EXPANDERS, GHOSTS, AND EXACTNESS

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Abstract. This document is based on a mini course given by Rufus Willett at the YMC*A at the Westfälische Wilhelms-Universität Münster, Germany, July 24–29, 2016. These notes were prepared by Eusebio Gardella.

Contents

Note from Rufus Willett: *As I hope is clear, essentially none of the results in the lectures are due to me. Having said that, the exposition is sometimes quite different to the original sources, and is largely based on joint work with Paul Baum, Erik Guentner, John Roe, Ján Špakula, and Guoliang Yu.*

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1. Introduction

Expanders are sparse, highly connected graphs with surprising applications in many parts of mathematics. This lecture series will introduce expanders and their relationship with group C^*-algebras and Kazhdan's property (T), as well as the so-called (by Yu) *ghost operators*. We will revisit and provide a unified viewpoint on some old(ish) (counter-)examples of Higson, Voiculescu, and Wassermann, connected to K-theory, quasi-diagonality, and exactness. Connections to coarse geometry and the infamous non-exact groups of Gromov will also be discussed.

2. The heat equation (on graphs)

Let $G = (V, E)$ be a finite, connected and undirected graph. Consider the *graph Laplacian* $\Delta_G \colon \ell^2(V) \to \ell^2(V)$ given by

$$\Delta_G(\delta_v) = \sum_{w \in V \colon \{v,w\} \text{ is an edge}} \delta_v - \delta_w \quad \text{for} \quad v \in V.$$

Remark 2.1. Compare Δ_G to $\Delta = -\frac{\partial}{\partial x^2} - \frac{\partial}{\partial y^2}$, for which

$$\Delta(f)(x) = \lim_{r\to 0} \frac{1}{4r^2}\left(f(x) - \frac{1}{\mathrm{vol}(S_r(x))}\int_{S_r(x)} f(y)\mathrm{d}y\right),$$

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where $S_r(x)$ denotes the sphere of radius r centered at x.

Let $f\colon V\times[0,\infty)\to\mathbb{C}$ be a *heat flow*, that is, a function satisfying $\frac{\partial f}{\partial t}+\Delta_G f=0$. The solution with initial condition $f_0\in\ell^2(V)$ is given by $t\mapsto e^{-t\Delta}f_0$. For $f\in\ell^2(V)$, we have

$$\langle\Delta_G(f),f\rangle=\sum_{w\in V\colon\{v,w\}\text{ is an edge}}|f(v)-f(w)|^2,$$

so Δ_G is positive, with eigenvalues $0<\lambda_1\leq\lambda_2\leq\cdots$. Its kernel is the space of constant functions on V. If we let $p_V\colon\ell^2(V)\to\ker(\Delta_G)$ denote the orthogonal projection, then

$$\|e^{-t\Delta_G}-p_V\|=e^{-\lambda_1 t}$$

for all $t>0$. Morally speaking, the eigenvalue λ_1 governs connectedness of G, in the sense that a large value of λ_1 indicates that G is highly connected.

In terms of matrices, we have

$$\Delta_G=\begin{pmatrix}\deg & -1 & 0 & & & \\ -1 & \deg & -1 & & & \\ 0 & -1 & \deg & & & \\ & & & \ddots & & \\ & & & & \deg & -1\\ & & & & -1 & \deg\end{pmatrix}\quad\text{and}\quad p_V=\frac{1}{|V|}\begin{pmatrix}1&1&1&\cdots&1&1\\1&1&1&\cdots&1&1\\1&1&1&\cdots&1&1\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\1&1&1&\cdots&1&1\\1&1&1&\cdots&1&1\end{pmatrix}.$$

3. Groups

Let Γ be a finitely generated group, and let $S=S^{-1}$ be a finite generating set in Γ. Denote by $\mathbb{C}[\Gamma]$ the group algebra of Γ, that is,

$$\mathbb{C}[\Gamma]=\left\{\sum_{g\in\Gamma}\lambda_g u_g\colon\lambda_g\in\mathbb{C},\lambda_g\neq0\text{ for finitely many }g\in G\right\},$$

with the obvious addition and multiplication, and adjoint given by $(\lambda u_g)^*=\overline{\lambda}u_{g^{-1}}$ for $\lambda\in\mathbb{C}$ and $g\in G$.

