NUCLEAR DIMENSION AND DYNAMICS

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Warning: little proofreading has been done.

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Introduction

Nuclear dimension plays an important role in the classification programme. It is therefore crucial to find good estimates for the nuclear dimension of crossed products. It has turned out that a certain dynamical property, the Rokhlin property is very useful in obtaining good estimates. Moreover, one can define a higher dimensional version of it which still gives good estimates and which for large classes of dynamical systems on compact spaces is automatically verified.

We will begin with some background in approximation properties related to nuclearity and exactness, and then define nuclear dimension and study some of its basic properties, including a number of examples related to group C^*-algebras and coarse geometry. Next we will survey the classical Rokhlin property and introduce higher dimensional versions and their applications to the theory of nuclear dimension. Most of the current results are restricted to actions of finite groups or the integers $\mathbb{Z}$.

1. Nuclear C^*-algebras

The notion of nuclearity arose when people were trying to endow a tensor product of C^*-algebras with a norm. Nuclearity is a regularity property for C^*-tensor products.

Let A and B be C^*-algebras, and assume that they are unital. One can form the algebraic tensor product of A and B, denoted $A \odot B$. There are natural embeddings of A and B into $A \odot B$ and one may ask whether there is a C^*-norm on $A \odot B$ extending the norms of A and B simultaneously. As it turns out, this is always possible. Indeed, if $A \hookrightarrow \mathcal{B}(\mathcal{H})$ and $B \hookrightarrow \mathcal{B}(\mathcal{K})$ are faithful representations, then one can faithfully represent $A \odot B$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. (Notice that $\mathcal{H} \otimes \mathcal{K}$ has an unambiguous definition.) The *minimal norm*, denoted by $\|\cdot\|_{min}$, is the norm induced on $A \odot B$, and it is independent of the faithful representation of A and B. One can then define the *minimal tensor product* of A and B by

$$A \otimes_{min} B = \overline{A \odot B}^{\|\cdot\|_{min}} \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{K}).$$

It can moreover be shown that $\|\cdot\|_{min}$ is the smallest C^*-norm on $A \odot B$ extending the norms of A and B (and sometimes it is the only one).

There is also a maximal norm constructed as follows. Let $\sigma\colon A \to \mathcal{B}(\mathcal{H})$ and $\rho\colon B \to \mathcal{B}(\mathcal{H})$ be faithful representations of A and B on the same Hilbert space with commuting ranges. There is an induced representation $\sigma \otimes \rho\colon A \odot B \to \mathcal{B}(\mathcal{H})$. One then defines $\mathcal{K}$ to be the sum of all such Hilbert spaces, and $\tau\colon A \odot B \to \mathcal{B}(\mathcal{K})$ to be the direct sum of all representations of the form $\rho \otimes \sigma$. The *maximal norm*], denoted by $\|\cdot\|_{max}$, is the norm induced on $A \odot B$ by τ. The *maximal tensor product* of A and B is then

$$A \otimes_{max} B = \overline{A \odot B}^{\|\cdot\|_{max}} \subseteq \mathcal{B}(\mathcal{K}).$$

These constructions naturally lead to the following definition.

Definition 1.1. A C^*-algebra A is said to be *nuclear* if $A \otimes_{min} B = A \otimes_{min} B$ for all C^*-algebras B, this is, if the minimal and the maximal norm on $A \odot B$ coincide for all C^*-algebras B.

Nuclear C^*-algebras form a very large class, containing all finite dimensional C^*-algebras (since $M_n \otimes B \cong M_n(B)$ has a unique norm), commutative C^*-algebras (since $C_0(X) \otimes B \cong C_0(X, B)$ has a unique norm), group C^*-algebras of amenable discrete groups, and more.

Nuclearity is closed under many natural operations: direct limits, quotients, extensions, passage to hereditary subalgebras, tensor products, crossed products by amenable groups. The class of nuclear C^*-algebra is *not* closed under passage to subalgebras. A subalgebra of a nuclear C^*-algebra is called an *exact* C^*-algebra.

There is an alternative description of nuclearity in terms of a certain approximation property that does not mention tensor products. We first define the class of maps by which nuclear C^*-algebras are approximated.

Definition 1.2. Let $\varphi\colon A \to B$ be a linear map.

