TOPOLOGICAL AND ALGEBRAIC REGULARITY PROPERTIES OF NUCLEAR C*-ALGEBRAS.

Course given by **Wilhelm Winter** at the
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Notes taken by Eusebio Gardella.

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Introduction.

There are several notions of "topological dimension" for noncommutative C^*-algebras, each with their own strengths. Recent work of Wilhelm Winter and collaborators has resulted in a new "nuclear" dimension that is very natural, generalizes classical covering dimension of spaces, and is known to be finite (in fact, strictly less than 6) for all simple, nuclear C^*-algebras that have been classified so far. This work is inspired by classification, but has led to substantial insights into the structure of arbitrary nuclear C^*-algebras (i.e., inspired by said work, Kirchberg has proved they enjoy a stronger approximation property than was previously known) and a new definition of "Rokhlin dimension" for homeomorphisms of compact spaces which is purely topological and extends the classical Rokhlin property (which is the zero-dimensional case).

Given the history of applications of operator algebras to dynamics, from Connes-Feldman-Weiss to Popa's groundbreaking work on the W^*-side and Giordano-Putnam-Skau and related results on the C^*-side, the above breakthroughs are particularly exciting.

1. Nuclearity: approximation and perturbation.

Throughout this section, A and B will C^*-algebras.

Definition 1.1. A linear map $\varphi : A \to B$ is said to be
(a) *contractive*, if $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$.
(b) *positive*, if $\varphi(a) \geq 0$ whenever $a \geq 0$.
(c) *completely contractive*, if $\varphi^{(n)} : M_n(A) \to M_n(B)$ is contractive for every $n \in \mathbb{Z}_{>0}$.
(d) *completely positive*, if $\varphi^{(n)} : M_n(A) \to M_n(B)$ is positive for every $n \in \mathbb{Z}_{>0}$.

Remark 1.2. If $\varphi : A \to B$ is completely positive and contractive, then it is also completely contractive.

Definition 1.3. Two positive elements c, c' in a C^*-algebra C are said to be *orthogonal*, written $c \perp c'$, if $cc' = 0$. A completely positive map $\varphi : A \to B$ is said to have *order zero*, if $\varphi(a) \perp \varphi(b)$ whenever $a \perp b$.

Theorem 1.4. A completely positive map $\varphi : A \to B$ has order zero if and only if there exists a homomorphism $\pi : A \to M(C^*(\varphi(A))) \leq B^{**}$ and $h \in M(C^*(\varphi(A)))^+ \cap \varphi(A)'$ such that $\varphi(a) = h \cdot \pi(a) = \pi(a) \cdot h$ for all $a \in A$.

In other words, completely positive maps of order zero are given by the product of a positive element that commutes with $\varphi(A)$ and a homomorphism.

Remark 1.5. If a completely positive map of order zero is unital (and hence contractive), then it is a homomorphism.

Theorem 1.6. There is a bijection between the completely positive contractive, surjective maps from A to B and $\mathrm{Hom}(C_0((0,1])\otimes A, B)$.

Theorem 1.7. (Functional Calculus for completely positive maps of order zero). Let $\varphi : A \to B$ be a completely positive map of order zero, and let h and π as in Theorem (1.4). Given $f \in C_0((0, \|\varphi\|])^+$, then

$$f(\varphi)(a) = f(h)\pi(a)$$

for $a \in A$, defines a completely positive map of order zero.

The following well-known theorem, due to Choi and Effros, asserts that a C^*-algebra A is nuclear if and only if it is amenable.

Theorem 1.8. A C^*-algebra A is nuclear if and only if it has the completely positive contractive approximation property, this is, for every finite subset $\mathcal{F} \subseteq A$ and every $\epsilon > 0$, there exist a finite dimensional C^*-algebra F and completely positive contractive maps $\phi : A \to F, L : F \to A$ such that $L \circ \varphi =_\epsilon \mathrm{id}_A$ on $\mathcal{F}$, that is,

$$\|L \circ \phi(a) - a\| < \epsilon$$

for every $a \in \mathcal{F}$.

It is difficult to extract hardly any information about the C^*-algebra from a given completely positive contractive approximation. For example, it is not easy to tell if the C^*-algebra is commutative just by looking at its approximation. The following characterization of nuclearity, due to Hirshberg, Kirchberg and ?, is somewhat more useful in this sense.

Theorem 1.9. A C^*-algebra A is nuclear if and only if it has completely positive contractive approximations of the form $A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A$, where each C^*-algebra F_λ is finite dimensional, and each φ_λ is a *convex* combination of completely positive contractive order zero maps.

The proof of this theorem uses, among others:

- every injective von Neumann algebra is hyperfinite.
- completely positive contractive maps of order zero whose domain is finite dimensional are projective (can be lifted).
- Hahn-Banach Theorem.

Remark 1.10. If A is commutative, one can show that the algebras F_λ are finitely dimensional *and* commutative.

Question 1.11. What happens if you ask for a uniform bound on the number of summands in the convex combination in Theorem (1.9)? One can get it for AF-algebras, the bound being 1 (the nuclear dimension of an AF-algebra is 0). Are there any other examples? We suspect that the answer is "no". This is related to the nuclear dimension.

The goal of the Elliott program is to classify nuclear C^*-algebras by K-theoretical data. One of the limitations of K-theory is that completely positive contractive approximations (that characterize nuclear C^*-algebras), are not compatible with K-theory. On the brighter side, K-theory is stable under small perturbations. Roughly speaking, this means that if two C^*-algebras act on the same Hilbert space, and their unit balls are close to each other, then their K-theory are isomorphic. There are more results in this direction:

Theorem 1.12. Separable AF-algebras are stable under small perturbations, meaning that if two separable AF-algebras act on the same Hilbert space, and their unit balls are close to each other, then they are isomorphic.

Theorem 1.13. Separable nuclear C^*-algebras are stable under small perturbations, meaning that if two separable nuclear C^*-algebras act on the same Hilbert space, and their unit balls are close to each other, then they are isomorphic.

2. Strongly self-absorbing C^*-algebras.

Definition 2.1. Let D be a separable, unital C^*-algebra, $D \neq \mathbb{C}$. Then D is said to be *strongly self absorbing* if there is an isomorphism $\varphi : D \to D \otimes D$ that is approximately unitarily equivalent to $\mathrm{id}_D \otimes 1_D$, where $\otimes$ means minimal tensor product.

Remark 2.2. Strongly self absorbing C^*-algebras are nuclear, so the choice of the tensor product in the above definition is irrelevant (some work is required to prove this, though).

Proposition 2.3. Let A be a separable, unital C^*-algebra. Then A is strongly self absorbing if and only if

(a) There exists $k > 1$ such that $D \cong D^{\otimes k}$ and
(b) There is a sequence $\{u_n\}_{n\in\mathbb{Z}_{>0}}$ of unitaries in $D \otimes D$ such that $u_n(\mathrm{id}_D \otimes 1_D)u_n^* \to 1_D \otimes \mathrm{id}_D$ in the point norm topology. This is called the *approximately half flip*.

Condition (a) above can be replaced by the condition that $D \cong D^{\otimes\infty}$.

The proof of $D \cong D^{\otimes\infty}$ uses a one-sided Elliott intertwining argument.

Corollary 2.4. If D is strongly self absorbing, then any unital endomorphism of D is approximately inner, via unitaries that represent the zero element in $K_1(D)$.

In fact, even more is known.

Proposition 2.5. If D is a strongly self absorbing C^*-algebra, then any endomorphism $\varphi : D \to D$ is strongly asymptotically inner, this is, for every $\epsilon > 0$ and every finite set $F \subseteq D$, there exists a path of unitaries $(u_t)_{t\in[0,1]}$ such that $t \mapsto \mathrm{Ad}(u_t) \circ \varphi$ connects up to ϵ the homomorphism φ with the identity map, on the set F, and each unitary u_t is homotopic to the unit.

To show that the unitaries can be chosen to be homotopic to the unit, one uses the fact that strongly self absorbing C^*-algebras are K_1-injective, this is, the canonical map $\mathcal{U}(D)/\mathcal{U}_0(D) \to K_1(D)$ is injective. In particular, if $u \in \mathcal{U}(D)$ is such that $[u]_1 = 0$, then in fact u is homotopic to the unit.

The following theorem, due to Effros and Rosenberg, is fundamental in the development of the theory.

Theorem 2.6. Every strongly self absorbing C^*-algebra is simple and nuclear.

Proof. **Simplicity.** Let J be an ideal of D. Then $J \otimes D$ and $D \otimes J$ are ideals of $D \otimes D$. Moreover, if $(u_n)_{n\in\mathbb{Z}_{>0}}$ is a sequence of unitaries in $D \otimes D$ that implements the approximate half flip, then

$$J \otimes D \subseteq \overline{\cup_{n\in\mathbb{Z}_{>0}} u_n(D \otimes J)u_n^*} \subseteq D \otimes J$$

(the first containment uses the approximate half flip, and the second containment uses the fact that $D \otimes J$ is an ideal). Similarly, $D \otimes J \subseteq J \otimes D$ and hence they are equal. Assume that J is a proper ideal, so that there exist two nonzero states τ and σ such that $\tau(J) = 0$ and $\sigma(J) \neq 0$. Then

$$0 = \tau(J) \otimes \mathbb{C} = \tau \otimes \sigma(J \otimes D) = \tau \otimes \sigma(D \otimes J) = \tau(D) \otimes \sigma(J) = \mathbb{C} \otimes \mathbb{C} = \mathbb{C},$$

which is a contradiction. Hence, D is simple.