Define the *group Laplacian* $\Delta_\Gamma=\sum_{s\in S}1-u_s\in\mathbb{C}[\Gamma]$. Since $\Delta_\Gamma=\frac{1}{2}\sum_{s\in S}(1-u_s)(1-u_s^*)$, it follows that Δ_Γ is positive in any $*$-representation of $\mathbb{C}[\Gamma]$. In particular, Δ_Γ is positive in the maximal completion $C^*(\Gamma)$ of $\mathbb{C}[\Gamma]$.

Let $H\leq\Gamma$ be a (normal) subgroup of finite index, and define a graph G by setting $V=\Gamma/H$, and declaring that $\{g_1H,g_2H\}$ is an edge if there exists $s\in S$ such that $sg_1H=g_2H$. Note that the $*$-representation $\mathbb{C}[\Gamma]\to\mathcal{B}(\ell^2(V))$ by permutation maps Δ_Γ to Δ_G.

Example 3.1. Let $\Gamma=\mathbb{Z}$ with $S=\{1,-1\}$, and $H=n\mathbb{Z}$. Then G is the cyclic graph C_n with n vertices and $n-1$ edges, and $\Delta_{\mathbb{Z}}=2-u_1-u_{-1}$. Moreover,

$$\begin{aligned}\mathrm{sp}(\Delta_{C_n})=\mathrm{sp}(\Delta_{\mathbb{Z}_n})&=\bigcup_{\pi\text{ irrep of }\mathbb{Z}_n}\mathrm{sp}(\pi(\Delta_{C_n}))\\&=\left\{2-e^{\frac{-2\pi ik}{n}}-e^{\frac{2\pi ik}{n}}\colon k=0,\ldots,n-1\right\}\\&=\{2-2\cos(2\pi k/n)\colon k=0,\ldots,n-1\}.\end{aligned}$$

Note that $\lambda_1=2-2\cos(2\pi/n)$, which tends to zero as n tends to infinity. Intuitively, this means that as n gets larger, it takes longer for heat to propagate on the graph C_n.

Definition 3.2. A group Γ is said to have *property (T)* if there exists $c>0$ such that $\mathrm{sp}(\pi(\Delta_\Gamma))\subseteq\{0\}\cup[c,\infty)$ in any $*$-representation π of $\mathbb{C}[\Gamma]$.

Example 3.3. $\mathrm{sp}_{C^*(\mathbb{Z})}(\Delta_{\mathbb{Z}})=[0,4]$, so $\mathbb{Z}$ does not have property (T).

Suppose that Γ has property (T), and let $(H_n)_{n\in\mathbb{N}}$ be a sequence of finite index normal subgroups such that $[\Gamma : H_n] \to \infty$ as $n \to \infty$. Then the sequence $V_n = \Gamma/H_n$ of graphs (obtained by prescribing a symmetric finite generating set $S \subseteq \Gamma$) satisfies:

(1) $|V_n| \to \infty$;
(2) all degrees (how many edges come out of a given vertex) are uniformly bounded;
(3) $\lambda_1(V_n)$ is *uniformly* bounded below.

Example 3.4. $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ and $\Gamma/H_n = \mathrm{SL}_3(\mathbb{Z}_n)$.

4. Expanders

Definition 4.1. A sequence $(V_n)_{n\in\mathbb{N}}$ of finite connected graphs is said to be an *expander* if conditions (1), (2) and (3) above are satisfied.

Given an expander $(V_n)_{n\in\mathbb{N}}$, set $\ell^2(V) = \bigoplus_{n\in\mathbb{N}} \ell^2(V_n)$, and $\Delta_V = \bigoplus_{n\in\mathbb{N}} \Delta_{V_n}$. Since $(V_n)_{n\in\mathbb{N}}$ is an expander, we conclude that $e^{-t\Delta_V}$ converges to $p_V := \bigoplus_{n\in\mathbb{N}} p_{V_n}$ not just strongly, but in *norm* (that is, the norm convergence on each block is uniform).

