DEFORMATION AND RIGIDITY THEORY, AND THE CLASSIFICATION OF II_1-FACTORS

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Abstract. These are lecture notes of a course given by **Stefaan Vaes** at the YMC*A at the University of Copenhagen, Denmark, August 17–21, 2015.

Warning: little proofreading has been done.

Contents

1. Classification of amenable factors

Recall that a von Neumann algebra M is said to be a *factor* if it has trivial center. Any von Neumann algebra can be written as a direct integral of factors over its center, so we therefore focus on factors. Thanks to Murray and von Neumann, these are classified into three types:

- **Type I:** there exists a minimal nonzero projection. In this case, $M \cong \mathcal{B}(\mathcal{H})$
- **Type II:** there does not exist a minimal nonzero projection, and there exists a finite projection. There are two subclasses:
  - **Type II_1:** 1 is a finite projection.
  - **Type II_∞:** 1 is an infinite projection.
- **Type III:** All nonzero projections are infinite.

We now turn to structural properties.

Theorem 1.1. A factor M is II_1 if and only if it has a unique tracial state. This trace is automatically normal (σ-weakly continuous on the unit ball) and faithful ($\tau(p) = 0$ implies $p = 0$ for all projections $p \in M$).

Theorem 1.2. Let M be a II_∞-factor. Then there exist a II_1-factor N and an infinite dimensional Hilbert space $\mathcal{H}$ such that $M \cong N \overline{\otimes} \mathcal{B}(\mathcal{H})$.

For quite some time, not much could be said about type III factors.

Definition 1.3. A von Neumann algebra M is said to be *hyperfinite* if there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of finite dimensional von Neumann subalgebras $A_n \subseteq M$, with $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, and such that $\bigcup\limits_{n\in\mathbb{N}} A_n$ is weak;y dense in M.

Theorem 1.4. There exists a unique hyperfinite II_1-factor (usually denoted by $\mathcal{R}$).

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Here is one way to construct $\mathcal{R}$: consider the direct system

$$\mathbb{C} \hookrightarrow M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow \cdots$$

with diagonal maps $a \mapsto \operatorname{diag}(a, a)$. The resulting direct limit $*$-algebra $\mathcal{R}_0$ has a trace τ (unique if moreover $\tau(1) = 1$). Define $\mathcal{H}$ to be the completion of $\mathcal{R}_0$ with respect to the inner product $\langle a, b\rangle = \tau(ab^*)$. Represent $\mathcal{R}_0$ by left multiplication on $\mathcal{H}$. The weak closure of $\mathcal{R}_0$ is then $\mathcal{R}$.

Beyond type II, we have the work of Tomita-Takesaki and Connes modular theory for type III factors: these are divided in type III_λ, for $0 \leq \lambda \leq 1$. Moreover, there is a notion of amenability for factors, and amenable factors can be completely classified:

Theorem 1.5. (Connes). Every amenable II_1-factor is hyperfinite. He could also handle the cases III_λ for $0 \leq \lambda < 1$.

Theorem 1.6. (Haagerup). Case III_1.

2. Beyond amenable factors

The factors we will focus on come from crossed product constructions.

Definition 2.1. Let Γ be a countable group acting on a tracial von Neumann algebra (P, τ) via trace preserving automorphisms α_g, for $g \in \Gamma$. The crossed product $P \rtimes_\alpha \Gamma$ is the unique tracial von Neumann algebra containing P unitally, and a unitary representation $(u_g)_{g\in\Gamma}$ of Γ satisfying $u_g a u_g^* = \alpha_g(a)$ for all $a \in P$ and all $g \in \Gamma$, and the trace is given by

$$\tau\left(\sum_{g\in\Gamma} a_g u_g\right) = \tau(a_1).$$

Here is a concrete construction of $P \rtimes_\alpha \Gamma$: consider the Hilbert space $L^2(P, \tau)$ constructed from P using τ, where P acts by left multiplication. Then $P \rtimes_\alpha \Gamma$ is represented on the Hilbert space $L^2(P) \otimes \ell^2(\Gamma)$ by

$$au_g \cdot (b \otimes \delta_h) = a\alpha_g(b) \otimes \delta_{gh}.$$

Finally, τ is given by the vector state of $1 \otimes \delta_1$.