Nuclearity. We will show that D has the completely positive contractive approximation property. Since $u_n \in D \otimes D$, we can approximate them by elements in the algebraic tensor product, so there exist elements $c_n = \sum_{j=1}^{m_j} x_{n,j} \otimes y_{n,j} \in D \odot D$ arbitrarily close to u_n, for every $n \in \mathbb{Z}_{>0}$. Choose a state on D $\varphi \in S(D)$, and define linear maps $T_n : D \to D$ by

$$T_n(d) = \varphi \otimes \mathrm{id}_D(c_n(d \otimes 1_D)c_n^*)$$

for $d \in D$. Notice that T_n is completely positive contractive, and it has finite rank, since its range is contained in the span of $\{\mathbb{C} \otimes y_{n,j}, \mathbb{C} \otimes y_{n,j}^* : j = 1, \ldots, m_j\}$ (for this we use that c_n is sufficiently close to u_n). Moreover, $T_n \to \mathrm{id}_D$ in the point norm topology, since $c_n(d \otimes 1_D)c_n^* \approx u_n(d \otimes 1_D)u_n^* \to 1_D \otimes d$. ☐

Notice that the proof really only uses the approximate half flip. Indeed, asking that the C^*-algebra be strongly self absorbing is overkill.

Definition 2.7. A projection p in a C^*-algebra A is said to be *infinite* if it is Murray-von Neumann equivalent to a proper subprojection of itself. Otherwise, p is called *finite*. If p is nonzero and if there are mutually orthogonal projections p_1, p_2 in A such that $p_1 + p_2 \leq p$ and $p \sim p_1 \sim p_2$, then p is called *properly infinite*.

A C^*-algebra A is called *infinite* if it contains an infinite projection, and it is called *finite* if it admits an approximate unit of projections and all projections in A are finite. If $A \otimes \mathcal{K}$ is finite, then A is called *stably finite*.

Remark 2.8. A simple C^*-algebra is stably finite if and only if it admits a (densely defined, possibly unbounded) quasitrace, and if a simple C^*-algebra is not stably finite, then some matrix algebra over it contains a properly infinite projection.

Definition 2.9. A C^*-algebra is said to be *purely infinite* if every hereditary C^*-subalgebra of it is infinite.

We will use the following characterization of purely infiniteness.

Proposition 2.10. A C^*-algebra A is purely infinite if and only if for every nonzero positive element $a \in A^+$, there is a stable subalgebra of $\mathrm{Her}(a) = \overline{aAa}$.

The following dichotomy theorem, due to Kirchberg, holds in much more generality. We present this less general statement because its proof is easier and involves some of the techniques that are commonly used when dealing with strongly self absorbing C^*-algebras.

Theorem 2.11. Let D be a strongly self absorbing C^*-algebra. Then D is either purely infinite or stably finite with a unique trace.

Proof. We only show the dichotomy, namely that if a strongly self absorbing C^*-algebra is not stably finite, then it is purely infinite. We only mention that if a strongly self absorbing C^*-algebra is stably finite, then it has a quasitrace. Since it is nuclear, the quasitrace extends to a trace, and uniqueness of the trace is easy.

Assume D is not stably finite, so that there exists $n \in \mathbb{Z}_{>0}$ with $M_n \otimes D$ not finite, that is, $M_n \otimes D$ contains an infinite projection. It follows that $M_n \otimes D$ contains the Toeplitz algebra as a unital subalgebra, and notice that $\mathcal{K}$, being an ideal in the Toeplitz algebra, is a (stable) subalgebra of $M_n \otimes D$. We use (a variant of) proposition (2.10) and will show that for every nonzero positive element $d \in D^+$, there is a stable subalgebra of $\overline{dDd} \otimes D^{\otimes(n+1)}$.

Using the fact that D is simple and infinite dimensional, one can find nonzero orthogonal positive elements $e_1, \dots, e_n \in D^+$ with $\|e_j\| \leq 1$. Define

$$\begin{aligned} f_1 &= d \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_n \otimes 1_D \\ f_2 &= d \otimes e_n \otimes e_1 \otimes \cdots \otimes e_{n-1} \otimes 1_D \\ &\vdots \\ f_n &= d \otimes e_n \otimes e_{n-1} \otimes \cdots \otimes e_1 \otimes 1_D. \end{aligned}$$

Notice that $f_j \in \overline{dDd} \otimes D^{\otimes(n+1)}$ for all $j = 1, \dots, n$, and that they are all unitarily equivalent to each other (need to use the extra copy of D at the end when using the approximate half flip). This gives us a homomorphism

$$\Phi : C_0((0,1]) \otimes M_n \otimes D \to \overline{dDd} \otimes D^{\otimes(n+1)}.$$

Since $\mathcal{K} \leq M_n \otimes D$ is stable, it follows that $\Phi(C_0((0,1]) \otimes \mathcal{K})$ is a stable subalgebra of $\overline{dDd} \otimes D^{\otimes(n+1)}$. □

Notice that it is difficult to come up with many examples of strongly self absorbing C^*-algebras. Actually, we are more interested in algebras that tensorially absorb strongly self absorbing C^*-algebras.

Theorem 2.12. Let A, D be separable C^*-algebras, D strongly self absorbing. Then the following are equivalent:
(a) A is D-stable, meaning that $A \otimes D \cong A$.
(b) There exists a unital homomorphism $\varphi : D \to M(A)_\omega \cap A'$.
(c) There exists a homomorphism $\sigma : A \otimes D \to A_\omega$ such that

$$\sigma \circ (\mathrm{id}_A \otimes 1_D) = \iota_A,$$

where $\iota_A : A \to A_\omega$ is the natural inclusion of A as the subalgebra of A_ω consisting of constant sequences.

The proof of this theorem uses a one-sided approximate intertwining argument, which we state below.

Proposition 2.13. (One-sided approximate intertwining.) Let A, B be separable, $\varphi : A \to B$ injective, and $(v_n)_{n \in \mathbb{Z}_{>0}} \subseteq \mathcal{U}(B_\omega^+) \cap \varphi(A)'_\omega$ a sequence of unitaries such that

$$\mathrm{dist}(v_n^* b v_n, \varphi(A)_\omega) \to 0^1$$

for every $b \in B$. Then $A \cong B$ via an isomorphism that is approximately unitarily equivalent to φ.

Corollary 2.14. D-stability passes to hereditary subalgebras, quotients and extensions (assuming that both the ideal and the quotient are D-stable).

Corollary 2.15. Assume that $D = \lim_\to D_n$ is a strongly self absorbing C^*-algebra, and let A be another C^*-algebra. Then A is D-stable if and only if there exist maps $\varphi_n : D_n \to M(A)_\omega \cap A'$.

Notice that in the above corollary there is no coherence condition on the maps φ_n. Indeed, this will follow from D being strongly self absorbing.

Definition 2.16. Given $p, q \in \mathbb{Z}_{>0}$, let

$$\mathcal{Z}_{p,q} = \{f \in C([0,1], M_p \otimes M_q) \colon f(0) \in 1_p \otimes M_q, f(1) \in M_p \otimes 1_q\}.$$

The *Jiang-Su algebra* $\mathcal{Z}$, is the inductive limit $\mathcal{Z} = \lim_\to \mathcal{Z}_{p_n,q_n}$, where p_n and q_n are coprime for every $n \in \mathbb{Z}_{>0}$, and the connecting maps are defined such that the resulting limit is simple.

Remark 2.17. If p and q are relatively prime, then $\mathcal{Z}_{p,q}$ is projectionless. In particular, $\mathcal{Z}$ is projectionless as well.

Examples 2.18. The following are all the known examples of strongly self absorbing C^*-algebras.
(1) UHF-algebras of infinite type (every prime appears with power either 0 or ∞; necessity of this condition follows from $D \cong D^{\otimes\infty}$). In particular, the universal UHF-algebra $\mathcal{Q}$ (characterized by the fact that $K_0(\mathcal{Q}) = \mathbb{Q}$), is strongly self absorbing.
(2) The Jiang-Su algebra $\mathcal{Z}$.
(3) The infinite Cuntz algebra $\mathcal{O}_\infty$.
(4) The Cuntz algebra $\mathcal{O}_2$.
(5) $\mathcal{O}_\infty \otimes D$, where D is any UHF-algebra of infinite type.

These are all the known examples of strongly self absorbing C^*-algebras, and we expect this list to be complete. It is not even known whether a strongly self absorbing C^*-algebra must satisfy the UCT. However, one can show that if a strongly self absorbing C^*-algebra satisfies the UCT, then its K-theory coincides with the K-theory of one of the above listed algebras.