(1) The map φ is said to be *positive* if $a \geq 0$ implies $\varphi(a) \geq 0$.
(2) The map φ is said to be *completely positive* if $\varphi \otimes \mathrm{id}_{M_n}\colon M_n(A) \to M_n(B)$ is positive for all $n \in \mathbb{N}$.

The following may be the most cited theorem in Operator Algebras.

Theorem 1.3. (Stinespring) Let $\varphi\colon A \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exist a Hilbert space $\mathcal{K}$, a homomorphism $\pi\colon A \to \mathcal{B}(\mathcal{K})$, and a bounded operator $V\colon \mathcal{H} \to \mathcal{K}$ such that

$$\varphi(a) = V^*\pi(a)V$$

for all $a \in A$. If moreover $\varphi(1) = 1$, then $V^*V = 1_{\mathcal{H}}$ and hence $\mathcal{H}$ can e regarded as a closed subspace of $\mathcal{K}$. We then have that φ is a cut-down of π, this is:

$$\pi(a) = \begin{pmatrix} \varphi(a) & * \\ * & * \end{pmatrix}$$

for all $a \in A$. If φ is not unital, we still have $\varphi(a) = h^{1/2}P_{\mathcal{H}}\pi(a)P_{\mathcal{H}}h^{1/2}$ for all $a \in A$, where $h = \varphi(1)$.

Theorem 1.4. Let A be a C^*-algebra. Then the following are equivalent:

(1) The C^*-algebra A is nuclear.
(2) There exists a net $(\psi_\lambda, F_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$, where for all $\lambda \in \Lambda$, the maps $\psi_\lambda\colon A \to F_\lambda$ and $\varphi_\lambda\colon F_\lambda \to A$ are completely positive with $\|\psi_\lambda\|$ and $\|\varphi_\lambda\|$ uniformly bounded, and
$$\lim_{\lambda\to\infty} \|\varphi_\lambda \circ \psi_\lambda(a) - a\| = 0$$
for all $a \in A$.
(3) The bidual A^{**} is hyperfinite, this is, it is a von Neumann inductive limit of finite dimensional algebras.

Proof. Most of these implications are highly non-trivial, except for (2) implies (1). Indeed, given an approximation as in (2), one can construct systems

$$A \otimes_{min} B \to F_\lambda \otimes_{min} B = F_\lambda \otimes_{max} B \to A \otimes_{max} B$$

approximating $A \otimes_{max} B$. Since the maps involved are contractive, it follows that $\|\cdot\|_{max} \leq \|\cdot\|_{min}$, and thus they are equal. □

Let us show how to construct approximations in a concrete example.

Example 1.5. Let X be a compact Hausdorff space and set $A = C(X)$. Let

$$\Lambda = \{\mathcal{U}\colon \mathcal{U} \text{ is a finite open cover of } X\}$$

with the partial order given by $\mathcal{U} \succeq \mathcal{V}$ if for all $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$. Given $\mathcal{U} \in \Lambda$, there are functions $f_U\colon X \to [0, \infty)$ for $U \in \mathcal{U}$ such that $\mathrm{supp}(f_U) \subseteq U$ and $\sum_{U\in\mathcal{U}} f_U(x) = 1$ for all $x \in X$. For each $U \in \mathcal{U}$, choose a probability measure ν_U supported on U. Define

$$\psi_{\mathcal{U}}\colon C(X) \to \mathbb{C}^{\mathcal{U}} \quad \text{by} \quad \psi_{\mathcal{U}}(f) = (\nu_U(f))_{U\in\mathcal{U}}$$

for all $f \in C(X)$. It follows that $\psi_{\mathcal{U}}$ is completely positive. Define

$$\varphi_{\mathcal{U}}\colon \mathbb{C}^{\mathcal{U}} \to C(X) \quad \text{by} \quad \varphi_U\left((\alpha_U)_{U\in\mathcal{U}}\right) = \sum_{U\in\mathcal{U}} \alpha_U f_U$$

for all $(\alpha_U)_{U\in\mathcal{U}}$ in $\mathbb{C}^{\mathcal{U}}$. One checks that $\varphi_{\mathcal{U}}$ is also completely positive and that $\varphi_{\mathcal{U}} \circ \psi_{\mathcal{U}}(f) \to f$ as $\mathcal{U} \to \infty$, for all $f \in C(X)$. It follows that $C(X)$ is nuclear.

Looking at this approximation of $C(X)$, it is actually possible to read off the covering dimension of X.