Example 2.2. Take $P = \mathbb{C}$. Then $M = L(\Gamma)$ is the group von Neumann algebra generated by a unitary representation $(u_g)_{g\in\Gamma}$ with $\tau(u_g) = \delta_{g,e}$.

Example 2.3. When $P \cong L^\infty(X, \mu)$ is abelian, a trace preserving action corresponds to a probability measure preserving action on X.

We turn to factoriality of $P \rtimes_\alpha \Gamma$:

Theorem 2.4. $L(\Gamma)$ is a factor if and only if Γ is ICC.

Theorem 2.5. Let Γ act on (X, μ) by probability measure preserving automorphisms, and set $M = L^\infty(X, \mu) \rtimes \Gamma$. Then $L^\infty(X, \mu)' \cap M = L^\infty(X, \mu)$ (the smallest possible) if and only if the action is essentially free, that is, for $g \in \Gamma \setminus \{1\}$ we have

$$\mu\left(\{x \in X \colon g \cdot x = x\}\right) = 0.$$

When the action is essentially free, then $Z(L^\infty(X, \mu) \rtimes \Gamma)$ consists of the Γ-invariant functions on $L^\infty(X, \mu)$. Consequently, $L^\infty(X, \mu) \rtimes \Gamma$ is a factor if and only if the action is ergodic. (No characterization is known for actions that are not essentially free.)

Example 2.6. Bernoulli shift: if Γ acts on (X_0, μ_0), consider $X = X_0^\Gamma = \prod_{g\in\Gamma} X_0$ with $(g \cdot x)_h = x_{g^{-1}h}$.

Definition 2.7. For $N \subseteq M$, a conditional expectation is a completely positive unital map $E \colon M \to N$ satisfying $E(axb) = aE(x)b$ for all $a, b \in N$ and for all $x \in M$.

Theorem 2.8. If M has a trace τ and $N \subseteq M$, then there exists a unique trace preserving conditional expectation $E_N \colon M \to N$.

Corollary 2.9. If M is hyperfinite, then it is amenable.

Proof. Take $\psi_n\colon M \to A_n$ to be $\psi_n = E_{A_n}$, for an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of finite dimensional subalgebras $A_n \subseteq M$ with weakly dense union. $\square$

If (M,τ) is a tracial von Neumann algebra, we set $\|x\|_2 = \tau(x^*x)^{1/2}$ for $x \in M$.

We state here, as a fact, that every element $x \in P \rtimes_\alpha \Gamma$ can be uniquely written as a Fourier Series $x = \sum_{g\in\Gamma} x_g u_g$, where the convergence is in the $\|\cdot\|_2$-norm; take $x_g = E(xu_g^*)$.

This minicourse will focus on families of crossed products of the form $L^\infty(X,\mu) \rtimes \Gamma$, the main application being to $\Gamma = \mathbb{F}_n$.

