ACTIONS OF FINITE GROUPS ON KIRCHBERG ALGEBRAS

Abstract. We consider the classification of pointwise outer actions os finite groups on Kirchberg algebras. The best results are for cyclic groups of prime order. Ingredients include equivariant semiprojectivity, equivariant versions of Kirchberg's absorption theorems, and Köler's universal coefficient theorem for equivariant KK-theory for cyclic groups of prime order.

These are lecture notes of a course given by **Chris Phillips** at the SFB Miniworkshop on *C^*-Algebras, C^*-Bundles, and Group Actions* at the University of Münster, November 25-29, 2013. Notes taken by Eusebio Gardella.

Warning: little proofreading has been done.

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1. Introduction and Motivation.

The target is the classification of pointwise outer finite group actions on unital Kirchberg algebras. There are four main steps to the (intended) proof:

(1) Equivariant versions of Kirchberg absorption theorems:
- $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ whenever A is a simple, separable, nuclear, unital C^*-algebra, and
- $\mathcal{O}_\infty \otimes A \cong A$ whenever A is a purely infinite, simple, separable, nuclear C^*-algebra.

(2) If A is a unital Kirchberg algebra, G a finite group, $\alpha\colon G \to \mathrm{Aut}(A)$ a pointwise outer action, D any unital C^*-algebra, and $t \mapsto \varphi_t$ is an equivariant asymptotic morphism $A \to \mathcal{O}_\infty \otimes D$ (this is, φ_t is a unital completely positive map for all t in $\mathbb{R}$ and moreover
$$\lim_{t\to\infty} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| = 0$$
for all a, b in A), then there exists a continuous path $t \mapsto u_t$ of invariant unitaries and an equivariant homomorphism $\psi\colon A \to \mathcal{O}_\infty \otimes D$ such that
$$\lim_{t\to\infty} \|\varphi_t(a) - u_t\psi(a)u_t^*\| = 0$$
for all a in A.

(3) If A and B are unital Kirchberg algebras with pointwise outer actions of a finite group G, then $KK_*^G(A, B)$ is essentially given by asymptotic unitary equivalence classes of equivariant "full" homomoprhisms $A \to B$.

(4) Universal Coefficient Theorem for equivariant KK-theory, due to Köler. Needs G cyclic of prime order. (There is a candidate for the general case involving orbit categories, but this is still open.)

We take $\mathbb{N} = \{1, 2, \ldots\}$. The p-adic integers will not appear in this document, so for n in $\mathbb{N}$, we denote the cyclic group of order n by $\mathbb{Z}_n$.

2. Semiprojectivity without group action.

The following lemma was known even before semiprojectivity was formally introduced.

Lemma 2.1. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that whenever A is a C^*-algebra and a in A satisfies
$$\|a^2 - a\| < \delta \quad \text{and} \quad \|a^* - a\| < \delta,$$
then there exists a projection p in A such that $\|p - a\| < \varepsilon$.

Proof. Assume that $a = a^*$. Then the condition $\|a^2 - a\| < \delta$ implies that $1/2$ is not in the spectrum of a, and hence $p = \chi_{[1/2,\infty)}(a)$ is a projection in A that can be shown to be close to a. ☐

In modern language, Lemma 2.1 states that any approximate homomorphism $\mathbb{C} \to A$ is close to a genuine homomorphism. More generally, for all n in $\mathbb{N}$, approximate homomorphisms $\mathbb{C}^n \to A$ can be perturbed to obtain an honest homomorphism $\mathbb{C}^n \to A$.

Lemma 2.2. Let $\varepsilon > 0$ and n in $\mathbb{N}$. Then there exists $\delta > 0$ such that whenever A is a C^*-algebra and $a_1, \ldots, a_n$ in A satisfy

$$\|a_j^2 - a_j\| < \delta, \quad \|a_j^* - a_j\| < \delta \quad \text{and} \quad \|a_j a_k\| < \delta,$$

for all distinct $j, k = 1, \ldots, n$, then there exist orthogonal projections $p_1, \ldots, p_n$ in A such that $\|p_j - a_j\| < \varepsilon$ for all $j = 1, \ldots, n$.

Proof. Assume $n = 2$. (The rest is an inductive argument.) Assume that $a_j = a_j^*$ for all $j = 1, 2$. Use Lemma 2.1 to find a projection p_1 in A close to a_1. Replace a_2 by $b_2 = (1-p_1)a_2(1-p_1)$. Since p_1 is close to a_1 and $c = (1-a_1)a_2(1-a_1)$ is close to a_2 (which can be seen by multiplying this expression out), one gets that $\|b_2^2 - b_2\|$ is small. Also, b_2 is self-adjoint and is close to a_2. Use Lemma 2.1 in $(1-p_1)A(1-p_1)$ to find a projection p_2 in $(1-p_1)A(1-p_1)$ such that $\|p_2 - b_2\|$ is small. Then p_1 and p_2 are orthogonal and

$$\|p_2 - a_2\| \leq \|p_2 - b_2\| + \|b_2 - a_2\|,$$

so p_2 is close to a_2. □

Remark 2.3. Fix $\varepsilon > 0$ and for n in $\mathbb{N}$ denote by $\delta(n)$ the positive number given by Lemma 2.2. Then $\delta(n) \to 0$ fairly fast as $n \to \infty$.