The operators Δ_V and p_V are block diagonals; Δ_V has nonzero entries only very close to the diagonal, while p_V has nonzero entries everywhere within each block, and the entries are always $1/|V_n|$, so they vanish at infinity.

5. Uniform Roe algebras

A metric space X is said to be of *bounded geometry* if $\sup_{x\in X} |B_r(x)| < \infty$ for every $r > 0$.

Example 5.1. Given an expander $(V_n)_{n\in\mathbb{N}}$, set $X = \bigsqcup_{n\in\mathbb{N}} V_n$ and define a metric d on it by

$$d(v,w) = \begin{cases} \text{length of the shortest path}, & \text{if such a path exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Then X has bounded geometry by the assumption of bounded degree.

Definition 5.2. Given $T \in \mathcal{B}(\ell^2(X))$, its *propagation* is defined as $\mathrm{prop}(T) = \sup\{d(x,y) \colon T_{x,y} \neq 0\}$. Define $\mathbb{C}_u[X] = \{T \in \mathcal{B}(\ell^2(X)) \colon \mathrm{prop}(T) < \infty\}$. Then the *uniform Roe algebra* of X is $C^*_u(X) = \overline{\mathbb{C}_u[X]} \subseteq \mathcal{B}(\ell^2(X))$.

Example 5.3. For an expander $(V_n)_{n\in\mathbb{N}}$ and $X = \bigsqcup_{n\in\mathbb{N}} V_n$, we have $\mathrm{prop}(\Delta_V) = 1$, so $\Delta_V \in \mathbb{C}_u[X]$. On the other hand, $p_V \in C^*_u(X) \setminus \mathbb{C}_u[X]$.

Note that any $T \in \mathbb{C}_u[X]$ can be written as $T = \sum_{j=1}^{N} f_j v_j$, where:

(a) $f_j \in \ell^\infty(X)$;
(b) each v_j is a translation, that is, a partial isometry with entries in $\{0,1\}$, with at most one 1 in each row or column.

There are many operators in $\mathcal{B}(\ell^2(X))$ that do not belong to $\mathbb{C}^*_u(X)$. For instance, consider $\sigma \in \mathcal{B}(\ell^2(\mathbb{N}))$ given by $\sigma(\delta_n) = \delta_{2^n}$ for all $n \in \mathbb{N}$.

6. Exactness

Definition 6.1. A *Cartan subalgebra* of a C^*-algebra A is a maximal abelian subalgebra $C \subseteq A$ with a conditional expectation $E \colon A \to C$, such that the normalizer $\mathcal{N}_A(C) = \{a \in A \colon aCa^* \subseteq C\}$ generates A as a C^*-algebra.

An open subset $U \subseteq \widehat{C}$ is said to be *invariant* if $aC_0(U)a^* \subseteq C_0(U)$ for all $a \in \mathcal{N}_A(C)$.

Example 6.2. Let X be a metric space of bounded geometry, and set $A = C_u^*(X)$ and $C = \ell^\infty(X) \subseteq A$. There is a conditional expectation $E\colon A \to C$ given by deleting the off-diagonal entries. One can also check that

$$\mathcal{N}_A(C) = \{fv\colon f \in \ell^\infty(X), v \text{ is a translation}\},$$

so it generates A. We conclude that C is a Cartan subalgebra of A. Furthermore, we have $\widehat{C} = \beta X$ (Stone-Čech compactification), and the open set $U = X \subseteq \beta X$ is invariant.

Given $U \subseteq \widehat{C}$ invariant, associate to it the ideal I_U of A given by

$$I_U = \{fa\colon f \in C_0(U), a \in \mathcal{N}_A(C)\}.$$

Define B_U as the Hausdorff completion of A in the seminorm

$$\|a\|_{B_U} = \sup\{\|\pi_x(a)\|\colon x \in \widehat{C} \setminus U\},$$

where $\pi_x\colon A \to \mathcal{B}(\mathcal{H})$ is the GNS representation associated to the state $\mathrm{ev}_x \circ E\colon A \to C \to \mathbb{C}$. One gets a short sequence

$$0 \to I_U \to A \to B_U \to 0,$$

which is exact except possibly at A: the kernel of $A \to B_U$ may strictly contain the ideal I_U.