3. Deformation / approximation properties and rigidity

Group Γ	Tracial von Neumann algebra (M,τ)
Unitary representation $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$	Bimodule (or correspondence, à la Connes) ${}_M\mathcal{K}_M$
Coefficient functions $\varphi\colon \Gamma \to \mathbb{C}$ given by $\varphi(g) = \langle \pi(g)\xi, \xi\rangle$ for fixed $\xi \in \mathcal{H}$	Completely positive maps $\psi\colon M \to M$ with $\psi\circ\tau \leq \tau$ and $\psi(1) \leq 1$ (these conditions can be arranged). They all arise as $\tau(\psi(x)y) = \langle x\xi y, \xi\rangle$ for some $\xi \in_M \mathcal{K}_M$.
Amenability: there exist positive definite functions $\varphi_n\colon \Gamma \to \mathbb{C}$ with finite support and $\varphi_n \to 1$ pointwise	Amenability: there exist completely positive maps $\psi_n\colon M \to M$ with finite dimensional rank (once regarded as maps $L^2(M,\tau) \to L^2(M,\tau)$) and $\lim_{n\to\infty} \|\psi_n(x) - x\|_2 = 0$ for all $x \in M$. This notion is somewhat strong: there is a unique amenable II_1-factor.
Haagerup property: there exist positive definite functions $\varphi_n\colon \Gamma \to \mathbb{C}$ in $c_0(\Gamma)$ and $\varphi_n \to 1$ pointwise	Haagerup property: there exist completely positive maps $\psi_n\colon M \to M$ that are compact (once regarded as maps $L^2(M,\tau) \to L^2(M,\tau)$) and $\lim_{n\to\infty} \|\psi_n(x) - x\|_2 = 0$ for all $x \in M$
Relative property (T) for $\Lambda \leq \Gamma$ (rigidity): if $\varphi_n\colon \Gamma \to \mathbb{C}$ are positive definite and converge to 1 pointwise, then they converge to 1 uniformly on Λ	Relative property (T) for $P \subseteq M$ (rigidity): if $\psi_n\colon M \to M$ are completely positive maps with $\psi_n \to \mathrm{id}$ pointwise in 2-norm, then $\psi_n \to \mathrm{id}$ uniformly in 2-norm over the norm unit ball of P.
Γ has property (T) if $\Gamma \leq \Gamma$ has relative property (T)	M has property (T) if $M \subseteq M$ has relative property (T)

As the terminology indicates, we have

Theorem 3.1. Let $\Lambda \leq \Gamma$ be a subgroup, and set $M = L(\Gamma)$ and $P = L(\Lambda)$. Then $\Lambda \leq \Gamma$ has relative property (T) if and only if $P \subseteq M$ has relative property (T).

(Relative) property (T) and the Haagerup property (or amenability) are rather orthogonal:

Proposition 3.2. If $\Lambda \leq \Gamma$ has the Haagerup property (in particular, if it is amenable) and has relative property (T), then Λ is finite. Likewise, if $P \subseteq M$ has the Haagerup property (in particular, if it is amenable) and has relative property (T), then P is a (possibly infinite) direct sum of matrix algebras.

Examples 3.3. (1) $\mathbb{F}_n$ has the Haagerup property, but is not amenable if $n \geq 2$.
(2) $\mathbb{Z}^2 \leq SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ has relative property (T).
(3) $SL_n(\mathbb{Z})$ has property (T) for $n \geq 3$.

Our first example of deformation rigidity, due to Popa, is

$$A = L(\mathbb{Z}^2) \cong L^\infty(\mathbb{T}^2) \subseteq M = L(SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2) \cong L^\infty(\mathbb{T}^2) \rtimes SL_2(\mathbb{Z}).$$

Proposition 3.4. For every $\alpha \in \mathrm{Aut}(M)$, there exists a unitary $u \in \mathcal{U}(M)$ such that $u\alpha(A)u^* = A$.

Proof. (Sketch) Since $\mathbb{F}_2$ is a subgroup of Γ of finite index, it follows that Γ has the Haagerup property. Choose positive definite functions $\varphi_n \colon \Gamma \to \mathbb{C}$ in $c_0(\Gamma)$ converging to 1 pointwise. Define $\psi \colon M \to M$ by $\psi_n(au_g) = \psi_n(g)au_g$. Then $\psi_n \to \mathrm{id}$ pointwise in $\|\cdot\|_2$-norm. Let $\alpha \in \mathrm{Aut}(A)$. Since $A \subseteq M$ has relative property (T), the same is true for $\alpha(A)$. Therefore ψ_n converges to id uniformly over the norm unit ball of $\alpha(A)$. In other words, given $\varepsilon > 0$, there exists n large enough such that $\|\psi_n(\alpha(a)) - \alpha(a)\|_2 < \varepsilon$ for all $a \in A$ with $\|a\| = 1$. Since $\varphi_n \in c_0(\Gamma)$, we have $|\varphi_n(g)| < \varepsilon$ outside a finite set $\mathcal{F}$ of Γ. One can show that

$$\mathrm{dist}_{\|\cdot\|_2}\left(\alpha(a), \mathrm{span}\{Au_g \colon g \in \mathcal{F}\}\right) \leq \sqrt{2\varepsilon}$$

for all $a \in A$ with $\|a\| = 1$. A result of Popa then implies that $\alpha(A)$ and A are unitarily conjugate. $\square$