It is possible to characterize some of these algebras within the class of strongly self absorbing C^*-algebras. More precisely:

[1] This condition is somehow saying that φ is almost surjective inside of the ultrapower when conjugated by v_n.

Remark 2.19. One can consider the category $\mathcal{SSAC}^* - \mathbf{alg}$ of strongly self absorbing C^*-algebras, with morphisms being either unital homomorphisms, or approximately unitarily equivalence classes of unital homomorphisms (this really gives us two categories, but what we will say next is true for both). Then $\mathcal{O}_2$ is the uniquely determined final object, that is, there is an injective morphism from every object in the category into $\emptyset_2$. This follows from Kirchberg's embedding theorem: every nuclear C^*-algebra embeds into $\mathcal{O}_2$, and strongly self absorbing C^*-algebras are necessarily nuclear (Theorem 2.6). This characterization is not intrinsic, since it depends on the "interaction" of $\mathcal{O}_2$ with the other strongly self absorbing C^*-algebras.

Remark 2.20. Another characterization of $\mathcal{O}_2$ is as follows: it is the unique C^*-algebra in $\mathcal{SSAC}^* - \mathbf{alg}$ with trivial K_0 group. This description *is* intrinsic.

Remark 2.21. It is also to characterize $\mathcal{Q}$ intrinsically: it is the unique C^*-algebra D in $\mathcal{SSAC}^* - \mathbf{alg}$ with $K_0(D) \cong \mathbb{Q}$ and with finite decomposition rank.

Question 2.22. Is there an intrinsic characterization of $\mathcal{O}_\infty$? Is there one for $\mathcal{Z}$?

Here is how the algebras in the category $\mathcal{SSAC}^* - \mathbf{alg}$ (at least the ones we know of) are mapped into each other:

$$\begin{tikzcd}
 & \mathcal{O}_2 & \\
\mathcal{O}_\infty \otimes \mathcal{Q} \arrow[ur] & & \mathcal{Q} \arrow[ul] \\
\mathcal{O}_\infty \otimes \mathrm{UHF} \arrow[u, dashed] & & \mathrm{UHF} \arrow[u, dashed] \\
\mathcal{O}_\infty \arrow[u] & & \mathrm{UHF} \arrow[u] \\
 & \mathcal{Z} \arrow[ul] \arrow[ur] & .
\end{tikzcd}$$

3. Topological dimension.

Definition 3.1. Let X be a compact metrizable topological space. We say that X has *dimension at most n*, written $\dim X \leq n$, if for every open cover $\mathcal{V}$ of X, there is an open subcover $\mathcal{U} = \left\{ U_k^{(j)} : j = 1, \ldots, n, k = 1, \ldots, K^j \right\}$ such that $U_k^{(j)} \cap U_{k'}^{(j)} = \emptyset$ whenever $k \neq k'$.

Example 3.2. $\dim(I^n) = n$. More generally, if M is an n-dimensional manifold, then $\dim(M) = n$.

One can carry this definition on to C^*-algebras, interpreting completely positive contractive approximations as partitions of unities. Indeed, one can restate the preceding definition in terms of partitions of unities in X, where the open sets being disjoint corresponds to the product of the corresponding functions being zero (the functions are orthogonal).

Definition 3.3. Let A be a C^*-algebra. Then we say that the *nuclear dimension of A is at most n*, written $\dim_{\mathrm{nuc}} A \leq n$, if for every $\epsilon > 0$ and every finite subset $\mathcal{F} \subseteq A$, there are an n-decomposable finite dimensional algebra $F = F^{(1)} \oplus \cdots \oplus F^{(n)}$, where each $F^{(j)}$ is non-zero, and completely positive contractive maps $\psi^{(j)} : A \to F^{(j)}$ and $\varphi^{(j)} : F^{(j)} \to A$ for $j = 1, \ldots, n$, $\varphi^{(j)}$ of order zero, such that if $\psi = \oplus_{j=1}^n \psi^{(j)} : A \to F$ and $\varphi = \sum_{j=1}^n \varphi^{(j)} : F \to A$,

$$A \xrightarrow{\psi = \oplus_{j=1}^n \psi^{(j)}} F = F^{(1)} \oplus \cdots \oplus F^{(n)} \xrightarrow{\varphi = \sum_{j=1}^n \varphi^{(j)}} A,$$

then $\varphi \circ \psi =_\epsilon \mathrm{id}_A$ on $\mathcal{F}$, this is, $\|\varphi(\psi(a)) - a\| < \epsilon$ for all $a \in \mathcal{F}$.

We moreover say that the *decomposition rank of A is at most n*, written $\mathrm{dr}(A) \leq n$, if we in the above paragraph can choose φ to be completely positive contractive (notice that $\psi = \oplus_{j=1}^n \psi^{(j)}$ is automatically completely positive contractive).

Here is the relation between these three notions of dimensions in the commutative case: they all agree.

Theorem 3.4. Let X be a compact metrizable topological space. Then $\dim X = \dim_{\mathrm{nuc}} C(X) = \mathrm{dr} C(X)$.

Proposition 3.5. Let A be a C^*-algebra. Then $\dim_{\mathrm{nuc}} A = 0$ if and only if A is an AF-algebra.

Remark 3.6. Permanence properties: hereditary subalgebras of finite decomposition rank or finite nuclear dimension algebras also have finite decomposition rank or finite nuclear dimension. Decomposition rank and nuclear dimension are Morita invariant (they are invariant under tensoring with $M_n(\mathbb{C})$).

Proposition 3.7. Let A be a C^*-algebra, $\mathcal{F} \subseteq A^+$ a finite subset with $\|a\| = 1$ for all $a \in F$, and $\eta > 0$.

(a) Suppose $\mathrm{dr} A \leq n$ and choose an n-decomposable completely positive contractive approximation (F, ψ, φ) for $\mathcal{F} \cup \mathcal{F}^2$ within η. Then, for every central projection $p \in F^{(j)}$ we have that
$$a\varphi(p) \approx_{4\eta^{1/2}} \varphi^{(j)}\left(\psi^{(j)}(a)p\right).$$
In particular, by choosing p to be the unit of $F^{(j)}$, we get that
$$a\varphi(1_{F^{(j)}}) \approx_{4\eta^{1/2}} \varphi^{(j)}\left(\psi^{(j)}(a)\right).$$
In other words, one can approximately recover the composition $\varphi^{(j)} \circ \psi^{(j)}$ on $\mathcal{F}$ by multiplying by the image under $\varphi^{(j)}$ of some positive element in $F^{(j)}$.

(b) Suppose that A is unital and $\dim_{\mathrm{nuc}} A \leq n$. Choose an n-decomposable completely positive contractive approximation (F, ψ, φ) for $\mathcal{F} \cup \mathcal{F}^2$ within η. Then,
$$a\varphi(\psi^{(j)}(1_A)) \approx_{4\eta^{1/2}} \varphi^{(j)}\left(\psi^{(j)}(a)\right).$$

Proposition 3.8. Let A be a C^*-algebra.

(a) Assume that A has finite decomposition rank, say $\mathrm{dr} A \leq n$. Then there is a system of n-decomposable completely positive contractive approximations $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ such that
$$A \xrightarrow{\overline{\psi}} \prod_{\lambda \in \Lambda} F_\lambda \Big/ \bigoplus_{\lambda \in \Lambda} F_\lambda$$
is a C^*-algebra homomorphism. In particular, A is quasi-diagonal (in fact, strongly quasi-diagonal).

(b) Assume that A has finite nuclear dimension, say $\dim_{\mathrm{nuc}} A \leq n$. Then there is a system of n-decomposable completely positive contractive approximations $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ such that
$$A \xrightarrow{\overline{\psi}} \prod_{\lambda \in \Lambda} F_\lambda \Big/ \bigoplus_{\lambda \in \Lambda} F_\lambda$$
is a completely positive contractive map of order zero.

Question 3.9. Is there a version of the above proposition for general nuclear C^*-algebras?

Lemma 3.10. Let A, B be C^*-algebras, $\psi : A \to B$ and $\varphi : B \to A$ completely positive contractive maps, $A \xrightarrow{\psi} B \xrightarrow{\varphi} B$. Given $a \in A^+$, let $\eta > 0$ such that $\|\varphi(\psi(a)) - a\| < \eta$ and $\|\varphi(\psi(a^2)) - a^2\| < \eta$. Then
$$\|\varphi(\psi(a)b) - \varphi(\psi(a))\varphi(b)\| < 3\eta^{1/2}\|b\|,$$
for every $b \in B$. In other words, $\psi(a)$ is approximately in the multiplicative domain of φ.

Proof. Stinespring's representation theorem. □

The following theorem was proved by Winter, Zacharias and Endlers, using Kirchberg-Phillips classification theorem for Kirchberg algebras.

Theorem 3.11. If A is a Kirchberg algebra satisfying the UCT, then $\dim_{\mathrm{nuc}} A \leq 2$.

Question 3.12. What is the exact value? (one or two?) It might be the case that torsion in $K_*(A)$ forces A to have nuclear dimension equal to 2.