Definition 6.3. We say that the pair (A, C) is *exact* if for all invariant open sets $U \subseteq \widehat{C}$, the short sequence above is exact.

We will check that expanders are not exact. So let $(V_n)_{n\in\mathbb{N}}$ be an expander, and set $X = \bigsqcup\limits_{n\in\mathbb{N}} V_n$. We let $A = C_u^*(X)$, $C = \ell^\infty(X)$ and $U = X$. Then $I_U = \mathcal{K}(\ell^2(X)) \cap C_u^*(X)$.

Definition 6.4. Adopt the notation from above. We write $G^*(X) = \ker(A \to B_u)$, and call it the *ghost ideal*. Its elements are called *ghost operators*.

We claim that $G^*(X)$ also equals $\{T \in C_u^*(X)\colon T_{x,y} \to 0 \text{ as } x, y \to \infty\}$. This follows from the fact that

$$G^*(X) = \bigcap_{x\in\beta X\setminus X} \ker(\pi_x) = \{a \in C_u^*(X)\colon (\mathrm{ev}_x \circ E)(bac) = 0 \text{ for all } b, c \in A \text{ and for all } x \in X\}.$$

By choosing b and c above to be translations, the condition $(\mathrm{ev}_x \circ E)(bac) = 0$ for all $b, c \in A$ and for all $x \in X$ is seen to be equivalent to every diagonal of a being in c_0. As a is a limit of finite propagation operators, it is automatically c_0 in every off-diagonal direction.

From this description, it follows that $\mathcal{K}(\ell^2(X)) \subseteq G^*(X)$, since the finite rank operators are in $G^*(X)$. Finally, it is clear that the averaging projection p_V is in $G^*(X) \setminus \mathcal{K}(\ell^2(X))$, so expanders are *not* exact.

Some consequences of the existence of non-exact pairs include:

1. Non (inner) exact groupoids exist (Skandalis);
2. There exists a quasidiagonal C^*-algebra $A \subseteq \mathcal{B}(\ell^2)$ such that $A/A\cap\mathcal{K}$ is not quasidiagonal (Wassermann);
3. There exists a quasidiagonal operator which cannot be approximated by operators that generate finite dimensional C^*-algebras (Voiculescu);
4. Examples where Ext is not a group (Anderson, Wassermann);
5. Counterexamples to Baum-Connes type conjectures (Higson, Higson-Lafforgue-Skandalis); this is the subject of the following section.

7. Coarse Baum-Connes conjecture

Let $(V_n)_{n\in\mathbb{N}}$ be an expander, and set $X = \bigsqcup\limits_{n\in\mathbb{N}} V_n$. Assume that $\mathrm{girth}(V_n) \to \infty$ as $n \to \infty$. Geometrically, this implies that V_n looks more and more like a tree. (To construct such an expander, use either property (τ) for a free group[1], or counting arguments.)

[1]Given a finitely generated discrete group Γ and a decreasing chain $\{H_n\}_{n\in\mathbb{N}}$ of finite index normal subgroups satisfying $\bigcap_{n\in\mathbb{N}} H_n = \{e\}$, we say that Γ has *property (τ) with respect to* $\{H_n\}_{n\in\mathbb{N}}$ if there exists $c > 0$ such that $\mathrm{sp}(\pi(\Delta_\Gamma)) \subseteq \{0\} \cup [c, \infty)$ for every $*$-representation π of $\mathbb{C}[\Gamma]$ that factors through $\mathbb{C}[\Gamma/H_n]$ for some $n \in \mathbb{N}$. Clearly property (T) implies property (τ)

Recall that a trace τ on a unital C^*-algebra A induces a group homomorphism $\tau_0\colon K_0(A) \to \mathbb{C}$ given by $\tau_0([p]) = (\tau \otimes \mathrm{tr}_n)(p)$ for $p \in A \otimes M_n$ (here, tr_n is the unnormalized trace on M_n). Similar remarks apply to "traces" $A \to \mathbb{C}_\infty = \prod_{n\in\mathbb{N}} \mathbb{C} / \bigoplus_{n\in\mathbb{N}} \mathbb{C}$ (algebraic sequence algebra).