4. Cartan subalgebras in II_1-factors

Definition 4.1. $A \subseteq M$ is a *Cartan subalgebra* if it is a maximal abelian subalgebra and it is regular, that is,

$$\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \colon uAu^* = A\}$$

generates M.

Example 4.2. For an essentially free ergodic probability measure preserving action of Γ on (X, μ), the subalgebra $L^\infty(X, \mu) \subseteq L^\infty(X, \mu) \rtimes \Gamma$ is Cartan.

Theorem 4.3. (Singer, 1950). For an essentially free ergodic probability measure preserving action of Γ on (X, μ), the inclusion $L^\infty(X, \mu) \subseteq L^\infty(X, \mu) \rtimes \Gamma$ contains the same data as the orbit equivalence relation

$$\mathcal{R}(\Gamma \curvearrowright X) = \{(x, g \cdot x) \colon x \in X, g \in \Gamma\} \subseteq X \times X.$$

In other words, there is a one-to-one correspondence between isomorphisms

$$\alpha \colon L^\infty(X, \mu) \rtimes \Gamma \to L^\infty(Y, \nu) \rtimes \Lambda$$

satisfying $\alpha(L^\infty(X, \mu)) = L^\infty(Y, \nu)$ and measurable functions $\Delta \colon X \to Y$ satisfying $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$ for all $x \in X$.

An important question regarding Cartan subalgebras is whether they are unique (up to unitary conjugacy), and whether they exist at all! It turn out that they do not always exist.

Theorem 4.4. (Voiculescu). $L(\mathbb{F}_n)$ has no Cartan subalgebras.

Theorem 4.5. (Ozawa-Popa, 2006). Constructed the first examples of II_1-factors with a unique Cartan subalgebra.

Theorem 4.6. (Popa-Vaes, 2011). If $\mathbb{F}_n \curvearrowright X$ is essentially free and ergodic, then $L^\infty(X) \subseteq L^\infty(X) \rtimes \mathbb{F}_n$ is the unique Cartan subalgebra.

Corollary 4.7. For essentially free ergodic actions $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_m \curvearrowright Y$, if there exists an isomorphism

$$L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m,$$

then $n = m$.

Proof. By uniqueness of the Cartan subalgebra, one gets orbit equivalence, and actions of $\mathbb{F}_n$ and $\mathbb{F}_m$ cannot be orbit equivalent unless $n = m$. $\square$

5. A particular case of Ozawa-Popa

Recall that a subalgebra $A \subseteq M$ of a von Neumann algebra is said to be *diffuse* if it contains no minimal projections. In this section, we will prove:

Theorem 5.1. (Ozawa-Popa). Let $A \subseteq L(\mathbb{F}_n)$ be an amenable diffuse subalgebra. Then $\mathcal{N}_{L(\mathbb{F}_n)}(A)''$ is amenable. In other words, $L(\mathbb{F}_n)$ is strongly solid.

Corollary 5.2. $L(\mathbb{F}_n)$ has no Cartan subalgebras if $n > 1$.

Proof. If $A \subseteq L(\mathbb{F}_n)$ is a commutative subalgebra, then it is amenable. (Diffusenessf) By the theorem above, $N_{L(\mathbb{F}_n)}(A)''$ is amenable. Since $\mathbb{F}_n$ is not amenable, we must have $N_{L(\mathbb{F}_n)}(A)'' \neq L(\mathbb{F}_n)$, so A is not Cartan. ☐

To prove the theorem, we need to review some facts about the completely bounded and completely metric approximation properties for groups and tracial von Neumann algebras.