Definition 1.6. Let X be a compact space. Given $n \in \mathbb{N}$, we say that X has *covering dimension at most* n, written $\dim(X) \leq n$, if for every open cover $\mathcal{U}$ of X, there is a finite refinement $\mathcal{V}$ such that whenever $V_0, \ldots, V_{n+1}$ in $\mathcal{V}$ are different, then $V_0 \cap \cdot \cap V_{n+1} = \emptyset$.

Finally, the *covering dimension of* X is the smallest integer n such that $\dim(X) \leq n$.

A variant of the covering dimension is the decomposition dimension.

Definition 1.7. Given $n \in \mathbb{N}$, we say that X has *decomposition dimension at most* n, written $\mathrm{ddim}(X) \leq n$, if for every open cover $\mathcal{U}$ of X, there is a finite refinement $\mathcal{V}$ which is the disjoint union of subfamilies $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(n)}$ such that the elements of $\mathcal{V}^{(j)}$ are pairwise disjoint for all $j = 0, \ldots, n$.

Finally, the *decomposition dimension of* X is the smallest integer n such that $\mathrm{ddim}(X) \leq n$.

The covering dimension and the decomposition dimension agree for compact spaces, although they may be different for locally compact spaces.

2. Nuclear dimension

The connection between nuclearity and dimension of spaces is the fact that the maps $\varphi_{\mathcal{U}} \colon \mathbb{C}^{\mathcal{U}} \to C(X)$ have order n in some sense. This leads to the notion of completely positive rank of a C^*-algebra, denoted by $\mathrm{cpr}(A)$. This notion was introduced by Winter in 2002, and it is not a very well-behaved dimension theory.

On the other hand, the notion of decomposition dimension for compact spaces gives rise to a better-behaved dimension theory for C^*-algebras. If one has a decomposition $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(n)}$ as in the definition of decomposition dimension, then the maps $\varphi_{\mathcal{V}}|_{\mathbb{C}^{\mathcal{V}^{(j)}}} \colon \mathbb{C}^{\mathcal{V}^{(j)}} \to C(X)$ are order zero, this is, they preserve orthogonality.

For a general C^*-algebra, this notion leads to the concept of decomposition rank of a C^*-algebra A, denoted $\mathrm{dr}(A)$. This theory was introduced by Kirchberg-Winter in 2004. Further improvements of this theory lead to the notion of nuclear dimension of a C^*-algebra, denoted $\dim_{\mathrm{nuc}}(A)$, which was introduced by Zacharias-Winter in 2010.

We begin by formalizing the notion of order zero map.

Definition 2.1. (Winter-Zacharias) A completely positive map $\varphi \colon A \to B$ is said to have *order zero* if $ab = 0$ implies $\varphi(a)\varphi(b) = 0$ for elements a and b in A.

Completely positive order zero maps are in some sense close to being homomorphisms.

Theorem 2.2. (Winter-Zacharias) Let $\varphi \colon A \to B$ be a completely positive order zero map with A unital. Then there exists a homomorphism $\pi \colon A \to M(C^*(\varphi(A))) \subseteq M(B)$ such that

$$\varphi(a) = h\pi(a) = \pi(a)h = h^{1/2}\pi(a)h^{1/2}$$

for all $a \in A$, where $h = \varphi(1) \geq 0$.

We now turn to the definition of nuclear dimension and decomposition rank.

Definition 2.3. Let A be a C^*-algebra. Given $n \in \mathbb{N}$, we say that A has *decomposition rank at most* n, written $\mathrm{dr}(A) \leq n$, if there exists a completely positive approximation $(\psi_\lambda, F_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$, where for each λ, there is a decomposition $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\oplus\varphi_\lambda^{(j)}"'] & & A \\ & F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)} \arrow[ur, "\sum \psi_\lambda^{(j)}"'] & \end{tikzcd}$$

such that each $\varphi_\lambda^{(j)}$ is order zero and $\|\psi_\lambda\|, \|\varphi_\lambda\| \leq 1$ for all $\lambda \in \Lambda$ and all $j = 0, \ldots, n$.

Finally, the *decomposition rank* of A is the smallest integer n such that $\mathrm{dr}(A) \leq n$.