Conjecture 7.1. (Coarse Baum-Connes). The assembly map $\mu\colon \lim_{r\to\infty} K_*^u(P_r(X)) \to K_*(C_u^*(X))$ is an isomorphism in all degrees.

(The left-hand side of the Baum-Connes conjecture will not be explained; it is a topological object.)

Lemma 7.2. Any class in $K_0(C_u^*(X))$ that belongs to the image of μ can be represented by idempotents from $\mathbb{C}_u[X]$.

Proof. Straightforward, given the definitions of the objects involved. ☐

Theorem 7.3. (Higson). Let $(V_n)_{n\in\mathbb{N}}$ be an expander with $\mathrm{girth}(V_n) \to \infty$, and set $X = \bigsqcup_{n\in\mathbb{N}} V_n$. Then μ is not surjective. In particular, the coarse Baum-Connes conjecture fails for X.

Proof. The strategy is as follows: construct (not necessarily everywhere continuous) traces $\tau, \widetilde{\tau}\colon C_u^*(X) \to \mathbb{C}_\infty$ satisfying

$$\tau(p_V) = [(1,1,1,\dots)] \quad \text{and} \quad \widetilde{\tau}(p_V) = [(0,0,0,\dots)],$$

such that τ and $\widetilde{\tau}$ are defined on all of, and agree on, $\mathbb{C}_u[X]$. Once we do this, the result will follow, since we will conclude that $[p_V]$ does not belong to the image of μ.

Construction of τ. Define $\tau_n\colon C_u^*(X) \to \mathcal{B}(\ell^2(V_n)) \to \mathbb{C}$ to be the composition of tr_n with the compression by $\mathrm{id}_{\ell^2(V_n)}$. Set $\tau = \prod_{n\in\mathbb{N}} \tau_n\colon C_u^*(X) \to \mathbb{C}_\infty$, and observe that $\tau([p_V]) = [(1,1,\dots)]$, since p_V is compressed to p_{V_n}, which has rank one.

Construction of $\widetilde{\tau}$. Let $\widetilde{V}_n$ denote the (vertex set of) the universal cover of V_n; this is necessarily a tree. Denote by $\pi\colon \widetilde{V}_n \to V_n$ the quotient map, and let Γ_n be the group of deck transformations, so that $\widetilde{V}_n/\Gamma_n \cong V_n$. Observe that π_n restricts to an isometry on every ball of radius at most $\mathrm{girth}(V_n)/2$. (This is because the shortest path between two points in $\widetilde{V}_n$ is necessarily mapped to the shortest path between the images of the endpoints, since there can be no cycles in that ball.)

Define $\phi_n\colon \mathbb{C}_u[V_n] \to \mathbb{C}_u[\widetilde{V}_n]$ by

$$\phi(T)_{x,y} = \begin{cases} T_{\pi(x),\pi(y)}, & \text{if } d(x,y) < \mathrm{prop}(T); \\ 0, & \text{else.} \end{cases}$$

This map is linear and lands in the fixed point subalgebra $\mathbb{C}_u[\widetilde{V}_n]^{\Gamma_n}$. Define $\phi\colon \mathbb{C}_u[V] \to \prod_{n\in\mathbb{N}} \mathbb{C}_u[\widetilde{V}_n]^{\Gamma_n}$ by $\phi = \prod_{n\in\mathbb{N}} \phi_n$. This is not a $*$-homomorphism, but multiplication is preserved when $\mathrm{girth}(V_n) >> \mathrm{prop}(T)$. Therefore, the induced map on the algebraic quotients

$$\phi\colon \mathbb{C}_u[V] \longrightarrow \frac{\prod_{n\in\mathbb{N}} \mathbb{C}_u[\widetilde{V}_n]^{\Gamma_n}}{\bigoplus_{n\in\mathbb{N}} \mathbb{C}_u[\widetilde{V}_n]^{\Gamma_n}}$$

is a $*$-homomorphism (because π_n restricts to an isometry on balls of radius at most $\mathrm{girth}(V_n)/2$).