Definition 5.3. A function $\varphi\colon \Gamma \to \mathbb{C}$ is called a *Fourier multiplier* if the map $m_\varphi\colon L(\Gamma) \to L(\Gamma)$ given by $m_\varphi(u_g) = \varphi(g)u_g$, for $g \in \Gamma$, is completely bounded. In this case, we set $\|\varphi\|_{\mathrm{cb}} = \|m_\varphi\|_{\mathrm{cb}}$.

Definition 5.4. (Cowling-Haagerup). We say that Γ has the *completely bounded approximation property* if there exist Fourier multipliers $\varphi_n\colon \Gamma \to \mathbb{C}$ with finite support, with $\limsup\limits_{n\to\infty} \|\varphi_n\|_{\mathrm{cb}} < \infty$ and satisfying $\varphi_n \to 1$ pointwise. The minimum value of $\limsup\limits_{n\to\infty} \|\varphi_n\|_{\mathrm{cb}}$, where $(\varphi_n)_{n\in\mathbb{N}}$ ranges over all possible sequences of Fourier multipliers as above, is denoted by $\Lambda_{\mathrm{ch}}(\Gamma)$.

We say that Γ has the *completely metric approximation property* if $\Lambda_{\mathrm{ch}}(\Gamma) = 1$.

Remark 5.5. If Γ is amenable, then it has the completely metric approximation property. (Use that if $\varphi\colon \Gamma \to \mathbb{C}$ is positive definite, then $\|\varphi\|_{\mathrm{cb}} = |\varphi(1)|$.)

This notion ca be translated to von Neumann algebras as follows.

Definition 5.6. (Cowling-Haagerup). We say that a tracial von Neumann algebra (M,τ) has the *completely bounded approximation property* if there exist completely bounded linear maps $\psi_n\colon M \to M$ with finite rank, with $\limsup\limits_{n\to\infty} \|\psi_n\|_{\mathrm{cb}} < \infty$ and satisfying $\psi_n \to 1$ pointwise weakly. The minimum value of $\limsup\limits_{n\to\infty} \|\psi_n\|_{\mathrm{cb}}$, where $(\psi_n)_{n\in\mathbb{N}}$ ranges over all possible sequences of completely bounded maps as above, is denoted by $\Lambda_{\mathrm{ch}}(M,\tau)$.

We say that (M,τ) has the *completely metric approximation property* if $\Lambda_{\mathrm{ch}}(M,\tau) = 1$.

Not surprisingly, we have:

Theorem 5.7. (Cowling-Haagerup). $\Lambda_{\mathrm{ch}}(\Gamma) = \Lambda_{\mathrm{ch}}(L(\Gamma))$.

Example 5.8. (Haagerup) $\Lambda_{\mathrm{ch}}(L(\mathbb{F}_n)) = 1$.

Example 5.9. (Cowling-Haagerup) If $\Gamma \leq \mathrm{Sp}(n,1)$ is a lattice, then $\Lambda_{\mathrm{ch}}(L(\Gamma)) = 2n-1$. In particular, if $\Gamma_n \leq \mathrm{Sp}(n,1)$ and $\Gamma_m \leq \mathrm{Sp}(m,1)$ are lattices, then $L(\Gamma_n) \cong L(\Gamma_n)$ if and only if $n = m$.

Recall that if $\omega\colon M \to \mathbb{C}$ is a linear functional with $\|\omega\| \leq 1$ and such that $\omega(1)$ is close to 1, then ω is close to a state; in fact, it is close to the state $|\omega|/\|\omega\|$. If moreover $u \in \mathcal{U}(M)$ is such that $\omega(u)$ is close to 1, then $\omega \circ \mathrm{Ad}(u)$ is close to ω. (For the last statement, first $\omega \times u$ is close to ω, and then $u^* \cdot \omega$ is close to ω.)