Exercise 2.4. Work out the induction argument in Lemma 2.2

Exercise 2.5. Write down the details of the proof of Lemma 2.1 and Lemma 2.2 assuming self-adjointness.

There is a similar result for approximate homomorphism from M_n.

Lemma 2.6. Let $\varepsilon > 0$ and n in $\mathbb{N}$. Then there exists $\delta > 0$ such that whenever A is a C^*-algebra and $a_{j,k}$ in A for $j, k = 1, \ldots, n$ satisfy

$$\|a_{j,k}^* - a_{j,k}\| < \delta \quad \text{and} \quad \|a_{j,k}a_{l,m} - \delta_{k,l}a_{j,m}\| < \delta,$$

for all $j, k, l, m = 1, \ldots, n$, then there exist matrix units $f_{j,k}$ in A such that $\|f_{j,k} - a_{j,k}\| < \varepsilon$ for all $j, k = 1, \ldots, n$.

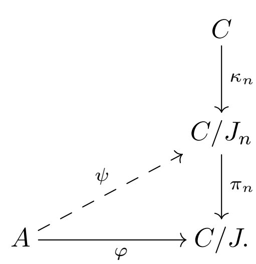
Proof. (Sketch) Use Lemma 2.2 to find projections $f_{j,j}$. To get $f_{1,j}$, let $x = f_{1,1}a_{1,j}f_{j,j}$ and set $f_{1,j} = x(x^*x)^{-1/2}$. Here $x^*x \in f_{j,j}Af_{j,j}$ is close to $f_{j,j}$, so it is invertible and this gives $(x^*x)^{-1/2}$. Now take $f_{j,k} = f_{1,j}^* f_{1,k}$. □

Exercise 2.7. Fill in the details in the proof of Lemma 2.6 with ε and δ.

Lemma 2.6 was first obtained by Glimm and was used in the classification of UHF-algebras. A generalization of this result to finite-dimensional C^*-algebras was proven by Bratteli.

All of these results can be expressed in terms of semiprojectivity.

Definition 2.8. A C^*-algebra A is said to be *semiprojective* if whenever C is a C^*-algebra, $J_1 \subseteq J_2 \subseteq \cdots A$ is an increasing sequence of ideals in C with $J = \overline{\bigcup_{n\in\mathbb{N}} J_n}$, and $\varphi\colon A \to C/J$ is a homomorphism, then there exist n in $\mathbb{N}$ and a homomorphism $\psi\colon A \to C/J_n$ such that $\pi_n \circ \psi = \varphi$:



If one can always choose the lift to be $\psi\colon A \to C$, then A is called *projective*.

For unital C^*-algebras, it does not make a difference if in the definition of semiprojectivity we require that the homomorphisms be unital. (See Proposition 2.15.)

Remark 2.9. If one assumes that everything is unital and commutative with $A = C(X)$, then the condition is equivalent to X being an absolute neighborhood retract.

The lemmas proved before can be rephrased by saying that the C^*-algebras $\mathbb{C}$, $\mathbb{C}^n$ and M_n are semiprojective. To see this explicitly, we need a technical lemma of independent interest.

Lemma 2.10. Let C be a C^*-algebra, let $J_1 \subseteq J_2 \subseteq \cdots \subseteq C$ be an increasing sequence of ideals in C and set $J = \overline{\bigcup_{n\in\mathbb{N}} J_k}$. For k in $\mathbb{N}$, denote by $\kappa_k\colon C \to C/J_k$ the partial quotient maps, and let $\kappa\colon C \to C/J$ be the total quotient map. Then

$$\|\kappa(a)\| = \lim_{k\to\infty} \|\kappa_k(a)\|$$

for all a in A.

Lemma 2.11. Let n in $\mathbb{N}$. Then $\mathbb{C}^n$ is semiprojective.

Proof. Let $\varphi\colon \mathbb{C}^n \to C/J$ be a homomorphism and let q_j in $\mathbb{C}^n$ be the j-th standard vector. Lift $\varphi(q_j)$ to c_j in C self-adjoint, so that $\kappa(c_j) = \varphi(q_j)$ for $j = 1,\ldots,n$. Then $\kappa(c_j^2 - c_j) = 0$, and hence by Lemma 2.10, for k large enough, we have that $\|\kappa_k(c_j)^2 - \kappa_k(c_j)\|$ is small for $j = 1,\ldots,n$. Similarly, $\|\kappa_k(c_j)\kappa_k(c_l)\|$ is small for $j \neq l$ if k is large enough. Apply Lemma 2.2 to find orthogonal projections p_j in C/J_k such that $\|p_j - \kappa_k(c_j)\|$ is small for $j = 1,\ldots,n$. This gives a homomorphism $\psi\colon \mathbb{C}^n \to C/J_k$. Since π commutes with functional calculus, one gets that $\pi_k(\psi(q_j)) = \pi_n(p_j) = \varphi(q_j)$ for all $j = 1,\ldots,n$, which concludes the proof. ☐

We would like to show that semiprojectivity for $\mathbb{C}^n$ implies the perturbation argument proved in Lemma 2.2.