Note that there exists a $*$-isomorphism

$$\mathbb{C}_u[\widetilde{V}_n]^{\Gamma_n} \to \mathbb{C}[\Gamma_n] \otimes_{\mathrm{alg}} \mathcal{B}(\ell^2(V_n)) \quad \text{given by} \quad T \mapsto \sum_{g\in\Gamma_n} \rho_g \otimes \chi_{D_n} u_g T \chi_{D_n},$$

where

- $\rho\colon \Gamma_n \to \mathbb{C}[\Gamma_n]$ is the right regular representation;

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for every family $\{H_n\}_{n\in\mathbb{N}}$. On the other hand, for every finitely generated free group $\mathbb{F}_m$, there exists a chain $\{H_n\}_{n\in\mathbb{N}}$ as above such that $\mathbb{F}_m$ has property (τ) with respect to it, while no free group has property (T). Finally, observe that property (τ) (with respect to a given chain) for a group Γ is precisely the condition needed to obtain an expander from the group using the Cayley graphs of the quotients by subgroups in the given chain.

- $D_n \subseteq \widetilde{V}_n$ is such that $\pi_n|_{D_n} \colon D_n \to V_n$ is an isomorphism;
- $u\colon \Gamma_n \to \mathcal{U}(\ell^2(\widetilde{V}_n))$ is the canonical action.

Observe that this isomorphism extends to the C^*-algebra completions since it is unitarily implemented.

Define $\widetilde{\tau}$ as the composition

$$\mathbb{C}_u[V] \xrightarrow{\phi} \frac{\prod_{n\in\mathbb{N}} \mathbb{C}_u[\widetilde{V}_n]^{\Gamma_n}}{\bigoplus_{n\in\mathbb{N}} \mathbb{C}_u[\widetilde{V}_n]^{\Gamma_n}} \xrightarrow{\cong} \frac{\prod_{n\in\mathbb{N}} \mathbb{C}[\Gamma_n]\otimes_{\mathrm{alg}}\mathcal{B}(\ell^2(V_n))}{\bigoplus_{n\in\mathbb{N}} \mathbb{C}[\Gamma_n]\otimes_{\mathrm{alg}}\mathcal{B}(\ell^2(V_n))} \xrightarrow{\prod_{n\in\mathbb{N}} \tau_{\Gamma_n}\otimes \mathrm{tr}_n} \mathbb{C}_\infty.$$

We claim that τ and $\widetilde{\tau}$ agree on $\mathbb{C}_u[V]$ (this is known as Atiyah's Γ-index theorem). This follows from the fact that $\widetilde{\tau}$, when restricted to the fundamental domain D_n, picks up the trace tr_n on $\mathcal{B}(\ell^2(V_n))$, which is how τ acts on V_n.

To finish the proof, we need to extend $\widetilde{\tau}$ to $C^*_u(V) \to \mathbb{C}_\infty$, and also show that $\widetilde{\tau}(p_V) = 0$. Both facts will be shown using the following claim.

Claim: for all $r > 0$ there exists $s > 0$ such that for all $T \in \mathbb{C}_u[V]$ with $\mathrm{prop}(T) \le r$, we have

$$\|\phi(T)\| \le 2 \limsup_{n\to\infty} \sup_{A\subseteq V_n, \mathrm{diam}(A)\le s} \|\chi_A T \chi_A\|.$$

The claim follows from the following (easy) fact: for all $r > 0$ there exists $s > 0$ such that for all $n \in \mathbb{N}$, there exists a decomposition $\widetilde{V_n} = A \sqcup B$ such that $A = \bigsqcup_{i\in I} A_i$ and $B = \bigsqcup_{j\in J} B_j$, with $\mathrm{diam}(A_i) \le s$, $\mathrm{diam}(B_j) \le s$ and $d(A_i, A_{i'}) > r$, $d(B_j, B_j{}') > r$ for all $i, i' \in I$ and $j, j' \in J$ with $i \ne i'$ and $j \ne j'$. (To show this, decompose the tree $\widetilde{V_n}$ into annuli with radii roughly $3r$.)