4. The Cuntz semigroup.

Definition 4.1. Let A be a C^*-algebra, $a, b \in A^+$. Then a is said to be *Cuntz dominated* by b, written $a \preceq b$, if there exists a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in A such that $a = \lim x_n^* b x_n$ (norm limit). Moreover, a and b are said to be *Cuntz equivalent*, written $a \sim b$, if $a \preceq b$ and $b \preceq a$.

Denote by $M_\infty(A)$ the $*$-algebra of matrices of arbitrary size over A, that is, $M_\infty(A) = \cup_{n \in \mathbb{Z}_{>0}} M_n(A)$, where the inclusion of $M_n(A)$ into $M_{n+1}(A)$ is given by $a \mapsto a \oplus 0$. Set $W(A) = M_\infty(A)/\sim$. Finally, define the *Cuntz semigroup* of A to be
$$Cu(A) = W(A \otimes \mathcal{K}).$$

Definition 4.2. Let A be a simple, unital C^*-algebra, and $m \in \mathbb{Z}_{>0}$. We say that

(a) A has *m-comparison*, if whenever $a, b_0, \ldots, b_m \in M_\infty(A)^+$ are such that for every quasitrace $\tau \in QT(A)$ we have that $d_\tau(a) < d_\tau(b_0), \ldots, d_\tau(b_m)$, then in $W(A)$ we have that
$$[a] \leq [b_0] + \cdots + [b_m].$$

(b) $W(A)$ is *m-almost unperforated*, if whenever $a, b_0, \ldots, b_m \in M_\infty(A)^+$ are such that there exists $n \in \mathbb{Z}_{>0}$ with $(n+1)[a] \leq n[b_0], \ldots, n[b_m]$ in $W(A)$, then in $W(A)$ we have that
$$[a] \leq [b_0] + \cdots + [b_m].$$

(c) $W(A)$ is *m-almost divisible*, if whenever $a \in M_\infty(A)^+$ and $n \in \mathbb{Z}_{>0}$, there exists $b \in M_\infty(A)$ such that in $W(A)$ we have that
$$n[b] \leq [a] \leq m(n+1)[b].$$

The next proposition is due to Rørdam.

Proposition 4.3. Let A be a simple and unital C^*-algebra, such that $QT(A) = T(A)$. Then A has m-comparison if and only if $Cu(A)$ has m-almost unperforation.

The following lemma gets used most of the times one has to compute Cuntz semigroups.

Lemma 4.4. Let A be a C^*-algebra, $a, b \in A^+$, $\|a\|, \|b\| \leq 1$, $\epsilon > 0$ with $\|a - b\| < \epsilon$. Then there exists $d \in A^+$, with $\|d\| \leq 1$ such that
$$(a - \epsilon)_+ = d^* b d,$$
where $(a - \epsilon)_+$ is defined using functional calculus: $(a - \epsilon)_+ = f(a)$, where $f(x) = (x - \epsilon)_+ = \max\{x - \epsilon, 0\}$ on the spectrum of a.

We have another stability under perturbation type of statement, this time for the Cuntz semigroup.

Theorem 4.5. $Cu(\cdot)$ is stable under small perturbations: if A and B are two C^*-algebras, such that $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ act on the same Hilbert space, and their unit balls are sufficiently close to each other, then $Cu(A) \cong Cu(B)$.

Theorem 4.6. Let A be a C^*-algebra with finite nuclear dimension, say $\dim_{\mathrm{nuc}} A \leq m$. Then A has m-comparison.

Proof. Let $a, b_0, \ldots, b_m \in M_\infty(A)^+$ and assume that there exists $n \in \mathbb{Z}_{>0}$ such that
$$(n+1)[a] \leq n[b_0], \ldots, n[b_m]$$
in $W(A)$. Choose a system of completely positive contractive approximations $A \xrightarrow{\psi_\lambda} F_\lambda \xrightarrow{\varphi_\lambda} A$, with φ_λ being m-decomposable. Consider the diagram
$$\begin{tikzcd} A \arrow[rr, "\iota_A"] \arrow[dr, "\overline{\psi^{(j)}}"'] & & \prod_{\lambda\in\Lambda} A \big/ \bigoplus_{\lambda\in\Lambda} A \\ & \prod_{\lambda\in\Lambda} F_\lambda \big/ \bigoplus_{\lambda\in\Lambda} F_\lambda \arrow[ur, "\overline{\varphi^{(j)}}"'] & \end{tikzcd}.$$

Notice that
- $\iota_A = \sum_{j=1}^m \varphi^{(j)} \circ \psi^{(j)}$.
- The Cuntz semigroup of $\prod_{\lambda\in\Lambda} F_\lambda \big/ \bigoplus_{\lambda\in\Lambda} F_\lambda$ is unperforated, using that each F_λ is finite dimensional.
- $\overline{\psi^{(j)}}$ and $\overline{\varphi^{(j)}}$ are completely positive contractive maps of order zero.
- Completely positive contractive maps of order zero induce maps between the Cuntz semigroups (completely positive contractive is not enough).

Fix $r, j \in \{1, \ldots, m\}$. Since $(n+1)[a] \leq k[b_r]$, applying $\overline{\psi^{(j)}}$ we get that $(k+1)\left[\overline{\psi^{(j)}}(a)\right] \leq k\left[\overline{\psi^{(j)}}(b_r)\right]$. Since the Cuntz semigroup of $\prod_{\lambda\in\Lambda} F_\lambda \big/ \bigoplus_{\lambda\in\Lambda} F_\lambda$ is unperforated, we get that $\left[\overline{\psi^{(j)}}(a)\right] \leq \left[\overline{\psi^{(j)}}(b_r)\right]$. Applying $\overline{\varphi^{(j)}}$ we get that $\left[\overline{\varphi^{(j)}} \circ \overline{\psi^{(j)}}(a)\right] \leq \left[\overline{\varphi^{(j)}} \circ \overline{\psi^{(j)}}(b_r)\right]$. Let $r = j$ and sum over j to get
$$[\iota_A(a)] \leq \sum_{j=1}^m [\iota_A(b_j)],$$
where this is happening in Cuntz semigroup of the bigger algebra $\prod_{\lambda\in\Lambda} A \big/ \bigoplus_{\lambda\in\Lambda} A$. One then has to do a little bit of work to show that this inequality can actually be pulled back to $Cu(A)$, to get $[a] \leq \sum_{j=1}[b_j]$. ☐

5. The Jiang-Su algebra $\mathcal{Z}$.

Recall (definition 2.16) the following.

Definition 5.1. Given $p, q \in \mathbb{Z}_{>0}$, let
$$\mathcal{Z}_{p,q} = \{f \in C([0,1], M_p \otimes M_q) \colon f(0) \in 1_p \otimes M_q, f(1) \in M_p \otimes 1_q\}.$$
The *Jiang-Su algebra* $\mathcal{Z}$, is the inductive limit $\mathcal{Z} = \lim_{to} \mathcal{Z}_{p_n, q_n}$, where p_n and q_n are coprime for every $n \in \mathbb{Z}_{>0}$, and the connecting maps are defined such that the resulting limit is simple.

Remark 5.2. It can be shown that $\mathcal{Z} = \lim_{\rightarrow} \mathcal{Z}_{p,p+1}$, where p runs over all prime numbers.

We will give alternative descriptions of the Jiang-Su algebra.

Definition 5.3. Let $p \in \mathbb{Z}_{>0}, p \geq 2$. Define the *universal dimension drop algebra* by

$$\mathcal{Z}^u_{p,p+1} = C^*\left(v, s_1, \ldots, s_p \colon s_1^* s_1 = s_j s_j^*, (s_j^* s_j)(s_k^* s_k) = \delta_{j,k}(s_j^* s_j)^2, v^* v = 1 - \sum_{j=1}^{p} s_j^* s_j, vv^* s_1^* s_1 = vv^* \right).$$

Notice that if the second relation was $(s_j^* s_j)(s_k^* s_k) = \delta_{j,k}(s_j^* s_j)$, then we would get the matrix unit for M_p. Hence, an alternative description of this algebra is

$$\mathcal{Z}^u_{p,p+1} = C^*\left(\Phi, \Psi \colon \begin{array}{l} \Phi \text{ is a completely positive contractive order zero map on } M_p \\ \Psi \text{ is a completely positive contractive order zero map on } M_2 \\ \Psi(e_{1,2}) = 1 - \Phi(1_{M_p}),\ \Psi(e_{1,1})\Phi(e_{1,1}) = \Psi(e_{1,1}) \end{array} \right).$$

Proposition 5.4. If $p \in \mathbb{Z}_{>0}, p \geq 2$, then $\mathcal{Z}^u_{p,p+1} \cong \mathcal{Z}_{p,p+1}$.

Remark 5.5. Of all pictures of $\mathcal{Z}_{p,p+1} \cong \mathcal{Z}^u_{p,p+1}$, the first one in definition 5.1 is very convenient when showing that it is weakly semiprojective.