Proof. Suppose the perturbation property fails. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists a C^*-algebra B_δ and elements $a_1^\delta,\ldots,a_n^\delta$ in B_δ such that

$$\|(a_j^\delta)^2 - a_j^\delta\| < \delta, \quad \|(a_j^\delta)^* - a_j^\delta\| < \delta \quad \text{and} \quad \|a_j^\delta a_k^\delta\| < \delta, \tag{1}$$

for all distinct $j,k = 1,\ldots,n$, but there are not orthogonal projections $p_1^\delta,\ldots,p_n^\delta$ in B_δ such that $\|p_j^\delta - a_j^\delta\| < \varepsilon$ for all $j = 1,\ldots,n$. Set $C = \prod_{m\in\mathbb{N}} B_{1/m}$, and for r in $\mathbb{N}$, take

$$J_r = \{(b_1,\ldots,b_r,0,\ldots)\colon b_j \in B_{1/j}\} \subseteq C.$$

Note that $C/J_r \cong \prod_{m=r+1}^\infty B_{1/m}$ and

$$J = \overline{\bigcup_{r\in\mathbb{N}} J_r} = \{(b_m)_{m\in\mathbb{N}} \in B\colon \lim_{m\to\infty} \|b_m\| = 0\}.$$

With $q_1,\ldots,q_n$ denoting the canonical projections in $\mathbb{C}^n$, set $\varphi(q_j) = \kappa(a_j^1, a_j^{1/2}, a_j^{1/3},\ldots)$. Then $\varphi\colon \mathbb{C}^n \to C/J$ is a homomorphism by the inequalities in (1). By semiprojectivity of $\mathbb{C}^n$, the homomorphism φ lifts to $\psi\colon \mathbb{C}^n \to C/J_r$ for some r in $\mathbb{N}$. One therefore gets projections $(p_j^{1/m})_{m\in\mathbb{N}}$ in the product $\prod_{m=r+1}^\infty B_{1/m}$ whose image in C/J is $\varphi(q_j)$ for all $j = 1,\ldots,n$. This gives $\|p_j^{1/m} - a_j^{1/m}\| \to 0$ as $m \to \infty$. Pick m large enough so that all of these are less than ε. This is a contradiction. ☐

There is a more general statement behind this fact.

Proposition 2.12. For all $\varepsilon > 0$ and for all n in $\mathbb{N}$, there exists $\delta > 0$ such that for all C^*-algebras B and D, for all homomorphisms $\eta\colon B \to D$ and for all positive elements $a_1,\ldots,a_n$ in B such that

$$\|a_j^2 - a_j\| < \delta, \quad \|a_j^* - a_j\| < \delta \quad \text{and} \quad \|a_j a_k\| < \delta$$

for all distinct $j,k = 1,\ldots,n$ and the elements $\eta(a_j)$ are orthogonal projections and satisfy the conditions exactly, then there are orthogonal projections $p_1,\ldots,p_n$ in B such that $\|p_j - a_j\| < \varepsilon$ and $\eta(p_j) = \eta(a_j)$ for all $j = 1,\ldots,n$.

Exercise 2.13. Prove Proposition 2.12.

Examples 2.14. Semiprojective and non-semiprojective C^*-algebras.

(1) Finite dimensional C^*-algebras are semiprojective.
(2) $C(S^1)$ is semiprojective (in the unital category – which is equivalent to being semiprojective in the non-unital category; see Proposition 2.15).

   Proof. We regard $C(S^1)$ as the universal C^*-algebra generated by a unitary. Given an almost unitary a in B, get a nearby unitary by setting $u = a(a^*a)^{-1/2}$. ☐

(3) The Cuntz algebras $\mathcal{O}_n$ for $2 \leq n \leq \infty$ are semiprojective. The result for $\mathcal{O}_\infty$ is trickier than for the other Cuntz algebras, and it is due to Blackadar.
(4) $C([0,1])$ is semiprojective.
(5) $C([0,1]^2)$ is not semiprojective, and neither is $C(X)$, where X is the Cantor set.
(6) (Sørensen-Thiel [6]) $C(X)$ is semiprojective if and only if X is an absolute neighborhood retract of dimension at most 1.
(7) Various dimension drop algebras are semiprojective.
(8) $C(S^1,F)$ and $C([0,1],F)$ are semiprojective for any finite dimensional C^*-algebra F.

(9) If F_n denotes the free group on n generators, then $C^*(F_n)$ is semiprojective for $1 \leq n < \infty$ and $C^*(F_\infty)$ is not.
(10) $\mathcal{K}$ is not semiprojective.

We now turn to the difference between the unital and non-unital versions of semiprojectivity.

Proposition 2.15. Let A be a unital C^*-algebra. Then A is semiprojective in the unital category if and only if it is semiprojective in the non-unital category.

Proof. If A is unitally semiprojective and we consider a lifting problem with non-unital maps, partially lift $\varphi(1)$ to $q_r \in C/J_r$. For $s > r$, let q_s be its image on C/J_s. Replace C/J_s by $q_s(C/J_s)q_s$.