Assuming the claim, let us finish the proof of the theorem. We immediately get $\|\phi\| \le 2$, so ϕ is bounded (and hence $\|\phi\| \le 1$ since it is a $*$-homomorphism). To show $\widetilde{\tau}(p_V) = 0$, it suffices to show $\phi(p_V) = 0$. Fix $\varepsilon > 0$ and find $T \in \mathbb{C}_u[V]$ with $\|T - p_V\| < \varepsilon$. In particular, we may assume that the coefficients of T are (eventually) at most 2ε. The expression

$$\sup_{A\subseteq V_n, \mathrm{diam}(A)\le s} \|\chi_A T \chi_A\|,$$

for n large enough, can be bounded by something that becomes arbitrarily small with ε. Indeed, relative to the size of V_n, the set A becomes small, and the entries of T are less than 2ε. So $\|\chi_A T \chi_A\|$ is smaller than 2ε for large enough n, and hence $\|\phi(T)\| \le 2\varepsilon$. Since ε is arbitrary, we conclude that $\phi(p_V) = 0$, as desired. This finishes the proof. ∎

8. Exact groups

Definition 8.1. Let Γ be a discrete, finitely generated group, and let $S = S^{-1} \subseteq \Gamma$ be a finite generating set. The *Cayley graph* of Γ (with respect to S) is the graph with vertex set Γ, such that $\{g, h\}$ is an edge if there exists $s \in S$ with $gs = h$.

The induced metric on this graph is called the *word metric* on Γ, and we write $|\Gamma|$ for Γ considered as a metric space with the word metric.

Remark 8.2. The metric space $|\Gamma|$ is independent of S, up to bi-Lipschitz equivalence.

Recall that a ghost operator $T \in \mathbb{C}^*_u(X)$ is one satisfying $T_{x,y} \to 0$ as $x, y \to \infty$; see Definition 6.4 and the comments after it.

Definition 8.3. ((With apologies to) Kirchberg-Wassermann). A discrete, finitely generated group Γ is said to be *exact* if $C^*_u(|\Gamma|)$ has no non-compact ghost operators.

Any finite group is (trivially) exact.

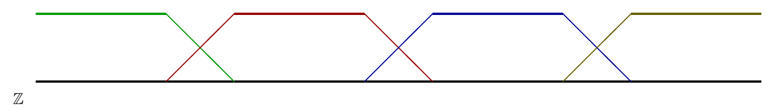
Theorem 8.4. The group $\mathbb{Z}$ is exact.

Proof. We claim that for every positive integer $N > 0$, there exist $S_N > 0$ and functions $\phi_i\colon \mathbb{Z} \to [0, 1]$, for $i \in I$, satisfying

(1) $\sum_{i\in I} \phi_i(x)^2 = 1$ for all $x \in \mathbb{Z}$;

(2) $\mathrm{diam}(\mathrm{supp}(\phi_i)) \le S_N$ for all $i \in I$;
(3) Whenever $x, y \in \mathbb{Z}$ satisfy $d(x,y) < N$, then $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| < \frac{1}{N}$.

To prove the claim, it suffices to scale and square-root the following picture, where $\mathbb{Z}$ is "zoomed out" enough so that it looks like the real line (in particular, the slopes cover "many" points in $\mathbb{Z}$):



Observe that condition (3) only involves at most two consecutive functions.

Write $m\colon \ell^\infty(\mathbb{Z}) \to \mathcal{B}(\ell^2(\mathbb{Z}))$ for the canonical inclusion as multiplication operators. Define $M_N\colon \mathcal{B}(\ell^2(\mathbb{Z})) \to \mathcal{B}(\ell^2(\mathbb{Z}))$ by $M_N(T) = \sum_{i\in I} m_{\phi_i} T m_{\phi_i}$ for $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$, where the convergence is in the strong operator topology. Note that:

(a) $\|M_N\| = 1$;
(b) $\mathrm{prop}(M_N(T)) \le S_N$ for all N (use (2) above);
(c) If $T \in \mathbb{C}_u[|\mathbb{Z}|]$, the fact that $\sum_{i\in I} \phi_i^2 = 1$ implies that

$$\|M_N(T) - T\| = \left\| \sum_{i\in I} [\phi_i, T]\phi_i \right\| \to 0 \quad \text{as} \quad N \to \infty.$$