The *nuclear dimension* for A is a slight weakening of the decomposition rank, where instead of requiring $\|\varphi_\lambda\| \leq 1$, we just require $\|\varphi_\lambda^{(j)}\| \leq 1$ for all $j = 0, \ldots, n$ and for all $\lambda \in \Lambda$. (Notice that this implies $\|\varphi_\lambda\| \leq n + 1$ for all $\lambda \in \Lambda$.)

Remark 2.4. If A is a C^*-algebra, then $\dim_{\mathrm{nuc}}(A) \leq \mathrm{dr}(A)$.

Proposition 2.5. Let A be a C^*-algebra with finite decomposition rank. Then A is quasidiagonal.

Proof. It follows from a perturbation argument that the completely positive approximation satisfies

$$\lim_{\lambda \to \infty} \|\varphi_\lambda(ab) - \varphi_\lambda(a)\varphi_\lambda(b)\| = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} \|\varphi_\lambda(a)\| = \|a\|$$

for all a and b in A. Since each F_λ is finite dimensional, it follows that A is quasidiagonal. ☐

The conclusion of the above statement can be strengthened: it follows that A is *strongly* quasidiagonal.

Proposition 2.6. Let X be a compact Hausdorff space. Then

$$\dim_{\mathrm{nuc}}(C(X)) = \mathrm{dr}(C(X)) = \dim(C(X).$$

Proof. One checks that the finite dimensional C^*-algebras $F_\lambda^{(j)}$ in the definition of the decomposition rank are necessarily commutative, yielding an approximation as in the definition of covering dimension. This shows that $\mathrm{dr}(C(X)) = \dim(X)$. Proving $\dim_{\mathrm{nuc}}(C(X)) = \dim(X)$ takes a bit more work but it is analogous. ☐

Proposition 2.7. Nuclear dimension enjoys the following properties:

(1) $\dim_{\mathrm{nuc}}(A \oplus B) = \max\{\dim_{\mathrm{nuc}}(A), \dim_{\mathrm{nuc}}(B)\}$.

(2) $\dim_{\mathrm{nuc}}(A\otimes B) \leq (\dim_{\mathrm{nuc}}(A)+1)(\dim_{\mathrm{nuc}}(B)+1)-1$, and $\dim_{\mathrm{nuc}}(A\otimes B) = \dim_{\mathrm{nuc}}(A)$ whenever B is an AF-algebra. (Notice that the natural guess $\dim_{\mathrm{nuc}}(A\otimes B) \leq \dim_{\mathrm{nuc}}(A)+\dim_{\mathrm{nuc}}(B)$ based on $\dim(X\times Y) = \dim(X)+\dim(Y)$ is in general not true.)
(3) $\dim_{\mathrm{nuc}}(\varinjlim A_n) \leq \liminf \dim_{\mathrm{nuc}}(A_n)$.
(4) If I is an ideal in A, then $\dim_{\mathrm{nuc}}(A/I) \leq \dim_{\mathrm{nuc}}(A)$.
(5) Given a short exact sequence
$$0 \to I \to A \to B \to 0,$$
one has
$$\dim_{\mathrm{nuc}}(A) \leq \dim_{\mathrm{nuc}}(I) + \dim_{\mathrm{nuc}}(B) + 1.$$
(It is not known if the $+1$ is needed. Also, the analogous statement for the decomposition rank is not true since an extension of a quasidiagonal C^*-algebra by a quasidiagonal C^*-algebra need not be quasidiagonal.)
(6) If B is hereditary in A, then $\dim_{\mathrm{nuc}}(B) \leq \dim_{\mathrm{nuc}}(A)$.
(7) $\dim_{\mathrm{nuc}}(A) = 0$ if and only if A is an AF-algebra.

Proof. Statements (1), (2) and (3) are easy to prove. To prove (4), use Choi-Effros lifting Theorem to choose a completely positive lift $\rho\colon A/I \to A$ and compose it with the approximation for A to obtain a decomposition for A/I with the same number of summands. Property (5) takes considerable work. For (6), assume first that A is separable and that B is full in A. It follows that $A\otimes\mathcal{K} \cong B\otimes\mathcal{K}$, and hence they have the same nuclear dimension. In the general case, B is stably isomorphic to an ideal of A. One checks using quasi-central approximate units for ideal that $\dim_{\mathrm{nuc}}(I) \leq \dim_{\mathrm{nuc}}(A)$. □

Example 2.8. Let $\mathcal{T}$ be the Toeplitz algebra. Then $\dim_{\mathrm{nuc}}(\mathcal{T}) \leq 2$.