Conversely, if A is unital and non-unitally semiprojective, consider a lifting problem with unital maps. Get a non-unital lift $\psi\colon A \to C/J_r$ for some r in $\mathbb{N}$. Then $\pi_r(\psi(1)) = 1$ because φ is unital, and thus $\|\pi_{s,r}(\psi(1)) - 1\|$ is small for $s > r$ big enough. In particular, if it is less than 1, then $\pi_{s,r}(\psi(1)) = 1$, being an invertible projection. Thus $\pi_{s,r} \circ \psi$ is a unital partial lift of φ. $\square$

Note that the argument breaks down for projectivity. In fact, the unital and non-unital versions of projectivity do not agree.

3. Equivariant semiprojectivity.

Equivariant semiprojectivity is defined similarly to usual semiprojectivity. Given a group G, take the category where the morphisms and objects in the diagram live to consist of all dynamical systems (G, A, α), where $\alpha\colon G \to \operatorname{Aut}(A)$ is a continuous action, with morphisms given by equivariant homomorphisms.

$$\begin{array}{ccc} & & (G, C, \gamma) \\ & & \downarrow \kappa_n \\ & \overset{\psi}{\nearrow} & (G, C/J_n, \gamma^{(n)}) \\ & & \downarrow \pi_n \\ (G, A, \alpha) & \xrightarrow[\varphi]{} & (G, C/J, \gamma^{(\infty)}). \end{array}$$

The reason why the argument in Proposition 2.15 works is that $\mathbb{C}$ is non-unitally semiprojective. This motivates the following example.

Example 3.1. Let G be any compact group, and let $\alpha\colon G \to \operatorname{Aut}(\mathbb{C})$ be the trivial action (there is no other choice). Then $(G, \mathbb{C}, \alpha)$ is equivariantly semiprojective.

Proof. Given $\varphi\colon (G, \mathbb{C}, \alpha) \to (G, C/J, \gamma^{(\infty)})$, lift $\varphi(1)$ to some self-adjoint element c_0 in C and set $c = \int_G \gamma_g(c_0)\, dg$. Then $\kappa(c) = \kappa(c_0) = \varphi(1)$ because φ is equivariant. Proceed as before, using that functional calculus on invariant elements gives invariant elements. Push c down far enough to form $\chi_{[1/2,\infty)}(\frac{1}{2}(c + c^*))$. $\square$

Examples 3.2. Let G be a compact group.

(1) If F is a finite dimensional C^*-algebra and $\alpha\colon G \to \operatorname{Aut}(F)$ is any continuous action, then (G, F, α) is equivariantly semiprojective.
(2) The system $(G, \mathcal{O}_n, \alpha)$, if $2 \leq n < \infty$ and α is a quasifree action, is equivariantly semiprojective.
(3) The system $(G, \mathcal{O}_\infty, \alpha)$, if G is finite and α is a quasifree action, is equivariantly semiprojective.
(4) Let n in $\mathbb{N}$ and let $\mathbb{Z}_{2^n}$ act on $C_0((0,1], C(\mathbb{Z}_{2^n}))$ by left translation on $C(\mathbb{Z}_{2^n})$. Then $(\mathbb{Z}_{2^n}, C_0((0,1], C(\mathbb{Z}_{2^n})), \texttt{lt})$ is equivariantly semiprojective. (This should be true for abitrary finite groups, but it is not known.)

The following result relates equivariant semiprojectivity of a system and semiprojectivity of the underlying algebra.

Theorem 3.3. (Phillips [4]) Suppose G is compact and (G, A, α) is semiprojective. Then A is semiprojective.

More generally:

Theorem 3.4. (Phillips-Sørensen-Thiel [5]) Let G be a locally compact group and let (G, A, α) be equivariantly semiprojective. If H is a subgroup of G such that G/H is compact, then $(A, H, \alpha|_H)$ is equivariantly semiprojective.

If in Theorem 3.4 we assume that G is compact and take H to be the trivial subgroup, then we recover Theorem 3.3. The result is known to fail if one does not assume that G/H is compact; see Remark 3.7.

As of equivariant projectivity, we have the following result on restrictions.

Theorem 3.5. Let G be a locally compact group and let (G, A, α) be equivariantly projective. If H is a subgroup of G that satisfies very weak conditions, then $(A, H, \alpha|_H)$ is equivariantly projective.

The weak conditions mentioned above are satisfied whenever G is discrete and H is arbitrary, or when G/H is compact, among others. It is not known whether the result is true without any assumptions on the subgroup H. This is, it may be true that an arbitrary restriction of an equivariantly projective system is again equivariantly projective.

As an example of an equivariantly semiprojective system with a non-compact group, we have the following.

Example 3.6. Let G be discrete and let A be the full group C^*-algebra on the generators a_g for g in G. Let G act on A by translating the generators: $\alpha_g(a_h) = a_{gh}$ for all g and h in G. Then (G, A, α) is equivariantly semiprojective in the unital category. Indeed, lift $\varphi(a_1)$ to some unitary u_1 in C/J_r for some r in $\mathbb{N}$. Now take $\psi(a_g)$ to be $\gamma_g(u_1) = u_g$.