From (a) and (c) we deduce that $M_N(T)$ converges to T in *norm*, as $N \to \infty$, whenever $T \in C_u^*[|\mathbb{Z}|]$. In particular, if T is a ghost, then so is $M_N(T)$. Since any ghost with finite propagation is necessarily compact, we deduce from (b) that $M_N(T)$ is compact. Since T is the norm limit of the $M_N(T)$, it is compact. We conclude that every ghost operator in $C_u^*(|\mathbb{Z}|)$ is compact, and hence $\mathbb{Z}$ is exact. $\square$

The method used in the proof of Theorem 8.4 is as general as it can be:

Theorem 8.5. (Dadarlat-Guentner, Ozawa, Sako). A finitely generated discrete group Γ is exact if and only if for every positive integer $N > 0$, there exist $S_N > 0$ and functions $\phi_i\colon \Gamma \to [0,1]$, for $i \in I$, satisfying

(1) $\sum_{i\in I} \phi_i(x)^2 = 1$ for all $x \in \Gamma$;
(2) $\mathrm{diam}(\mathrm{supp}(\phi_i)) \le S_N$ for all $i \in I$;
(3) Whenever $x, y \in \Gamma$ satisfy $d(x,y) < N$, then $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| < \frac{1}{N}$.

Corollary 8.6. Exactness is preserved under extensions, subgroups, free products, etc. In particular, free groups, linear groups (harder; due to Higson-Guentner-Weinberger) and amenable groups are exact.

Non-exact groups also exist! We sketch the construction of one family of such groups in the next section.

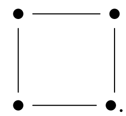
9. Gromov's monster groups

The general strategy for constructing non-exact groups is as follows. Let Γ be a finitely generated discrete group and let $(V_n)_{n\in\mathbb{N}}$ be an expander such that there are isometric embeddings $V_n \hookrightarrow \Gamma$ for all $n \in \mathbb{N}$ (the existence of such an expander is not obvious). Then there is an embedding

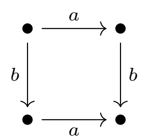
$$C_u^*\left(\bigsqcup_{n\in\mathbb{N}} V_n\right) \hookrightarrow C_u^*(|\Gamma|),$$

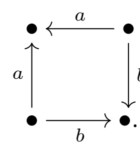
and since $C_u^*\left(\bigsqcup_{n\in\mathbb{N}} V_n\right)$ is known to contain non compact ghost operators (see Section 6), the same is true for $C_u^*(|\Gamma|)$. Hence Γ is not exact.

Expanders as above exist, but their existence is not easy to show. We sketch the idea. Start with a finite graph V. For example, it could be



Label the edges with arrows and letters from a finite alphabet S such as $S = \{a, b\}$, for example:





Define Γ to be the group generated by S with relations given by all words gotten by walking around cycles in V. For example, in the first example from above, we have $\Gamma = \langle a, b \colon aba^{-1}b^{-1} = 1\rangle$, so $\Gamma = \mathbb{Z}^2$. In the second case, we get $\Gamma = \langle a, b \colon abb^{-1}a^{-1} = 1\rangle$ (this is the trivial relation), so $\Gamma = \mathbb{F}^2$.

One gets a map $V \to \Gamma$ by picking a basepoint and following the arrows. In the first example, this gives an isometry, but in the second one it gives something degenerate (since there are no non-trivial cycles).

Theorem 9.1. (Gromov, Ollivier, Delzant-Arzhantseva, Osajda). Let $(V_n)_{n\in\mathbb{N}}$ be an expander such that there exists $c > 0$ with $\mathrm{girth}(V_n) > c \cdot \mathrm{diam}(V_n)$. (It is not obvious that such thing exists, but it can be constructed using free groups and property (τ).) Then one can choose labels on (a subsequence of) $(V_n)_{n\in\mathbb{N}}$ such that iterating the above procedure produces isometries $V_n \hookrightarrow \Gamma$.

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