Proof. This follows from the fact that there is a short exact sequence
$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0,$$
and using properties 5 and 7 above, together with Proposition 2.6. □

Question 2.9. Assume that the extension $0 \to I \to A \to B \to 0$ has a splitting $B \to A$. Does it follow that
$$\dim_{\mathrm{nuc}}(A) \leq \dim_{\mathrm{nuc}}(B) + \dim_{\mathrm{nuc}}(I)?$$

Example 2.10. Let $\theta \in \mathbb{R}$. Then
$$\dim_{\mathrm{nuc}}(A_\theta) = \begin{cases} 1, & \text{if } \theta \text{ is irrational;} \\ 2, & \text{if } \theta \text{ is rational.} \end{cases}$$

Proof. If θ is irrational, Elliott proved that A_θ is an A$\mathbb{T}$-algebra, and hence its nuclear dimension is 1. If θ is rational, then A_θ can be deformed into $C(\mathbb{T}^2)$, so it has nuclear dimension 2. □

Example 2.11. Denote by $\mathcal{Z}$ the Jiang-Su algebra. Then $\dim_{\mathrm{nuc}}(\mathcal{Z}) = 1$, since its building blocks have nuclear dimension 1.

The Jiang-Su algebra plays an important role in classification, and in fact one has the following.

Theorem 2.12. Let A be a simple, separable, non-elementary C^*-algebraof finite nuclear dimension. Then $A \cong A\otimes\mathcal{Z}$.

Remark 2.13. There are examples of C^*-algebras with $\dim_{\mathrm{nuc}}(A) < \infty$ and $\mathrm{dr}(A) = \infty$. For instance, any Kirchberg algebra, since they are not quasidiagonal and have nuclear dimension at most 3 in general, and at most 2 whenever they satisfy the UCT.

However, the following question remains open.

Question 2.14. Suppose A is a simple C^*-algebra such that $\mathrm{dr}(A) < \infty$. Does it follow that $\dim_{\mathrm{nuc}}(A) = \mathrm{dr}(A)$?

There is an example of a unital C^*-algebra A with a faithful tracial state such tat $\dim_{\mathrm{nuc}}(A) < \infty$ and $\mathrm{dr}(A) = \infty$. This algebra is quasidiagonal and non-simple. The proof consists in showing that A is not strongly quasidiagonal.

3. Kirchberg algebras

We begin with the Cuntz algebras. Note that $\mathcal{O}_1 = C(S^1)$.

Theorem 3.1. Let $n \in \mathbb{N}$. Then $\dim_{\mathrm{nuc}}(\mathcal{O}_n) = 1$. Moreover, $\dim_{\mathrm{nuc}}(\mathcal{O}_\infty) \leq 2$. (It is now known that $\dim_{\mathrm{nuc}}(\mathcal{O}_\infty) = 1$, but the proof is more involved.)

Proof. Fix $n \in \mathbb{N}$. Denote by $\mathcal{T}_n$ the Toeplitz-Cuntz algebra, this is, the universal C^*-algebra generated by isometries $s_1, \ldots, s_n$ such that $s_j^* s_k = \delta_{j,k}$ for all $j,k \in \{1,\ldots,n\}$. Then there is a short exact sequence
$$0 \to \mathcal{K} \to \mathcal{T}_n \to \mathcal{O}_n \to 0.$$
Use the Fock representation of $\mathcal{T}_n$ to obtain completely positive maps $\mathcal{T}_n \to M_N$ and $\mathcal{T}_n \to M_N'$. Compose these with a Choi-Effros lift $\mathcal{O}_n \to \mathcal{T}_n$ and construct from this a completely positive contractive order zero approximation with 2 summands. One has to use, among other things, that any unital endomorphism of $\mathcal{O}_n$ is approximately inner.

It also follows that $\dim_{\mathrm{nuc}}(\mathcal{T}) \leq 2$. Since $\mathcal{O}_\infty$ is a direct limit of Cuntz-Toeplitz algebras, we conclude that $\dim_{\mathrm{nuc}}(\mathcal{O}_\infty) \leq 2$. □

We now know that the estimate in the following theorem can be improved to 2. When the algebra A is not assumed to satisfy the UCT, the finest estimate now available is $\dim_{\mathrm{nuc}}(A) \leq 3$.