Proposition 5.6. Let A be a unital, separable C^*-algebra. Then A is $\mathcal{Z}$-stable if for all (in fact, some) $p \geq 2$, there exist completely positive contractive order zero maps $\Phi : M_p \to A_\omega \cap A'$ and $\Psi : M_2 \to A_\omega \cap A'$ satisfying

$$\Psi(e_{1,2}) = 1 - \Phi(1_{M_p}), \Psi(e_{1,1})\Phi(e_{1,1}) = \Psi(e_{1,1}).$$

The following definition generalizes 2.16 to algebras whose fibers are UHF-algebras, rather than simply matrix algebras.

Definition 5.7. Let p, q be supernatural numbers. Denote by M_p and M_q the UHF-algebras of type p and q respectively, and set

$$\mathcal{Z}_{p,q} = \{f \in C([0,1], M_p \otimes M_q) \colon f(0) \in 1_p \otimes M_q, f(1) \in M_p \otimes 1_q\}.$$

Sometimes one needs to use that $\mathcal{Z}$ has subalgebras isomorphic to $\mathcal{Z}_{p,p+1}$. It is sometimes useful to know that $\mathcal{Z}$ also has subalgebras isomorphic to $\mathcal{Z}_{2^\infty,3^\infty}$.

Theorem 5.8. Let A be a separable, unital, simple, stably finite C^*-algebra, and let $\alpha : A \otimes \mathcal{Z}_{2^\infty,3^\infty} \to A_\omega$ be such that

$$\alpha \circ \left(id_A \otimes 1_{\mathcal{Z}_{2^\infty,3^\infty}} \right) = \iota_A.$$

Then A has stable rank one, is K_1-injective, and $Cu(A)$ is almost unperforated.

One can define an universal version of $\mathcal{Z}$, that will turn out to be isomorphic to $\mathcal{Z}$, as follows:

$$\mathcal{Z}^u = C^*\left(\Phi^{(k)}, \Psi^{(k)} \colon \text{ for every } k \in \mathbb{Z}_{>0}, \begin{array}{l} \Phi^{(k)} \text{ is a completely positive contractive order zero map on } M_{p_k} \\ \Psi^{(k)} \text{ is a completely positive contractive order zero map on } M_2 \\ \Psi^{(k)}(e_{1,2}) = 1 - \Phi^{(k)}(1_{M_{p_k}}),\ \Psi^{(k)}(e_{1,1})\Phi^{(k)}(e_{1,1}) = \Psi^{(k)}(e_{1,1}) \end{array} \right),$$

where $\{p_k\}_{k \in \mathbb{Z}_{>0}}$ is any increasing sequence of prime numbers.

Theorem 5.9. $\mathcal{Z}^u \cong \mathcal{Z}$. In particular, it is possible to write $\mathcal{Z}$ as the universal C^*-algebra on countably many generators and relations.

Leonel Robert has classified all homomorphisms between one-dimensional noncommutative CW-complexes (satisfying some conditions), in terms of the associated map in the Cuntz semigroup. More concretely, maps between the Cuntz semigroups (with some mild restrictions) can be lifted to maps between between one-dimensional noncommutative CW-complexes, and the lift is unique up to approximate unitary equivalence.

Proposition 5.10. There is a unital, trace-collapsing[2] homomorphism $\delta : \mathcal{Z}_{2^\infty,3^\infty} \to \mathcal{Z}_{2^\infty,3^\infty}$.

Given δ as in the above proposition, set $\mathcal{Z}^\delta = \lim_{\to} (\mathcal{Z}_{2^\infty,3^\infty}, \delta)$.

Theorem 5.11. $\mathcal{Z}^\delta$ is a separable, simple, unital C^*-algebra with a unique trace.

Proof. The crucial point is to show that $\mathcal{Z}^\delta$ has a unique trace, and for this one uses the fact that δ is trace-collapsing. □

Remark 5.12. The results of Jiang and Su already imply that $\mathcal{Z}^\delta \cong \mathcal{Z}$.

Theorem 5.13. The Jian-Su algebra $\mathcal{Z}$ is strongly self absorbing.

Proof. We show that $\mathcal{Z}^\delta$ is strongly self absorbing. One can show that $\left(\mathcal{Z}^\delta\right)^{\otimes\infty}$ has stable rank one and strict comparison, using Theorem 5.8. Using Robert's classification result again, $\left(\mathcal{Z}^\delta\right)^{\otimes\infty}$ has the approximate inner half flip: the embeddings into one and the other factor induce the same map on the Cuntz semigroup, and hence they are approximately unitarily equivalent. By Theorem (2.12), one gets that $\mathcal{Z}^\delta$ is $\left(\mathcal{Z}^\delta\right)^{\otimes\infty}$-stable, and this finishes the proof. □

The Jiang-Su algebra is "minimal" among all strongly self absorbing C^*-algebras, as the following theorem shows. The proof is due to Dadarlat and Rørdam in the case of the existence of a nontrivial projection in D, and Winter proved it in the general case. We sketch the proof in the first case.

[2] This is, the induced map on the trace space maps everything to a single trace. In fact, one can choose this trace, as long as it doesn't have atoms.

Theorem 5.14. If D is strongly self absorbing, then $D \cong D \otimes \mathcal{Z}$.

Proof. Assume $p \in D$ is a nontrivial projection. Then $q = p \otimes p + (1-p) \otimes (1-p) \in D \otimes D$ is a nontrivial projection. Set

$$\begin{aligned} e_0 &= 1_D^{\otimes 2n} \otimes p \otimes (1-p) \\ e_1 &= 1_D^{\otimes 2n-2} \otimes q \otimes p \otimes (1-p) \\ &\vdots \\ e_0 &= q^{\otimes n} \otimes p \otimes (1-p) \end{aligned}$$

and

$$\begin{aligned} e_0' &= 1_D^{\otimes 2n} \otimes (1-p) \otimes p \\ e_1' &= 1_D^{\otimes 2n-2} \otimes q \otimes (1-p) \otimes p \\ &\vdots \\ e_0' &= q^{\otimes n} \otimes (1-p) \otimes p. \end{aligned}$$

Notice that $e_0, \ldots, e_n$ and $e_0', \ldots, e_n'$ are two families of mutually orthogonal projections in $D^{\otimes 2n+2}$, and hence $e = \sum_{j=0}^{n} e_j$ and $e' = \sum_{j=0}^{n} e_j'$ are projections in $D^{\otimes 2n+2}$. Since D is strongly self absorbing, it has the approximate half flip, so that e_j unitarily equivalent to a projection that is arbitrarily close e_j'. In particular, e_j is Murray-von Neumann equivalent to e_j', and hence e and e' are Murray-von Neumann equivalent.

Now, if n is large enough, a calculation in the Cuntz semigroup shows that

$$1 - (e + e') = q^{\otimes (n+1)} \prec 1 - q^{\otimes (n+1)} \otimes 1.$$

Based on this, one can construct completely positive contractive order zero maps Φ and Ψ like in the picture of $\mathcal{Z}$, and use this to show that $D \otimes \mathcal{Z} \cong D$. □

6. $\mathcal{Z}$-stable classification of C^*-algebras.

Recall that the *Elliott invariant* of a separable, simple, unital, nuclear C^*-algebra A is given by

$$\mathrm{Ell}(A) = \left(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A : T(A) \to S(K_0(A))\right).$$

Definition 6.1. Let $\mathcal{E}$ be a class of separable, simple, unital, nuclear C^*-algebras. We say that $\mathcal{E}$ *satisfies the Elliott conjecture*, if whenever $A, B \in \mathcal{E}$ and $\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is an isomorphism, then there is an isomorphism of C^*-algebras $\varphi : A \to B$ such that $\mathrm{Ell}(\varphi) = \Lambda$.

If p, q are relatively prime supernatural numbers, recall that the generalized dimension drop algebra $\mathcal{Z}_{p,q}$ is an algebra over $C([0,1])$.

Definition 6.2. Let p, q be relatively prime supernatural numbers. A unital homomorphism $\varphi : A \otimes \mathcal{Z}_{p,q} \to B \otimes \mathcal{Z}_{p,q}$ that respects the $C([0,1])$-algebra structure (that is, maps fibers to fibers), is said to be *unitarily suspended*, if there is a norm continuous path of unitaries $(u_t)_{t \in [0,1)}$ in $\mathcal{U}(B \otimes M_p \otimes M_q)$ such that

$$\varphi_t = \mathrm{Ad}(u_t) \circ (\varphi_0 \otimes \mathrm{id}_{M_q})$$

for all $t \in [0,1)$, where $\varphi_t = (\mathrm{id}_B \otimes \epsilon_t) \circ \varphi$.

Given $\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ and a supernatural number r, one can use the Künneth formula to get a morphism $\Lambda_{M_r} : \mathrm{Ell}(A \otimes M_r) \to \mathrm{Ell}(B \otimes M_r)$.

Definition 6.3. A morphism $\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ between the Elliott invariants of A and B is said to be *lifted along* $\mathcal{Z}_{p,q}$, if there is a unitarily suspended homomorphism $\varphi : A \otimes \mathcal{Z}_{p,q} \to B \otimes \mathcal{Z}_{p,q}$ such that $\mathrm{Ell}(\varphi_0) = \Lambda_{M_p}$ and $\mathrm{Ell}(\varphi_1) = \Lambda_{M_q}$.