Remark 3.7. If $|G| = \infty$, then $A \cong C^*(F_\infty)$ itself is not semiprojective. (See (9) in Example 2.14.) It follows that Theorem 3.3 is not true if G is not compact, and that Theorem 3.4 is not true if G/H is not compact.

We mention two applications of equivariant semiprojectivity (not using Kirchberg algebras).

Theorem 3.8. (Gardella [1]) If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a finite group G with the Rokhlin property, then α is the dual of some coaction on A^G.

This uses semiprojectivity of the system $(G, C(G), \mathtt{lt})$ for a finite group G.

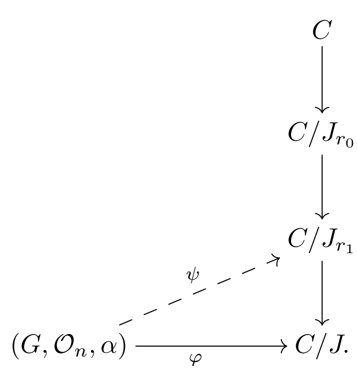
Theorem 3.9. (Pasnicu-Phillips [3]) Let A be a non-necessarily simple purely infinite C^*-algebra (in the sense of Kirchberg-Rørdam) with the ideal property (ideals are generated by their projections), an let $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ be any action. Then $A \rtimes_\alpha G$ is purely infinite and has the ideal property.

The result above uses equivariant semiprojectivity of the system $(\mathbb{Z}_2, C_0((0,1], C(\mathbb{Z}_2)), \mathtt{lt})$. The result should be true for arbitrary finite groups.

It is not known, and it is probably false, whether pure infiniteness by itself is preserved under formation of crossed products by arbitrary actions of $\mathbb{Z}_2$. It is known that the ideal property is *not* preserved in general.

Example 3.10. Let G be a finite group, let m in $\mathbb{N}$, and set $n = m|G|$. Consider $\mathcal{O}_n$ with generators $s_{g,j}$ for $j = 1, \ldots, m$ and g in G, and let G act on $\mathcal{O}_n$ by $\alpha_g(s_{h,j}) = s_{gh,j}$ for g, h in G and $j = 1, \ldots, m$. Then $(G, \mathcal{O}_n, \alpha)$ is equivariantly semiprojective.

Proof. (Sketch) Assume $m = 1$ so $n = |G|$ and G acts on $\mathcal{O}_n$ by translation of the generators s_g for g in G. Equivariantly partially lift $\mathbb{C}^n = \mathrm{span}(\{s_g s_g^*\colon g \in G\})$ to C/J_{r_0} and denote the lifts by $q_1, \ldots, q_n$. (Since the algebra is finite dimensional and the group is compact, the system is equivariantly semiprojective; see Theorem 3.22.) Then partially lift the isometry s_1 to have range equal to the corner in C/J_{r_1} given by the image of the lift of $s_1 s_1^*$. (We are using that the Toeplitz algebra is semiprojective – this is standard.)



Denote by $e_1, \ldots, e_n \in C/J_{r_1}$ the images of $q_1, \ldots, q_n$. Get t_1 such that $t_1^* t_1 = 1$ and $t_1 t_1^* = e_1$. Take $t_g = \gamma_g(t_1)$ for g in G. Then $\{t_g\colon g \in G\}$ satisfies the relations defining the Cuntz algebra $\mathcal{O}_n$, and hence generate a copy of $\mathcal{O}_n$. This is the partial lift. □

Remark 3.11. The argument works for $\mathcal{O}_{mn}$, and using Blackadar's method, also for $\mathcal{O}_\infty$. This particular action is the one on $\mathcal{O}_\infty$ needed for Kirchberg's equivariant stability.

We collect some facts about equivariant semiprojectivity. Many will appear in the paper [5]. The first one is an equivariant analog of Proposition 2.15.

Theorem 3.12. If G is a compact group and A is a unital C^*-algebra, then (G, A, α) is equivariantly semiprojective in the unital category if and only if it is equivariantly semiprojective in the non-unital category.

The result is not true for non-compact groups. Indeed, the system $(G, C^*(F_G), \alpha)$ constructed in Example 3.6 is equivariantly semiprojective in the unital category but not in the non-unital category. This follows from the following result.

Theorem 3.13. If (G, A, α) is equivariantly semiprojective in the non-unital category and G is not compact, then $A^G = \{0\}$.

In particular, if G is not compact and A is unital, no action of G on A can be equivariantly semiprojective in the non-unital category. This shows that Example 3.6 is not equivariantly semiprojective in the non-unital category.

The following is another example of an algebra that is not semiprojective itself but is equivariantly semiprojective with a suitable action. It turns out that although one can not partially lift the algebra by itself, the action helps one find a partial lift.

Example 3.14. Let G be a discrete group and let $A = \ast_{g \in G} \mathbb{C}$ be the full free product of copies of $\mathbb{C}$ indexed by G (not amalgamated even over the unit). Let G act on A via the "free Bernoulli shift": translation on the indices. Then (G, A, α) is equivariantly semiprojective. To see this, lift one of the projections, say the projection indexed by $1 \in G$, to, say, p_1 in C/J_r for some r in $\mathbb{N}$. We determine the other projections by setting $p_g = \gamma_g^{(r)}(p_1)$ for all g in G.