Theorem 3.2. Let A be a unital Kirchberg algebra satisfying the UCT. Then $\dim_{\mathrm{nuc}}(A) \leq 5$.

Proof. By classification, A is isomorphic to an inductive limit of algebras of the form

$$(M_{m_1}(\mathcal{O}_{n_1}) \oplus \cdots \oplus M_{m_k}(\mathcal{O}_{n_k})) \otimes C(\mathbb{T}),$$

for some $k, m_1, \ldots, m_k \in \mathbb{N}$ and some $n_1, \ldots, n_k \in \mathbb{N} \cup \{\infty\}$. Such algebras have nuclear dimension at most 5, and the estimate passes to the limit. □

4. Uniform Roe algebras

Let (X, d) be a metric space. The topology of X refers to the small scale structure. In contrast, the coarse geometry refers to the large scale structure. For instance, the integers $\mathbb{Z}$ are equivalent to a line in this sense. For this reason, one mostly deals with discrete spaces in coarse geometry, since every metric space is coarse equivalent to one such space.

Definition 4.1. A discrete metric space (X, d) is said to be of *bounded geometry* if for all $R > 0$ and all $x \in X$, the cardinality of $B_R(x)$ is uniformly bounded on x.

One can associate a C^*-algebra to every discrete metric space of bounded geometry as follows. Consider infinite matrices $(\alpha_{x,y})_{x,y\in X}$ with complex coefficients such that

(1) There exists $R > 0$ such that $\alpha_{x,y} = 0$ whenever $d(x, y) > R$, and
(2) There exists $M \geq 0$ such that $|\alpha_{x,y}| \leq M$ for all $x, y \in X$.

Consider the Hilbert space

$$\ell^2(X) = \{(\beta_x)_{x\in X} \colon \sum_{x\in X} |\beta_x|^2 < \infty\}.$$

Then each matrix $(\alpha_{x,y})_{x,y\in X}$ as above defines a bounded operator on $\ell^2(X)$. The set of all such matrices is a $*$-subalgebra of $\mathcal{B}(\ell^2(X))$, which we denote by $\mathcal{UC}(X)$.

Definition 4.2. Define the *uniform Roe algebra* of a discrete metric space (X, d) by

$$\mathcal{UC}^*_R(X) = \overline{\mathcal{UC}(X)} \subseteq \mathcal{B}(\ell^2(X)).$$

Example 4.3. Let Γ be a discrete group, with length function $\ell(x) = \min\{n \colon x = s_1 \cdots s_n, s_j \in S\}$, where S is a set of generators with $S^{-1} = S$. Now, $d(x, y) = \ell(x^{-1}y)$ defines a metric on Γ of bounded geometry. Then,

$$\mathcal{UC}^*_R(X) \cong \ell^\infty(\Gamma) \rtimes_{\mathtt{Lt},r} \Gamma \subseteq \mathcal{B}(\ell^2(\Gamma)).$$

Definition 4.4. Two metric spaces X and Y are said to be *coarse equivalent* if there exist $M > 0$ and continuous maps $f \colon X \to Y$ and $g \colon Y \to X$ such that

$$d(f \circ g(y), y) < M \quad \text{and} \quad d(g \circ f(x), x) < M$$

for all $x \in X$ and all $y \in Y$.

It turns out that the uniform Roe algebra is a complete invariant for coarse equivalence of discrete metric spaces. In other words:

Theorem 4.5. Let X and Y be discrete metric spaces. Then X and Y are coarse equivalent if and only if $\mathcal{UC}^*_R(X) \cong \mathcal{UC}^*_R(Y)$.

Although it seems like the Roe algebras will not be nuclear in most cases, it actually is in many situations of interest.

Theorem 4.6. (Guentner-Karmintnar-Ozawa, 2000) Let Γ be a discrete finitely generated group. Then the following are equivalent:

(1) The group Γ is exact.
(2) The C^*-algebra $\mathcal{UC}^*_R(X)$ is nuclear.

The class of groups covered in this Theorem is very large. Presumably every discrete group is exact!

We introduce the notion of asymptotic dimension for a discrete metric space (X, d).

Definition 4.7. A *uniform cover* of X is a cover by sets of uniformly bounded diameter. (Notice that a uniform cover on an infinite space must be infinite.)