Theorem 6.4. Let $\mathcal{E}$ be a class of separable, simple, unital and nuclear C^*-algebras. Suppose that for any $A, B \in \mathcal{E}$ and any isomorphism $\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ can be lifted along $\mathcal{Z}_{p,q}$, for some pair of relatively prime supernatural numbers p, q. Then the class

$$\mathcal{E}^{\mathcal{Z}} = \{A \otimes \mathcal{Z} \colon A \in \mathcal{E}\}$$

satisfies the Elliott conjecture. In particular, if $\mathcal{E}$ consists of C^*-algebras that absorb the Jiang-Su algebra, then $\mathcal{E}$ itself satisfies the Elliott conjecture.

Proof. Uses a two-sided intertwining argument. □

The following theorem is due to Huaxin Lin.

Theorem 6.5. Let $\mathcal{E}$ be the class of unital, separable, nuclear C^*-algebras, satisfying the UCT and such that $A \otimes \mathcal{Q}$ is tracially approximately an interval algebra (TAI). Then $\mathcal{E}^{\mathcal{Z}}$ satisfies the Elliott conjecture.

Proof. The crucial difficulty is the following: given an isomorphism $\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ between the Elliott invariants of $A, B \in \mathcal{E}$, use the classification of TAI algebras to lift $\Lambda_{M_p} : \mathrm{Ell}(A \otimes M_p) \to \mathrm{Ell}(B \otimes M_p)$ and $\Lambda_{M_q} : \mathrm{Ell}(A \otimes M_q) \to \mathrm{Ell}(B \otimes M_q)$ to φ_0 and φ_1 respectively (one needs to show that if $C \in \mathcal{E}$, then $C \otimes M_r$ is a TAI algebra for every supernatural number r, and then use the fact that the class $\mathcal{E}'$ of unital, separable, nuclear, TAI C^*-algebras, satisfying the UCT satisfies the Elliott conjecture– this was proved by Huaxin Lin, too).

We would like to show that φ_0 and φ_1 are asymptotically unitarily equivalent, but this turns out not to be true in general. However, it is possible to identify the obstruction and modify these maps to get the desired result. ☐

There are two natural directions for further work:

Problem 6.6. Confirm TAI after tensoring with $\mathcal{Q}$ for interesting classes of C^*-algebras.

Problem 6.7. Generalize TAI classification.

7. The regularity conjecture.

Theorem (6.5) suggests the following strategy for classifying interesting classes of C^*-algebras:

(1) Classify algebras of the form $A \otimes \mathcal{Z}$, for A in some interesting class.
(2) Show that $A \otimes \mathcal{Z} \cong A$ for every A un such a class.

Indeed, Theorem (6.5) deals with the first of these points, asserting that it is enough to classify algebras of the form $A \otimes \mathcal{Q}$, for A in such a class. The second point, namely, proving $\mathcal{Z}$-stability of interesting classes of C^*-algebras, is a difficult task, and is the motivation of this section.

Conjecture 7.1. Let A be a separable, simple, unital, non-elementary, nuclear C^*-algebra. Then the following are equivalent:

(a) $\dim_{\mathrm{nuc}} A < \infty$.
(b) A is $\mathcal{Z}$-stable.
(c) A has strict comparison of positive elements.
(d) A has strict comparison of positive elements and almost divisibility ($Cu(A)$ has almost divisibility).[3]
(e) A has almost divisibility.
(f) A has property (SI): roughly speaking, given two sequences $(a_n)_{n\in\mathbb{Z}_{>0}}, (b_n)_{n\in\mathbb{Z}_{>0}}$ of positive, quasi-central elements in A such that $\tau(a_n) \to 0$ and $\tau(b_n)$ is bounded, then one can conjugate $(a_n)_{n\in\mathbb{Z}_{>0}}$ underneath $(b_n)_{n\in\mathbb{Z}_{>0}}$ by quasi-central unitaries.

If A is moreover assumed to be stably finite, one can strengthen the conjecture by replacing $\dim_{\mathrm{nuc}} A < \infty$ with $\mathrm{dr} A < \infty$ in (a). None of the implications are easy, except for the trivial ones: (d) implies (c) and (e).

Theorem 7.2. The following are the known implications. A full arrow means that the implication holds in full generality (this is, without any further conditions besides the ones mentioned in the conjecture), while a dotted arrow means that the implication is known to hold under some additional conditions (we specify these conditions below). The labeling on each arrow stands for the the initial(s) of the author(s) of the proof, with the following convention: W: Winter, R: Rørdam, M-S: Matui-Sato, D-T: Dadarlat-Toms.

$$\begin{tikzcd}
(a) \arrow[r, bend left, "W"] & (b) \arrow[l, dashed, bend left, "W"] \arrow[r, bend left, "R"] \arrow[d, bend left, "M-S"] & (d) \arrow[l, dashed, bend left, "W"] \arrow[r] \arrow[d, bend left] & (e) \\
 & (f) \arrow[u, dashed, "M-S"] & (c) \arrow[l, dashed, "M-S"] \arrow[u, dashed, "D-T"] &
\end{tikzcd}$$

Dotted arrows:

- (d) implies (b): under the additional assumption that A has locally finite nuclear dimension (this is, for every finite subset $F \subseteq A$ and every $\epsilon > 0$, there is a C^*-subalgebra $B \leq A$ with finite nuclear dimension, such that every element of F is within ϵ of an element of B). It is not known if there exists a nuclear C^*-algebra that does not have locally finite nuclear dimension.
- (c) implies (d): under the additional assumption that $\partial_e T(A)$ (the extreme points in the boundary) is compact and has finite topological dimension, in the sense of definition 3.1.
- (c) implies (f): under the additional assumption that $\partial_e T(A)$ is finite.
- (f) implies (b): under the additional assumption that $\partial_e T(A)$ is finite.
- (b) implies (a): under either of the following additional assumptions:
  - If we have classification and finite nuclear dimensional models.
  - If we can use Gong's reduction Theorem.
  - If A is locally homogeneous.

[3] In the lecture, this condition was called (c'), and the following ones were (d) and (e).

Remark 7.3. If A has locally finite nuclear dimension, then $A\otimes\mathcal{Z}\cong A$ if and only if $Cu(A\otimes\mathcal{Z})\cong Cu(A)$.

Matui-Sato's proof of the implications (c)$\Rightarrow$ (f) $\Rightarrow$ (b) (provided that $\partial_e T(A)$ is finite), is a unified version of Kirchberg's $\mathcal{O}_\infty$ absorption theorem (it unifies the finite and infinite case).

Problem 7.4. Using Matui-Sato's technique, come up with a classification theorem for nuclear, simple, unital, separable C^*-algebras wit finitely many (or none at all) traces, generalizing Kirchberg-Phillips' and Lin's classification theorems.

Question 7.5. Does (d) imply (e) and (f), and does (e)+(f) imply (b), without the assumption that $\partial_e T(A)$ is finite?

Theorem 7.6. Let X be a compact, metrizable space, not necessarily of finite topological dimension. Then

$$1\leq \mathrm{dr}\,(C(X)\otimes\mathcal{Z})\leq 2.$$

We suspect that as long as $\dim X>0$ (this is, X is not finite), then $\mathrm{dr}\,(C(X)\otimes\mathcal{Z})=2$.

Proof. We will show that $\mathrm{dr}\,(C(X)\otimes M_{2^\infty})\leq 2$. Based on this, standard techniques can be used to show that $\mathrm{dr}\,(C(X)\otimes\mathcal{Z})\leq 5$, but one can show that the bound is actually 2.

Recall a result from Voiculescu, that asserts that $C_0((0,1])\otimes\mathcal{O}_2$ is quasi-diagonal, and hence $C_0((0,1])\otimes\mathcal{O}_2\hookrightarrow(M_{2^\infty})_\omega$. Moreover, the image of this embedding is "tracially small". One in fact has some additional control over the embedding, and this is will be crucial in the proof.

Find an approximation of the embedding $C(X)\hookrightarrow C(X)\otimes M_{2^\infty}$ of the form

$$\begin{tikzcd} C(X) \arrow[rr] \arrow[dr, "\psi=\psi^{(0)}\oplus\psi^{(1)}"'] & & C(X)\otimes M_{2^\infty} \\ & \mathbb{C}^k\oplus C(Y) \arrow[ur, "\varphi=\varphi^{(0)}+\varphi^{(1)}"'] & \end{tikzcd},$$

with $Y\subseteq X$ and $\varphi^{(0)},\varphi^{(1)}$ completely positive contractive order zero maps, such that $\varphi^{(1)}$ has "tracially small" image. Notice that a completely positive contractive order zero map on $C(Y)$ corresponds to a homomorphism on the cone of $C(Y)$. We get a map the map φ in the sequence

$$C_0((0,1]\times Y)\xrightarrow{\cong}C_0((0,1])\otimes C(Y)\xrightarrow{\varphi}C(X)\otimes M_{2^\infty}\ .$$

Since we have some freedom in the choice of the embedding $C_0((0,1])\otimes\mathcal{O}_2\hookrightarrow(M_{2^\infty})_\omega$, we can make it match the cone above to get

$$C_0((0,1]\times Y)\hookrightarrow C_0((0,1])\otimes C(Y)\otimes\mathcal{O}_2\to(C(X)\otimes\mathcal{O}_2)_\omega.$$

Kirchberg-Rørdam showed that in this situation, there exists a graph Γ and maps

$$\begin{tikzcd} C_0((0,1]\times Y) \arrow[rr, hook] \arrow[dr] & & C_0((0,1])\otimes C(Y)\otimes\mathcal{O}_2 \arrow[r] & (C(X)\otimes\mathcal{O}_2)_\omega \\ & C_0(\Gamma)\otimes\mathcal{O}_2 \arrow[ur] & & \end{tikzcd}.$$

This uses the fact that $\mathcal{U}(C(S^1,\mathcal{O}_2))$ is connected.

