the lemma. Take μ to be *any* measure without atoms, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in Γ converging to ∞, and let $s \in \Gamma$. Set $\nu = s_*\mu$. Let $x, y \in X$ as in the definition of convergence group for $(t_n)_{n\in\mathbb{N}}$. Upon passing to a subsequence, it follows that both $(t_n)_*\mu$ and $(t_n)_*\nu$ converge to δ_y in the weak-$*$ topology; all we need here is that x is not an atom for either μ or ν. In particular,

$$0 = \lim_n (t_n)_*\mu - (t_n)_*\nu = \lim_n (t_n)_*\mu - (t_n)_* s_* \mu,$$

as desired. $\square$

Theorem 3.9. If Γ is a lattice in a noncompact semisimple Lie group, then Γ is properly proximal. In particular, $SL_3(\mathbb{Z})$ is properly proximal.

Biexactness is a rank-one phenomenon (in the Lie-group setting), while proper proximality allows for commutativity in the Lie group (that is, higher rank). Also, proper proximality is closed under direct products, while biexactness is not.

We have seen that groups with property Γ are not properly proximal. In fact a similar argument shows the following:

Proposition 3.10. If Γ is inner amenable, then Γ is not properly proximal.

For a long time this was the only known obstruction; in the next section we will see another one (being vNE to an inner amenable group) which shows the following (to be discussed in the upcoming section).

Example 3.11. $SL_3(F_p[t^{-1}]) \ltimes F_p[t,t^{-1}]^3$ is not properly proximal and not inner amenable (but ME, so vNE, to an inner amenable group).

We list some open problems:

Question 3.12. Are acylindrically hyperbolic groups properly proximal?

Question 3.13. Is $\mathrm{Out}(\mathbb{F}_n)$ properly proximal?

Question 3.14. Are ICC property (T) groups properly proximal?

We now discuss the higher-rank case.

Example 3.15. Let $\Gamma \subseteq SL_3(\mathbb{R})$ be a lattice. We consider the action of Γ (via $SL_3(\mathbb{R})$) on $\mathbb{PR}^2$ given by matrix multiplication, and we identify $\mathbb{PR}^2$ with the homogeneous space given as the quotient of $SL_3(\mathbb{R})$ by the subgroup of matrices of the form $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. This is not a convergence action (and $SL_3(\mathbb{R})$ is not a convergence group), but it has a similarly-looking property, which we will explore through the so-called KAK-decomposition.

Let $K \subseteq SL_3(\mathbb{R})$ be the maximal compact subgroup, namely $K = SO(3)$. Let A_+ denote the set of all diagonal matrices of the form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Then every element in $SL_3(\mathbb{R})$ can be expressed as kak', where $k, k \in K$ and $a \in A_+$. To see this, given a matric $g \in SL_3(\mathbb{R})$, consider the polar decomposition $g = vb$, where b is a positive operator and v is a unitary. Since all positive 3x3 matrices are diagonalizable, there is another unitary u such that $ubu^* =: a$ belongs to A_+, and then $g = (vu)au^*$ has the desired factorization.

The KAK-decomposition gives a nice description for the action on $\mathbb{PR}^2$: the elements of $SO(3)$ are just rotations, and the stretching comes from the element from A_+. Given a sequence $(g_n)_{n\in\mathbb{N}}$ in $SL_3(\mathbb{R})$, decompose g_n as $g_n = k_n a_n g'_n$. Since $SO(3)$ is compact, we may assume by passing to a subsequence that $k_n \to k$ and $k'_n \to k'$. Write a_n as $\mathrm{diag}(\lambda_1^{(n)}, \lambda_2^{(n)}, \lambda_3^{(n)})$. For a vector $\xi \in \mathbb{R}^3 \setminus \{0\}$, denote by $[\xi]$ the induced element in $\mathbb{RP}^2$. If $\lambda_1^{(n)}/\lambda_2^{(n)} \to \infty$ and ξ does not have a zero in its first coordina, then

$$k^{-1} g_n k'^{-1}[\xi] \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

so also in this setting we get that the space collapses to a point. (Like we had for convergence actions.) Naturally one doesn't always have the condition $\lambda_1^{(n)}/\lambda_2^{(n)} \to \infty$, but as soon as the sequence $(g_n)_{n\in\mathbb{N}}$ is not precompact, then upon passing to a subsequence we have either $\lambda_1^{(n)}/\lambda_2^{(n)} \to \infty$ or $\lambda_2^{(n)}/\lambda_3^{(n)} \to \infty$ (or both).