Given $n \in \mathbb{N}$, we say that X has *asymptotic dimension at most* n, written $\mathrm{asdim}(X) \leq n$, if for every uniform cover $\mathcal{U}$ of X, there exists another uniform cover $\mathcal{V}$ which is refined by $\mathcal{U}$ (this is, $\mathcal{V}$ is coarser than $\mathcal{U}$) such that $\mathcal{V}$ has order n, this is, no point in X is in more than $n + 1$ elements of $\mathcal{V}$.

Finally, the *asymptotic dimension* of X is the smallest integer n such that $\mathrm{asdim}(X) \leq n$.

Examples 4.8. Some computations of asymptotic dimensions:

(1) For the real line, one has $\operatorname{asdim}(\mathbb{R}) = 1$.
(2) For the integers, one has $\operatorname{asdim}(\mathbb{Z}) = 1$. Indeed, the uniform cover $\{(n, n+1) \colon n \in \mathbb{Z}\}$ shows that $\operatorname{asdim}(\mathbb{Z}) \neq 0$.
(3) If K is a compact space, then $\operatorname{asdim}(K) = 0$.

Theorem 4.9. (Winter-Zacharias) Let X be a discrete metric space. Then

$$\dim_{\mathrm{nuc}}(\mathcal{UC}^*_R(X)) \leq \operatorname{asdim}(X).$$

There is evidence that equality may hold.

5. Crossed products

There are in general no good estimates for the nuclear dimension of $A \rtimes G$ in terms of the nuclear dimension of A, even for $G = \mathbb{Z}$ or G finite. A useful dynamical property in this situation is the Rokhlin property. This is best seen in the finite group case.

Definition 5.1. Let A be a unital C^*-algebra, let G be a finite group, and let $\alpha \colon G \to \operatorname{Aut}(A)$ be an action. We say that α has the *Rokhlin property* if for all finite subsets $F \subseteq A$ and all $\varepsilon > 0$, there are projections e_g in A for $g \in G$ such that

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all g and h in G.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) $\sum_{g \in G} e_g = 1$.

Theorem 5.2. Let $\alpha \colon G \to \operatorname{Aut}(A)$ be a finite group action with the Rokhlin property. Then

$$\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq \dim_{\mathrm{nuc}}(A).$$

Proof. Let $n = |G|$. One may think of $A \rtimes_\alpha G$ as a subalgebra of $M_n(A)$ via

$$a u_g \mapsto \sum_{h \in G} e_{gh,h} \otimes \alpha^{-1}(a) \in M_n \otimes A.$$

Recall that $\dim_{\mathrm{nuc}}(A) = \dim_{\mathrm{nuc}}(M_n(A))$, and denote this value by N. One has a diagram

$$\begin{tikzcd}
A \rtimes_\alpha G \arrow[dr] & & & \\
& M_n(A) \arrow[dr] & & M_n(A). \\
& & F^{(0)} \oplus \cdots \oplus F^{(N)} \arrow[ur] &
\end{tikzcd}$$

Given a finite subset $F \subseteq A$ and $\varepsilon > 0$, choose a family of Rokhlin projections $e_g \in A$ for $g \in G$ and define $\rho \colon M_n(A) \to A \rtimes_\alpha G$ by

$$\rho(e_{g,h} \otimes a) = e_h u_g a u_h^* e_h.$$

Then

- ρ is approximately order zero on $M_n((F)$.
- $\rho(a u_g) \approx a u_g$ for $a \in F$ and $g \in G$.

Given a finite set $\widetilde{F} \subseteq A \rtimes_\alpha G$, let $F \subseteq A$ be the set of all matrix coefficients of elements of $\widetilde{F}$ when regarded as a subset of $M_n(A)$. This yields the diagram

$$\begin{tikzcd}
A \rtimes_\alpha G \arrow[rrrr, "\mathrm{id}_A"] \arrow[dr] & & & & A \rtimes_\alpha G \\
& M_n(A) \arrow[dr] & & M_n(A) \arrow[ur, "\rho"'] & \\
& & F^{(0)} \oplus \cdots \oplus F^{(N)} \arrow[ur, "\oplus\varphi^{(j)}"'] & &
\end{tikzcd}.$$

The maps $\rho \circ \varphi^{(j)}$ are approximately order zero, and hence can be perturbed to get an N-decomposable system. $\square$

Remark 5.3. A similar argument shows that in the presence of the Rokhlin property, one has $\dim_{\mathrm{nuc}}(A^\alpha) \leq \dim_{\mathrm{nuc}}(A)$. Moreover, the analogous statements for the decomposition rank are true.