The algebra A itself is not semiprojective in general, unless G is finite. The problem is that when one finds lifts p_g in $C/J_{r(g)}$ for g in G, one could have $\sup_{g \in G} r(g) = \infty$ and hence may not be able to lift them all to a finite stage.

Remark 3.15. Example 3.14 above works for any semiprojective C^*-algebra in place of $\mathbb{C}$.

Remark 3.16. If in Example 3.14 one replaces $\mathbb{C}$ with a projective C^*-algebra, then one obtains an equivariantly projective system (G, A, α). In this case, though, the algebra A itself is projective since one has $r(g) = 0$ for all g in G.

For G acting on $A = \oplus_{g \in G} B$ with G discrete acting on A by translation of the summands, and such that A is (semi)projective, one would like to conclude that (G, A, α) is equivariantly (semi)projective. This is only known for $G = \mathbb{Z}_{2^n}$, and open in all other cases.

The following is one relation between equivariant semiprojectivity of a system and semiprojectivity of the crossed product.

Theorem 3.17. If G is discrete and $C^*(G)$ is semiprojective (for example, if G is either finite or $\mathbb{Z}$, but not if G is Z^2), and (G, A, α) is semiprojective, then $A \rtimes_\alpha G$ is semiprojective.

For G compact and infinite, the trivial action on $\mathbb{C}$ is equivariantly semiprojective but $C^*(G)$ is not semiprojective.

There exists a non-equivariantly semiprojective action of $\mathbb{Z}_2$ on $\mathcal{O}_2$.

Question 3.18. If F is finite dimensional with some action of a compact group G, is any of

$$C_0((0,1]) \otimes F \quad C(S^1) \otimes F \quad \text{or} \quad C([0,1]) \otimes F,$$

with the trivial action on the first factor, equivariantly semiprojective? This is not known even if G is finite, $F = C(G)$, and the action is by translation (except when $G = \mathbb{Z}_{2^n}$).

More generally,

Question 3.19. If F is finite dimensional with some action γ of a compact group G, and (G, A, α) is equivariantly semiprojective, is $(G, A \otimes F, \alpha \otimes \gamma)$ equivariantly semiprojective? (This is true without the actions.)

The relationship between equivariant semiprojectivity and the Rokhlin property is still unclear.

Question 3.20. Are Rokhlin actions of finite groups on $\mathcal{O}_2$ equivariantly semiprojective?

The question above is known to have an affirmative answer for $\mathbb{Z}_2$. By Izumi's results, up to conjugacy there is a unique action of $\mathbb{Z}_2$ on $\mathcal{O}_2$ with the Rokhlin property, and there is one such action that is quasifree. Now, arbitrary quasifree actions of compact groups on $\mathcal{O}_n$ with $n < \infty$ are equivariantly semiprojective.
The following more general question was asked by George Elliott.

Question 3.21. (Elliott) If G is finite, A is semiprojective, and $\alpha \colon G \to \mathrm{Aut}(A)$ has the Rokhlin property, is (G, A, α) equivariantly semiprojective?

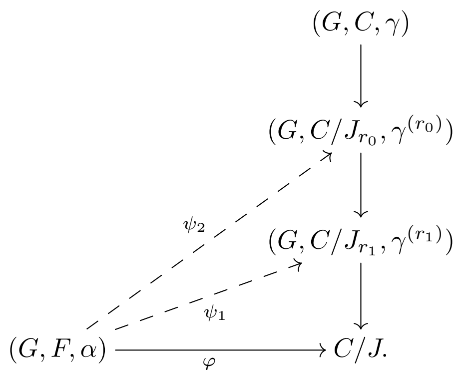
The following result was used to show that the dynamical system constructed in Example 3.10 is equivariantly semiprojective, and has other important applications.

Theorem 3.22. Let G be a compact group, let F be a finite dimensional C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(F)$ be any action. Then (G, F, α) is equivariantly semiprojective.

Proof. (Sketch) Find a non-equivariant lift of $F \to C/J$ to $\psi_0\colon C/J_{r_0}$. Push down to $\psi_1\colon A \to C/J_{r_1}$ to get approximate equivariance, this is,

$$\|\psi_{r_1}(\alpha_g(a)) - \gamma_g^{(r_1)}(\psi_{r_1}(a))\| < \varepsilon\|a\|$$

for all a in A. (We use that A is finite dimensional to get the uniform bound.)



Now we look at the respective unitary groups, and restrict ψ_1 to $T_1\colon \mathcal{U}(A) \to \mathcal{U}(C/J_{r_1})$. Since G is compact, one can average to get a nearby map $S_0\colon \mathcal{U}(A) \to C/J_{r_1}$ that is exactly equivariant, nearly unitary, nearly multiplicative. For u in $\mathcal{U}(A)$, set

$$\rho_0(u) = S_0(u)[S_0(u)^* S_0(u)]^{-1/2}.$$

Then ρ_0 is close to S_0, so it is nearly multiplicative, exactly unitary, and exactly equivariant. Set

$$\rho_1(u) = \exp\left(\int_G \log(\rho_0(v)^*\rho_0(vu)\rho_0(u)^*)dv\right)\rho_0(v).$$

(The measure dv is the normalized Haar measure on $\mathcal{U}(A)$. The expression inside the logarithm is close to 1 because ρ_0 is approximately multiplicative, and hence there is no problem in the choice of the branch of the logarithm.) It follows that ρ_1 is exactly unitary, exactly equivariant, and nearly multiplicative. The point is that ρ_1 is much closer to being multiplicative than ρ_0.