Suppose that $\lambda_1^{(n)}/\lambda_2^{(n)} \to \infty$. Then the vectors that can be collapsed to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are those *not* in the span of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This subspace has dimension 2, and thus measure zero in $\mathbb{R}^3$. The upshot is that if we take any sequence $(g_n)_{n\in\mathbb{N}}$ in our lattice $\Gamma \subseteq SL_3(\mathbb{R})$, and if μ denotes the Haar measure on $\mathbb{RP}^2$, if $\lambda_1^{(n)}/\lambda_2^{(n)} \to \infty$ then the weak-$*$ limit of $g_n(\mu - t_*\mu)$ is zero.

This naturally motivates the following definition, where instead of having proper proximality in the whole space, we only have it on some subspace.

Definition 3.16. A *boundary piece* for Γ is a closed subset $X \subseteq \beta\Gamma \setminus \Gamma$ which is invariant under left and right translations.

Example 3.17. Let $\Gamma \subseteq SL_3(\mathbb{R})$ be a lattice. In the context of Example 3.15, the set X of all limit points of sequences $\{g_n\}_{n\in\mathbb{N}}$ in $SL_3(\mathbb{Z})$ which satisfy $\lambda_1^{(n)}/\lambda_2^{(n)} \to \infty$, is a boundary piece for Γ. Similarly, the set Y of all limit points of sequences $\{g_n\}_{n\in\mathbb{N}}$ in $SL_3(\mathbb{Z})$ which satisfy $\lambda_2^{(n)}/\lambda_3^{(n)} \to \infty$, is also a boundary piece for Γ. Note that $X \cup Y = \beta\Gamma \setminus \Gamma$, because in every precompact sequence either the ratio of the first two eigenvalues goes to infinity, or the ratio of the second two eigenvalues goes to infinity.

This gives us a notion of "converging to infinity" relative to X.

Definition 3.18. If X is a boundary piece for Γ, we say that a sequence $(g_n)_{n\in\mathbb{N}}$ in Γ *converges to* ∞ *relative to* X, written $g_n \to_X \infty$, if any limit point of $\{g_n\}_{n\in\mathbb{N}}$, which exist in $\beta\Gamma$, belongs to X.

Remark 3.19. When $X = \beta\Gamma \setminus \Gamma$, then $g_n \to_X \infty$ if and only if $g_n \to \infty$.

In turn, we can define the notion of "vanishing at infinity" relative to X. We say that a function $f \in \ell^\infty(\Gamma)$ *vanishes at infinity relative to* X, written $f \in I_X$, if $f(t_n) \to 0$ whenever $t_n \to_X \infty$. In other words, $I_X = \{f \in C(\beta\Gamma)\colon f|_X = 0\}$. Set

$$A_X(\Gamma) = \{f \in \ell^\infty(\Gamma)\colon f - \mathtt{Rt}_t(f) \in I_X \ \text{ for all } t \in \Gamma\}.$$

Definition 3.20. We say that Γ is *properly proximal relative to* X if there does not exist a left invariant state on $A_X(\Gamma)$.

Example 3.21. For $X = \beta\Gamma \setminus \Gamma$, we get the usual definition of proper proximality.

Theorem 3.22. (Ozawa). If Γ is properly proximal relative to X and Y separately, then it is properly proximal with respect to $X \cup Y$.

Corollary 3.23. Any lattice $\Gamma \subseteq SL_3(\mathbb{R})$ is properly proximal.