The Rokhlin property is quite restrictive, and it is therefore necessary to introduce a more flexible notion.

Definition 5.4. Let A be a unital C^*-algebra, let G be a finite group, and let $\alpha \colon G \to \operatorname{Aut}(A)$ be an action. Given $n \in \mathbb{N}$, we say that α has *Rokhlin dimension at most n*, written $\operatorname{Rok}_{\dim}(\alpha) \leq n$, if for all finite subsets $F \subseteq A$ and all $\varepsilon > 0$, there is a family of positive elements $\left(f_g^{(j)}\right)_{g \in G, j = 0, \ldots, n}$ in A such that

(1) $\|f_g^{(j)} f_h^{(j)} - f_h^{(j)} f_g^{(j)}\| < \varepsilon$ for all g and h in G and all $j = 0, \ldots, n$.
(2) $\|\alpha_g\left(f_h^{(j)}\right) - f_{gh}^{(j)}\| < \varepsilon$ for all g and h in G and all $j = 0, \ldots, n$.
(3) $\|f_g^{(j)} a - a f_g^{(j)}\| < \varepsilon$ for all $g \in G$, all $j = 0, \ldots, n$ and all $a \in F$.
(4) $\|\sum_{g \in G, j=0,\ldots,n} f_g^{(j)} - 1\| < \varepsilon$.

Finally, the *Rokhlin dimension* of α is thee smallest integer n such that $\mathrm{Rok}_{\mathrm{dim}}(\alpha) \leq n$.

Once again, we have estimates for the nuclear dimension of the completely positive.

Theorem 5.5. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action. Then

$$\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq (\mathrm{Rok}_{\mathrm{dim}}(\alpha) + 1)(\dim_{\mathrm{nuc}}(A) + 1) - 1.$$

Proof. The argument is now similar, except that one gets $n+1$ maps $\rho^{(0)}, \ldots, \rho^{(n)}$ at the last stage, each of which comes from a different tower. □

We turn to automorphisms.

Definition 5.6. An action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ is said to have *Rokhlin dimension at most* n, written $\mathrm{Rok}_{\mathrm{dim}}(\alpha) \leq n$, if for all finite subsets $F \subseteq A$, for all $p \in \mathbb{N}$ and all $\varepsilon > 0$, there is a family of positive elements $\left(f_k^{(j)}\right)_{k=0,\ldots,p-1, j=0,\ldots,n}$ in A such that

(1) $\|f_k^{(j)} f_\ell^{(j)} - f_h^{(j)} f_g^{(j)}\| < \varepsilon$ for all $k, \ell = 0, \ldots, p-1$ and all $j = 0, \ldots, n$.
(2) $\|\alpha\left(f_k^{(j)}\right) - f_{k+1}^{(j)}\| < \varepsilon$ for all $k = 0, \ldots, p-1$ and all $j = 0, \ldots, n$, where the indices are taken modulo p.
(3) $\|f_k^{(j)} a - a f_k^{(j)}\| < \varepsilon$ for all $k = 0, \ldots, p-1$, all $j = 0, \ldots, n$ and all $a \in F$.
(4) $\|\sum_{k=0,\ldots,p-1, j=0,\ldots,n} f_k^{(j)} - 1\| < \varepsilon$.

Finally, the *Rokhlin dimension* of α is thee smallest integer n such that $\mathrm{Rok}_{\mathrm{dim}}(\alpha) \leq n$.

Theorem 5.7. Let A be a unital C^*-algebra and let α be an automorphism of A. Then

$$\dim_{\mathrm{nuc}}(A \rtimes_\alpha \mathbb{Z}) \leq 2(\mathrm{Rok}_{\mathrm{dim}}(\alpha) + 1)(\dim_{\mathrm{nuc}}(A) + 1) - 1.$$

Automorphisms with finite Rokhlin dimension are plentiful: they are generic on $\mathcal{Z}$-stable C^*-algebras. One moreover has

Theorem 5.8. Let X be a finite dimensional compact metric space and let $\alpha \in \mathrm{Aut}(C(X))$ be given by a minimal homeomorphism of X. Then

$$\mathrm{Rok}_{\mathrm{dim}}(\alpha) \leq 2\dim(X) + 1.$$