Notice that $\varphi^{(0)}$ uses two colors, while $\varphi^{(1)}$ uses one color. Hence there is a total of 3 colors, which corresponds to dimension at most 2. $\square$

8. Minimal dynamical systems.

Throughout this section, X will be a compact, metric space, $T\colon X\to X$ a minimal homeomorphism (that is, X has no non-trivial closed T-invariant subsets). Denote by $\alpha\colon C(X)\to C(X)$ the automorphism induced by T, this is, $\alpha(f)=f\circ T^{-1}$ for $f\in C(X)$. Recall that the crossed product of $C(X)$ by α is given by

$$A=C(X)\rtimes_\alpha\mathbb{Z}=C^*\left(C(X),u\colon u^*u=uu^*=1,ufu^*=\alpha(f)\right).$$

Definition 8.1. If $Y\subseteq X$ is closed, denote

$$_Y=C^*\left(C(X),uC_0(X\setminus Y)\right)\subseteq C(X)\rtimes_\alpha\mathbb{Z}.$$

If $Y=\{y\}$, it is customary to write A_y for $A_{\{y\}}$.

The following is a collection of theorems due to Putnam, Lin and Phillips.

Proposition 8.2. Adopt the notation of the above definition.

(a) If $\{Y_n\}_{n\in\mathbb{Z}_{>0}}$ is a decreasing sequence of closed subspaces and $Y=\cap_{n\in\mathbb{Z}_{>0}}Y_n$, then

$$A_Y=\varinjlim A_{Y_n}.$$

(b) If Y contains at most one point from each orbit (in particular, Y has to be finite), then A_Y is simple. In particular, A_y is simple for every $y\in Y$.

(c) We have the following isomorphisms

$$\begin{aligned} K_0(A_y) &\cong K_0(A) \\ T(A_y) &\cong T(A) \\ K_1(A_y) \oplus \mathbb{Z} &\cong K_1(A). \end{aligned}$$

(d) If $Y \subseteq X$, has non-empty interior, then A_Y is a RSH-algebra with a canonical decomposition. In particular, it is not simple.

The following theorem is a generalization by Winter of a theorem by Lin and Phillips. Their theorem didn't include the UHF-algebra, but assumed real rank zero beforehand.

Theorem 8.3. Let B be a UHF-algebra. And suppose that for some $y \in Y$, $A_y \otimes B$ is TAF/TAI. Then $A \otimes B$ is TAF/TAI.

Theorem 8.4. If $\dim X < \infty$, then $\dim_{\mathrm{nuc}} A < \infty$, and hence A is $\mathcal{Z}$-stable.

Proof. The following is the crucial point. Given $x, y \in X$ in different orbits, one should be able to find an approximation

$$A \xrightarrow{\psi} A_x \oplus A_y \xrightarrow{\iota_x \oplus \iota_y = \varphi} A.$$

To construct the map ψ, one would like to write $a = a_x + a_y$ for $a_x \in A_x$ and $a_y \in A_y$, for all $a \in A$. This is not possible, though one can do it approximately: $a \approx a_x + a_y$.

Find a positive function $f \in C(X)$ with $\|f\|_\infty \leq 1$ which is almost invariant under α, and such that $f(x) = 1$ and $f(y) = 0$. Compressing by this function gives a map $A \to A_x$, and compressing by $1-f$ gives a function $A \to A_y$. Now we need to approximate these maps by decomposable systems. Notice that A_x and A_y are inductive limits of RSH-algebras which have finite decomposition rank. Actually, it is true that if a RSH-algebra has base space of dimension at most n, then its decomposition rank is at most n too. □

Combining several results, we get the following classification theorem. It is an example of how $\mathcal{Z}$-stability can be used to obtain a classification result.

Theorem 8.5. Let $\mathcal{E}$ be the class of all C^*-algebras of the form $C(X) \rtimes_\alpha \mathbb{Z}$, where X is an infinite, compact, metric space of finite dimension, and α is induced by a uniquely ergodic minimal homeomorphism of X. Then $\mathcal{E}$ satisfies the Elliott conjecture.

Proof. If $A \in \mathcal{E}$, then A_y is simple, has a unique trace, real rank zero, ASH and hence TAF. By Theorem (8.3), $A \otimes B$ is TAF for any UHF-algebra B. By Theorem (6.5), $\mathcal{E}^{\mathcal{Z}}$ satisfies the Elliott conjecture. Finally, by theorem 8.4, $\mathcal{E} = \mathcal{E}^{\mathcal{Z}}$. □

This proof works if the projections of A separate traces. The natural questions is then

Question 8.6. What happens if the projections of A don't separate traces?

9. Dynamics and dimension.

Let (X, T) be as before.[4] The following definitions are due to Hirshberg, Winter and Zacharias.

Definition 9.1. We say that the *Rokhlin dimension of* (X,T) *is at most* n, written $\dim_{\mathrm{Rok}}(X,T) \leq n$, if for every $L \in \mathbb{Z}_{>0}$ there are open subsets $U_\ell^{(m)} \subseteq X, m = 0, \ldots, n, \ell = 1, \ldots, L$, satisfying:

(a) $T^{-1}\left(U_\ell^{(m)}\right) = U_{\ell+1}^{(m)}$ for $m = 0, \ldots, n$ and $\ell = 1, \ldots, L-1$ (we do not require that this condition hold for the last stage, $\ell = L$, of each Rokhlin tower).

(b) $U_\ell^{(m)} \cap U_{\ell'}^{(m)} = \emptyset$ if $\ell \neq \ell'$.

(c) $\left\{U_\ell^{(m)}\right\}_{m,\ell}$ covers X.

The following definition is similar to the Rokhlin dimension, but the open cover is supposed to refine a given cover, in a specific way (see condition (d) below).

Definition 9.2. We say that the *dynamical dimension of* (X,T) *is at most* n, written $\dim(X,T) \leq n$, if for every $L \in \mathbb{Z}_{>0}$ and every open cover $\mathcal{U}$ of X, there are open subsets $U_{k,\ell}^{(m)} \subseteq X, m = 0, \ldots, n, k = 1, \ldots, K^{(m)}, \ell = 1, \ldots, L$, satisfying:

(a) $T^{-1}\left(U_{k,\ell}^{(m)}\right) = U_{k,\ell+1}^{(m)}$ for $m = 0, \ldots, n, k = 1, \ldots, K^{(m)}$ and $\ell = 1, \ldots, L-1$ (again, we do not require that this condition hold for the last stage, $\ell = L$, of each Rokhlin tower).

(b) $U_{k,\ell}^{(m)} \cap U_{k',\ell'}^{(m)} = \emptyset$ if $(k,\ell) \neq (k',\ell')$.

(c) $\left\{U_{k,\ell}^{(m)}\right\}_{m,k,\ell}$ covers X.

(d) $\left\{T^{-p}\left(U_{k,\ell}^{(m)}\right)\right\}_{m,k,\ell}$ refines $\mathcal{U}$ for every $p = 0, \ldots, L$.

[4] We suspect that the definitions that come next could be more general, but we have not obtained results in a more general setting.

Notice that condition (d) in the above definition does not follow from condition (a), because the last stage of each tower need not be cyclically permuted by T^{-1}.

We usually think as m as standing for the "color" of the tower(s), $K^{(m)}$ as the number of Rokhlin towers of color m (these towers are indexed by $k = 1, \ldots, K^{(m)}$), and L as the number of stages that each Rokhlin tower has (these stages are indexed by $\ell = 1, \ldots, L$). Rokhlin towers consist of disjoint sets, and Rokhlin towers of the *same color* are disjoint from each other, while this need not be true if the colors are different. Also, the different stages of each Rokhlin tower (except the last one), are cyclically permuted by T^{-1}.

Definition 9.3. We say that (X, T) has *slow dynamical dimension growth*, if for every ϵ and every open cover $\mathcal{U}$ of X, there n, L and open subsets $U_{k,\ell}^{(m)} \subseteq X, m = 0, \ldots, n, k = 1, \ldots, K^{(m)}, \ell = 1, \ldots, L$, satisfying (a), (b), (c) and (d) in the above definition, and such that

$$\frac{n+1}{L} < \epsilon.$$

In other words, a system has slow dynamical dimension growth if the height of the Rokhlin towers increases faster than the number of colors of the refinement of $\mathcal{U}$.