There are two points.

If everything were abelian, then ρ_1 would already be a homomorphism. If written additively: given a map $\rho_0\colon H \to V$ from a compact group H to a vector space V, set

$$\rho_1(g) = \int_H [\rho_0(g + x) - \rho_0(x)]\, dx$$

for g in H, where dx is the Haar measure on H. By using the change of variables $x \mapsto h + x$ in the first integral (using that the Haar measure is translation invariant), we get

$$\begin{aligned}\rho_1(g) + \rho_1(h) &= \int_H [\rho_0(g + x) - \rho_0(x)]\, dx + \int_H [\rho_0(h + x) - \rho_0(x)]\, dx \\ &= \int_H [\rho_0(g + h + x) - \rho_0(h + x) + \rho_0(h + x) - \rho_0(x)]\, dx \\ &= \rho_1(g + h)\end{aligned}$$

for all g and h in H. In general, if $\varepsilon_0 > 0$ is small enough, there exists a constant C depending on ε_0 such that for all $\varepsilon < \varepsilon_0$, whenever $\|u - 1\| \leq \varepsilon$, then

$$\|\log(u) - (u - 1)\| \leq C\varepsilon^2.$$

Similarly with exponentials, one gets that whenever $\|a\| \leq \varepsilon$ and $\|b\| \leq \varepsilon$, then

$$\left\|e^{a+b} - e^a e^b\right\| \leq C_0\varepsilon^2.$$

Analogoulsy,

$$\|\log(uv) - \log(u) - \log(v)\| \leq C_1\varepsilon^2.$$

This implies that if ρ_0 is ε-multiplicative, then ρ_1 is $C\varepsilon^2$-multiplicative. This is, if $\varepsilon > 0$ is such that

$$\|\rho_0(uv) - \rho_0(u)\rho_0(v)\| \leq \varepsilon$$

for all u and v in $\mathcal{U}(A)$. (Use compactness of $\mathcal{U}(A)$ to get the uniform bound.) Then there are universal constants C_2 and C_3 such that

$$\|\rho_1(u) - \rho_0(u)\| \leq C_2\varepsilon \quad \text{and} \quad \|\rho_1(uv) - \rho_1(u)\rho_1(v)\| \leq C_3\varepsilon^2.$$

(Note the ε^2 factor in the second inequality.) Iterate the process and take the limit, say ρ, of the resulting Cauchy sequence. This will be $\psi|_{\mathcal{U}(A)}$. One must still get back a homomorphism from A. This involves a relatively simple but somewhat lengthy trick and we omit it. □

We make a few comments about why conventional methods do not work. Suppose $G = \mathbb{Z}_3$ acts on a C^*-algebra B via β, and that there are approximate orthogonal approximate projections b_0, b_1, b_2 in B that are exactly permuted by G. The conventional approach is to first find a projection q_0 close to b_0. Next one would work in the corner $(1-q_0)B(1-q_0)$, but one does not in general get $\beta(q_0) \perp q_0$. If one instead gets q_1 in $(1-q_0)B(1-q_0)$ and then try to change q_0 to get $\beta(q_0) = q_1$, then one does not in general get $q_0 \perp q_1$.

4. Classification results.

We proceed to present some results relevant to the classification theorems for actions on Kirchberg algebras.

Lemma 4.1. Let (G, A, α) be equivariantly semiprojective, with G finite and A unital and separable. Suppose that (G, D, δ) is another dynamical system such that any two unital equivariant homomorphisms $A \to C([0,1], D)$ are approximately unitarily equivalent. Then any two asymptotic morphisms $A \to D$ are equivariantly asymptotically unitarily equivalent.

The most important case of this lemma is when $A = \mathcal{O}_\infty$ and α is the action of G given by $\alpha_g(s_{h,j}) = s_{gh,j}$ for all g, h in G and for all j in $\mathbb{N}$. The C^*-algebra D will be any G-algebra that absorbs $(G, \mathcal{O}_\infty, \alpha)$ equivariantly. (Note that $C([0,1], D)$ again has this form.)