Proof. (Of Ozawa-Popa). Set $M = L(\mathbb{F}_n)$. Fix a sequence $\psi_n\colon M \to M$ realizing the fact that M has the completely metric approximation property. For $P \subseteq M$ amenable, define functionals

$$\mu_n^P\colon P\overline{\otimes}P^{\mathrm{op}} \to \mathbb{C}$$

by $\mu_n^P(a \otimes b^{\mathrm{op}}) = \tau(\psi_n(a)b)$ for $a, b \in P$.

We claim that $\limsup_{n\to\infty} \|\mu_n^P\| \leq 1$. To see this, observe that there is a commutative diagram

$$\begin{tikzcd}
P\overline{\otimes}P^{\mathrm{op}} \arrow[r, "\psi_n\otimes\mathrm{id}"] \arrow[drr, "\mu_n^P"'] & M\overline{\otimes}P^{\mathrm{op}} \arrow[r, "\mathrm{leftmult}\otimes\mathrm{rightmult}"] & \mathcal{B}(L^2(M)) \arrow[d, "\langle\cdot\overline{1},\overline{1}\rangle"] \\
 & & \mathbb{C}.
\end{tikzcd}$$

By the completely metric approximation property, we have $\limsup_{n\to\infty} \|\psi_n \otimes \mathrm{id}\|_{\mathrm{cb}} = 1$. Since P is amenable, leftmult $\otimes$ rightmult is continuous (contractive) with respect to the spatial tensor product. Since the vector state $\langle\cdot\overline{1},\overline{1}\rangle$ has norm one, the claim follows.

By the comments before the beginning of the proof, there exist states $\omega_n^P\colon P\overline{\otimes}P^{\mathrm{op}} \to \mathbb{C}$ satisfying $\|\mu_n^P - \omega_n^P\| \to 0$. Moreover, one has

$$\omega_n^P(a\otimes b^{\mathrm{op}}) \to \tau(ab)$$

for all $a, b \in P$. For $u \in \mathcal{U}(P)$, set $\overline{u} = (u^*)^{\mathrm{op}}$. Then $\omega_n^P(u\otimes\overline{u}) \to 1$. In particular,

$$\omega_n^P \circ \mathrm{Ad}(u\otimes\overline{u}) - \omega_n^P\| \to 0$$

for all $u \in \mathcal{U}(P)$.

We claim that $\omega_n^A \circ \mathrm{Ad}(u\otimes\overline{u}) - \omega_n^A\| \to 0$ for all unitaries $u \in \mathcal{N}_M(A)$. To prove the claim, observe that, for a fixed unitary $u \in \mathcal{N}_M(A)$, the von Neumann algebra generated by A and u is again amenable (it is a certain $\mathbb{Z}$-crossed product of A). An inductive argument shows that $\mathcal{N}_M(A)$ is amenable. Apply the previous claim and the comments after it; the claim is proved.

Canonically implement $\omega_n^A\colon A\overline{\otimes}A^{\mathrm{op}} \to \mathbb{C}$ by $\xi_n \in L^2(A)\otimes L^2(A^{\mathrm{op}})$. We have thus found vectors

$$\xi_n \in L^2(A)\otimes L^2(A^{\mathrm{op}}) \subseteq L^2(M)\otimes L^2(M^{\mathrm{op}})$$

satisfying $\langle(a\otimes b^{\mathrm{op}})\xi_n, \xi_n\rangle \to \tau(ab)$ for all $a, b \in A$, and

$$\|(u\otimes\overline{u})\xi_n - \xi_n(u\otimes\overline{u})\| \to 0$$

for all $u \in \mathcal{N}_M(A)$.