Proof. This follows from the above theorem of Ozawa together with Example 3.17. ☐

One can also do this for lattices in $SL_n(\mathbb{R})$, except that one will get a decomposition of $\beta\Gamma \setminus \Gamma$ into $n-1$ boundary pieces.

Corollary 3.24. Proper proximality is closed under products. In particular, $\mathbb{F}_2 \times \mathbb{F}_2$ is properly proximal.

4. Measure equivalence, von Neumann equivalence, and W*-equivalence

We want to discuss another permanence property for proper proximality; see Theorem 4.5. We need some definitions.

Definition 4.1. For a group action $\Gamma \curvearrowright^\sigma (M, \mathrm{Tr})$ on a semifinite tracial von Neumann algebra, a *fundamental domain* is a projection $p \in M$ such that $\sum_{t\in\Gamma} \sigma_t(p) = 1$.

Definition 4.2. Let Γ and Λ be discrete groups. We say that Γ and Λ are:

1. *von Neumann equivalent*, written $\Gamma \sim_{\mathrm{vNE}} \Lambda$, if there exist commuting actions of Γ and Λ on the same semifinite tracial von Neumann algebra, which have (separately) fundamental domains of finite trace;
2. *measure equivalent*, written $\Gamma \sim_{\mathrm{ME}} \Lambda$, if they are von Neumann equivalent and the von Neumann algebra M as above can be chosen to be abelian;
3. *W*-equivalent*, written $\Gamma \sim_{\mathrm{W}^*} \Lambda$, if $L(\Gamma) \cong L(\Lambda)$.

It is obvious that measure equivalence implies von Neumann equivalence. The converse is open. Here is the only other known implication:

Proposition 4.3. If $\Gamma \sim_{\mathrm{W}^*} \Lambda$, then $\Gamma \sim_{\mathrm{vNE}} \Lambda$.

Proof. We assume that Γ (and thus Λ too) is ICC. Take $M = \mathcal{B}(\ell^2(\Gamma))$, and identify M with $\mathcal{B}(\ell^2(\Lambda))$ via the GNS construction of $L(\Lambda)$ associated to its unique trace. Then there exist natural commuting actions of Γ and Λ on M, namely λ_Γ and ρ_Λ. Let $p \in \mathcal{B}(\ell^2(\Gamma))$ be the projection onto the span of δ_e. Then $\tau(p) < \infty$ and $\sum_{t\in\Gamma} \lambda_t(p) = 1$. Under the identification of $\ell^2(\Gamma)$ with $\ell^2(\Lambda)$ given by the GNS construction, we also get $\sum_{s\in\Lambda} \rho_s(p) = 1$. We deduce that $\Gamma \sim_{\mathrm{vNE}} \Lambda$. ☐

It is known that von Neumann equivalence does not imply W*-equivalence.

Example 4.4. $\mathbb{F}_2 \sim_{\mathrm{ME}} \mathbb{F}_3$, and hence $\mathbb{F}_2 \sim_{\mathrm{vNE}} \mathbb{F}_3$. On the other hand, $\mathbb{F}_2 \nsim_{\mathrm{ME}} \mathbb{F}_\infty$, and it is not known whether $\mathbb{F}_2$¿ $\sim_{\mathrm{vNE}}^{?} \mathbb{F}_\infty$.

Theorem 4.5. (Ishan-Peterson-Ruth). If $\Gamma \sim_{\mathrm{vNE}} \Lambda$, then Γ is properly proximal if and only if Λ is.

The basic idea of the proof is to transfer non-proper proximality to actions on dual Banach spaces, using an amenability-type description. Given actions $\Gamma \curvearrowright (M, \mathrm{Tr}) \curvearrowleft \Lambda$ implementing a measure equivalence, one can induce isometric actions of Γ on dual operator spaces to actions of Λ: given an action $\Lambda \curvearrowright E = (E_*)^*$ by complete isometries, we consider $\Gamma \curvearrowright (M \overline{\otimes} E)^{\Lambda}$, induced by taking Γ to act trivially on E. This induced action shows that Γ is not properly proximal.

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