Proof. (Sketch) Using equivariant semiprojectivity, one can assume that φ_t and ψ_t are equivariant homomorphisms for all t in $[0, \infty)$. Also, assume that $t \mapsto \varphi_t$ is constant with value φ. Start by choosing $u_1(t)$ on $[0,1]$ such that

$$u_1(t)\psi_t(a)u_1(t)^*$$

for $0 \leq t \leq 1$, is close to $\varphi(a)$ for all a in some finite subset F of A. Extend u_1 to $[0,\infty)$ by setting $u_1(t) = u_1(1)$ for $t > 1$. Set

$$\gamma_t^{(1)}(a) = u_1(t)\psi_t(a)u_1(t)^*.$$

Replace $\gamma_t^{(1)}$ by γ_t, where $\gamma_t = \varphi$ for $t \in [0,1]$, and on $[1, 1+\rho]$ for some small $\rho > 0$, use a straight line homotopy from φ to $\gamma_{1+\rho}^{(1)}$, and then set $\sigma_t = \gamma_t^{(1)}$ for $t > 1+\rho$. (Note that $\gamma_1^{(1)}$ is close to φ, and that $\gamma_{1+\rho}^{(1)}$ is not a homomorphism but an approximate homomorphism.) Use equivariant semiprojectivity of (G, A, α) to perturb σ_t to get a path $t \mapsto \tau_t$ of equivariant homomorphisms which is the same as σ_t except on $[1, 1+\rho]$.

Apply approximate unitary equivalence for $\tau|_{[1,2]} \colon A \to C([1,2], D)$ getting $u_2(t)$, and proceed inductively. □

Recall the following definitions.

Definition 4.2. Let G be a finite group, let A be a unital C^*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action.

(1) We say that α has the *Rokhlin property* if there exist sequences $(e_g^{(n)})_{n\in\mathbb{N}}$ of projections in A for g in G, such that

$$\sum_{g\in G} e_g^{(n)} = 1 \ \text{ for all } \ n \in \mathbb{N}, \quad \lim_{n\to\infty} \|\alpha_g(e_h^{(n)}) - e_{gh}^{(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|e_g^{(n)}a - ae_g^{(n)}\| = 0 \ \text{ for all } \ a \in A.$$

(2) We say that α has the *continuous Rokhlin property* if one can choose continuously parametrized families $(e_g^{(t)})_{t\in[0,\infty)}$ of projections in A for g in G satisfying analogous conditions as above.

What is actually wanted is:

(1) For any G-algebra D, any two unital equivariant asymptotic morphisms $\mathcal{O}_2 \to \mathcal{O}_\infty \otimes D$ are equivariantly asymptotically unitarily equivalent. The action on $\mathcal{O}_2$ is the unique action of G with the Rokhlin property; see [2].
(2) For any G-algebra D, any two unital equivariant asymptotic morphisms $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes D$ are equivariantly asymptotically unitarily equivalent. The action on $\mathcal{O}_\infty$ is the one constructed based on the actions in Example 3.10. See also Remark 3.11.

We get useful results using the (continuous) Rokhlin property.

Theorem 4.3. Let G be a finite group and let A and B be G-algebras. If one of the actions has the Rokhlin property and if two equivariant homomorphisms $\varphi, \psi \colon A \to B$ are approximately unitarily equivalent, then they are *equivariantly* approximately unitarily equivalent.

There is an analogous statement using the continuous Rokhlin property and asymptotic morphisms.

Theorem 4.4. Let G be a finite group and let A and B be G-algebras. If one of the actions has the continuous Rokhlin property and if two equivariant asymptotic homomorphisms $(\varphi_t), (\psi_t)\colon A \to B$ are asymptotically unitarily equivalent, then they are *equivariantly* asymptotically unitarily equivalent.

It is known that any two unital asymptotic morphisms $\mathcal{O}_2 \to \mathcal{O}_\infty \otimes D$ are asymptotically unitarily equivalent. Since the unique action of G on $\mathcal{O}_2$ with the Rokhlin property in fact has the continuous Rokhlin property, one gets (1) using Theorem 4.4.

However, there is no action of any non-trivial finite group on $\mathcal{O}_\infty$ with the Rokhlin property. To get a result on approximate unitary equivalence for homomorphisms from $\mathcal{O}_\infty$, a trick from the non-equivariant case allows one to reduce to the case $[1] = 0$ in $K_0^G(\mathcal{O}_\infty \otimes D)$. We need another standard definition.

Definition 4.5. Let n in $\mathbb{N}$. Define the *extended Cuntz algebra* E_n to be the univeral C^*-algebra on generators $s_1, \ldots, s_n$ such that

$$s_j^* s_j = 1 \quad \text{for all } j \in \mathbb{N}, \quad \text{and} \quad s_j s_j^*, s_k s_k^* \quad \text{are orthogonal if } j \neq k.$$

(We omit the relation $\sum_{j=1}^n s_j s_j^* = 1$ in the definition of the standard Cuntz algebra $\mathcal{O}_n$.)

Remark 4.6. For all n in $\mathbb{N}$, there is an exact sequence $0 \to \mathcal{K} \to E_n \to \mathcal{O}_n \to 0$. Moreover, $\varinjlim E_n \cong \mathcal{O}_\infty$.

Take two unital equivariant homomorphisms $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes D$. Restrict these homomorphisms to the extended Cuntz algebra $E_{m|G|}$. Use $[1] = 0$ to extend to a homomorphism

$$\mathcal{O}_{(m+1)|G|} \to \mathcal{O}_\infty \otimes D.$$

The acton on $\mathcal{O}_{(m+1)|G|}$ *does* have the Rokhlin property. K-theory implies that the two homomorphisms are approximately unitarily equivalent. So the two homomorphisms are equivariantly approximately equivalent by Theorem 4.3.

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