For the remainder of the proof, we will need to use that the canonical representation $\mathbb{F}_n \times \mathbb{F}_n^{\mathrm{op}} \to \mathcal{B}(\ell^2(\mathbb{F}_n))$, by left and right multiplication, is contained, up to compact operators, in the left $\otimes$ right representation $\mathbb{F}_n \times \mathbb{F}_n^{\mathrm{op}} \to \mathcal{B}(\ell^2(\mathbb{F}_n)\otimes\ell^2(\mathbb{F}_n^{\mathrm{op}}))$. Roughly speaking this means that ${}_{L(\mathbb{F}_n)}\ell^2(\mathbb{F}_n)_{L(\mathbb{F}_n)}$ is contained in

$${}_{L(\mathbb{F}_n)\otimes 1}(\ell^2(\mathbb{F}_n)\otimes\ell^2(\mathbb{F}_n^{\mathrm{op}}))_{1\otimes L(\mathbb{F}_n)}$$

up to compact operators. Apply this to the first tensor factor of ξ_n, using diffuseness, to conclude that the trivial representation is contained in the left regular representation. Hence $\mathcal{N}_M(A)''$ is amenable, concluding the proof. $\square$

6. Popa's intertwining by bimodules

We introduce a notion of weak inclusion of subalgebras relative to an ambient II_1-factor.

Let M be a II_1-factor and let $A, B \subseteq M$. The following generalizes the existence of a unitary $u \in \mathcal{U}(M)$ such that $u^*Au \subseteq B$. We denote by $E_B\colon M \to B$ the unique trace preserving conditional expectation; it is the orthogonal projection of $L^2(M)$ onto $L^2(B)$.

Definition 6.1. We write $A \precsim_M B$ if there exist a natural number $n \in \mathbb{N}$, elements $x_1, \ldots, x_n \in M$, and a positive number $\delta > 0$, satisfying

$$\sum_{j,k=1}^{n} \|E_B(x_j^* a x_k)\|_2 \geq \delta$$

for all $a \in \mathcal{U}(A)$.

Theorem 6.2. (Popa, 2001). Let A and B be subalgebras of a II_1-factor M.

(1) $A \precsim_M B$ if and only if there exist projections $p \in A$ and $q \in B$, a nonzero partial isometry $v \in pMq$, and a normal homomorphism $\theta\colon pAp \to qBq$ such that $av = v\theta(a)$ for all $a \in pAp$.
(2) If A and B are Cartan, then $A \precsim_M B$ if and only if there exists a unitary $u \in \mathcal{U}(M)$ satisfying $uAu^* \subseteq B$ (and hence $uAu^* = B$ by maximality).

In the proof, one considers $\mathcal{L} = (B \cup A^{\mathrm{op}})''$ acting on ${}_B L^2(M)_{A^{\mathrm{op}}}$.
We now define Ozawa's class $\mathcal{S}$.

Definition 6.3. (Ozawa). A group Γ is said to be in the class $\mathcal{S}$ if it is exact and there exists an isometry $V\colon \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ such that $V \circ \lambda_g \circ \rho_h - (\lambda_g \otimes \rho_h) \circ V$ is compact for all $g, h \in \Gamma$.

Remark 6.4. If $\Gamma \in \mathcal{S}$, then

$$C_r^*(\Gamma) \otimes_{\min} C_r^*(\Gamma) \to \mathcal{B}(\ell^2(\Gamma))/\mathcal{K}(\ell^2(\Gamma))$$

is a homomorphism.

Example 6.5. Hyperbolic groups, in particular free groups, belong to the class $\mathcal{S}$.

An obstruction to belonging to the class $\mathcal{S}$ is the existence of an infinite subgroup $\Lambda \leq \Gamma$ with

$$C_\Gamma(\Lambda) = \{g \in \Gamma\colon gh = hg \text{ for all } h \in \Gamma\}$$

nonamenable. Indeed, take $h_n \in \Lambda$ such that $h_n \to \infty$. Consider

$$\xi_n = V(\delta_{h_n}) \in \ell^2(\Gamma) \otimes \ell^2(\Gamma).$$

Theorem 6.6. Let Γ be a group in the class $\mathcal{S}$, and let $\Lambda \leq \Gamma$ be an infinite subgroup. For $g \in C_\Gamma(\Lambda)$, we have

$$\lim_{n\to\infty} \|(\lambda_g \otimes \rho_g)\xi_n - \xi_n\| = 0.$$

In particular, $C_\Gamma(\Lambda)$ is amenable.

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