Remark 9.4. It is clear from the definition that if the dynamical system has a factor of finite Rokhlin dimension, then the dynamical system itself has finite Rokhlin dimension. The analogous statement for the dynamical dimension is in general not true.

Proposition 9.5. Let (X, T) be as before. Then $\dim(X, T) < \infty$ if and only if $\dim_{\text{Rok}}(X, T) < \infty$ and $\dim X < \infty$.

If $\mathcal{U}, \mathcal{V}$ are open covers of X, then $\mathcal{U} \cup \mathcal{V}$ denotes the open cover of X obtained by taking all the elements in $\mathcal{U}$ or $\mathcal{V}$, as well as the intersection of elements of $\mathcal{U}$ with elements of $\mathcal{V}$. Also, if $\mathcal{U}$ is an open cover of X, $D(\mathcal{U})$ is the minimum order of all refinements of $\mathcal{U}$ to an open subcover of X (since X is compact, this is guaranteed to be a (finite) number).

Remark 9.6. If X has finite dimension and $\mathcal{U}$ is an open cover, then $D\left(\mathcal{U} \cup T\mathcal{U} \cup \cdots \cup T^L\mathcal{U}\right) \leq \dim X$. In particular,

$$\lim_{L\to\infty} \frac{D\left(\mathcal{U} \cup T\mathcal{U} \cup \cdots \cup T^L\mathcal{U}\right)}{L} \leq \lim_{L\to\infty} \frac{\dim X}{L} = 0.$$

Recall the definition of *mean dimension* of the dynamical system (X, T):

$$\operatorname{mdim}(X, T) = \sup_{\mathcal{U} \text{ open cover of } X} \left(\lim_{L\to\infty} \frac{D\left(\mathcal{U} \cup T\mathcal{U} \cup \cdots \cup T^L\mathcal{U}\right)}{L} \right).$$

If X has finite dimension, remark 9.6 implies that $\operatorname{mdim}(X) = 0$, regardless of what the map T is! More is true:

Theorem 9.7. If (X, T) has small dynamical dimension growth, then $\operatorname{mdim}(X) = 0$. Conversely, if $\operatorname{mdim}(X) = 0$ and $\dim_{\text{Rok}}(X, T) < \infty$, then (X, T) has slow dynamical dimension growth.

It is not known whether the hypothesis that (X, T) have finite Rokhlin dimension is necessary for the converse, and presumably it is. Indeed, the reason why slow dynamical dimension growth may be stronger than mean dimension zero, is that in the definition of the former, one does not care about the dynamics of the system (X, T) once one obtains the cover $\mathcal{U} \cup T\mathcal{U} \cup \cdots \cup T^L\mathcal{U}$. On the other hand, in the definition of slow dynamical dimension growth, condition (d) implies that $\left\{U_{k,\ell}^{(m)}\right\}_{m,k,\ell}$ refines $\mathcal{U} \cup T\mathcal{U} \cup \cdots \cup T^L\mathcal{U}$, and this subcover reflects the dynamics of (X, T), at least more than the cover $\mathcal{U} \cup T\mathcal{U} \cup \cdots \cup T^L\mathcal{U}$ itself.

A natural question is: are the Rokhlin dimension and the dynamic dimension ever finite? The following theorem, due to Hirshberg, Winter and Zacharias, answers this question positively.

Theorem 9.8. If $\dim X < \infty$, then $\dim_{\text{Rok}}(X, T) < \infty$ and $\dim(X, T) < \infty$.

Proof. Uses first return times to closed subsets with non-empty interior and the RSH-algebra structure of A_y. $\square$

Definition 9.9. Let $U, V \subseteq X$ be open subsets. If $m \in \mathbb{Z}_{>0}$, we sat that U is *m-subequivalent* to V, written $U \prec_m V$, if whenever $Y \subseteq U$ is a compact subset, there are open sets $U_k^{(j)} \subseteq U, V_k^{(j)} \subseteq V$, for $j = 0, \ldots, m, k = 1, \ldots, K^{(j)}$ satisfying:

(a) For each (j, k), there is $r_{(j) \in \mathbb{Z}}$ such that $T^{r_k^{(j)}}\left(U_k^{(j)}\right) \subseteq V_k^{(j)}$. In other words, each $U_k^{(j)}$ is *eventually* (after $r_k^{(j)}$ iterations) transported inside of $V_k^{(j)}$ by T.

(b) For each j, $\left\{U_k^{(j)}\right\}_{k=1}^{K^{(j)}}$ are pairwise disjoint.

(c) $\left\{U_k^{(j)}\right\}_{j,k}$ covers Y.

Notice that condition (a) and (b) together imply that for each j, $\left\{V_k^{(j)}\right\}_{k=1}^{K^{(j)}}$ are pairwise disjoint.

The following is the dynamic analogue of Cuntz-subequivalence for positive elements (one uses this equivalence to construct the Cuntz semigroup).

Definition 9.10. Given $m \in \mathbb{Z}_{>0}$, we say that the system (X, T) has *m-comparison* if whenever $U, V \subseteq X$ are open subsets with $\mu(U) < \mu(V)$ for every regular, T-invariant, Borel measure μ on X, then $U \prec_m V$. 0-comparison is also called *strict comparison*.

Theorem 9.11. If $\dim(X, T) < m$, then (X, T) has m-comparison.

We would like to have a dynamical version of $\mathcal{Z}$-stability. The problem is that there is no dynamical system that plays the role of the Jiang-Su algebra $\mathcal{Z}$. However, we are only interested in $\mathcal{Z}$-stability and not in $\mathcal{Z}$ itself, and $\mathcal{Z}$-stability can be characterized in terms of completely positive contractive order zero maps (see section 5, and theorem 5.14 in particular). In turn, completely positive contractive order zero maps can be expressed in terms of the Cuntz subequivalence, whose dynamical analogue is defined above. Hence it should be possible to come up with a dynamical definition of $\mathcal{Z}$-stability, for which we expect the following "theorem" to be true:

Theorem 9.12. (**to be**). Suppose that (X, T) is $\mathcal{Z}$-stable. Then
(a) $\dim_{\mathrm{Rok}}(X, T) \leq 1$.
(b) $C(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable.
(c) (X, T) has comparison.

This motivates a lot of questions:

Question 9.13. (With the right definition of dynamical $\mathcal{Z}$-stability.) Is it true that $\mathcal{Z}$-stability of $C(X) \rtimes_\alpha \mathbb{Z}$ implies dynamical $\mathcal{Z}$-stability of (X, T)?

Question 9.14. Is it true that $\dim_{\mathrm{nuc}}(C(X) \rtimes_\alpha \mathbb{Z}) < \infty$ implies that $\dim(X, T) < \infty$?

Question 9.15. Is it true that comparison in $C(X) \rtimes_\alpha \mathbb{Z}$ implies comparison in (X, T)?

One would start by showing implications among the dynamical properties, and then try to relate them to the ordinary properties of the crossed product.

10. Outlook and open problems.

We briefly describe some open problems which should serve as a guideline for future developments.

Problems 10.1. This is a collection of problems or proposed generalizations.
(1) If A is finite and has finite nuclear dimension, does this imply that $\mathrm{dr} A < \infty$? This is predicted by classification.
(2) If A is nuclear, does it follow that it has locally finite nuclear dimension? This problem is suspected to be much harder than the previous one. Evidence in this direction would be, for example, having a result of the form *if $A \otimes \mathcal{Z}$ has locally finite nuclear dimension, then so does A itself.* But this is not known to be true.
(3) Prove a non-simple, non-unital version of the regularity conjecture (7.1).
(4) Prove a range result for the Cuntz semigroup of algebras that satisfy the regularity conjecture: which semigroups arise as the Cuntz semigroup of a C^*-algebra that satisfies conjecture 7.1?
(5) Prove a unified classification result for purely infinite and purely finite, $\mathcal{Z}$-absorbing, C^*-algebras.
(6) Classify TAS C^*-algebras.
(7) As of examples, Connes' odd spheres are not uniquely ergodic, and we don't know if the crossed products are isomorphic if and only if the trace spaces are homeomorphic.
(8) Generalize the results of the previous sections to free minimal actions of $\mathbb{Z}^d$. In particular, some replacement for the algebras A_y is needed.
(9) Is it possible to write $\mathcal{Z}$, or $\mathcal{Z} \otimes \mathcal{K}$, as a crossed product by a commutative C^*-algebra?
(10) Define a dynamical version of the Cuntz semigroup, and prove dynamic analogous of existing results involving the crossed product.
(11) The proof of theorem 9.8 heavily uses the RSH-algebra structure of A_Y. Is there a dynamical proof of this result?
(12) Are there applications of these dynamical results (to dynamical systems)?
(13) Interpret Giordano-Putnam-Skau's classification as the 0-th dimensional version of a higher dimensional phenomenon, not necessarily restricting to Cantor spaces.