Construction and basic properties of full and reduced crossed product

Paul Lunkenheimer

23.10.2019

Dynamical Systems

If (G, X) is a locally compact transformation group, we can define $\alpha : G \to Aut\ C_0(X)$ by

$$\alpha_s(f)(x) = f(s^{-1}x).$$

Lemma

α is a homomorphism and continuous into the point-norm topology.

Definition

If A is a C^*-algebra, G a loc. comp. group and $\alpha : G \to Aut\ A$ a continuous homomorphism into the point-norm topology, we call (A, G, α) a C^*-dynamical system.

Examples

- $h \in Homeo(X)$, $G = \mathbb{Z}$ and $n \cdot x = h^n(x)$
- G locally compact group, then G acts on G by multiplication and we get $(C_0(G), G, lt)$
- If $\beta \in Aut\, A$, we can define $(A, \mathbb{Z}, \alpha)$ where $\alpha_n(a) = \beta^n(a)$
- $(\mathbb{C}, G, \mathrm{id})$
- $(A, \{e\}, \mathrm{id})$

Proposition

If $(C_0(X), G, \alpha)$ is a dynamical system, there is a group action of G on X such that

$$\alpha_s(f)(x) = f(s^{-1} \cdot x).$$

Covariant Representations

Definition

If $\pi : A \to B(H)$ is a representation and $U : G \to \mathcal{U}(H)$ is a unitary representation, such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^*,$$

we call (π, U) a covariant representation of (A, G, α).

Proposition (regular representations)

If $\pi : A \to B(H)$ is a rep., we get a covariant representation $(\widetilde{\pi}, U)$ on $L^2(G, H)$ by defining

$$\widetilde{\pi}(a)(h)(s) = \pi(\alpha_s^{-1}(a))(h(s)) \text{ and } U_r(h)(s) = h(r^{-1}s).$$

U is faithful and if π is faithful, so is $\widetilde{\pi}$.

Proof.

$$\begin{aligned} U_s\widetilde{\pi}(a)U_s^*(h)(r) &= \widetilde{\pi}(a)U_s^*(h)(s^{-1}r) = \pi(\alpha_{r^{-1}s}(a))(U_s^*(h)(s^{-1}r)) \\ &= \widetilde{\pi}(\alpha_s(a))(h)(r) \end{aligned}$$

$\square$

Lemma

If π is nondegenderate, $\widetilde{\pi}$ is nondegenerate as well.

Examples

- $(C_0(G), G, lt)$ with the cov. rep. (M, λ) on $L^2(G)$, where
$$M_f(h) = fh$$
and λ is the left regular representation.
- $z \mapsto e^{2\pi i\theta} z$ is a Homeomorphism on $\mathbb{T}$ and we get the dynamical system $(C(\mathbb{T}), \mathbb{Z}, \alpha)$. Take a representation ev_z of $C(\mathbb{T})$, then
$$\widetilde{ev_z}(f)(h)(n) = ev_z(\alpha_{-n}(f))(h(n)) = f(e^{2\pi i\theta n} z)h(n)$$
is the induced regular representation on $L^2(G)$.
- covariant representations of $(A, \{e\}, \mathrm{id})$ are the representations of A
- nondegenerate covariant representations of $(\mathbb{C}, G, \mathrm{id})$ are the unitary representations of G

The *-Algebra $C_C(G, A)$

Proposition

We get a *-Algebra structure on the vector-space $C_C(G, A)$ by defining the convolution

$$f * g(s) = \int f(r)\alpha_r(g(r^{-1}s))d\,\mu(r)$$

and the involution

$$f^*(s) = \triangle(s^{-1})\alpha_s(f(s^{-1})^*).$$

Proof.

$$(g * f)^*(s) = \triangle(s^{-1})\alpha_s\Big(\int g(r)\alpha_r(f(r^{-1}s^{-1}))\,d\mu(r)^*\Big)$$

$$= \triangle(s^{-1})\int \alpha_{sr}(f((sr)^{-1})^*)\alpha_s(g(r)^*)\,d\mu(r)$$

$$= \int \triangle(r^{-1})\alpha_r(f(r^{-1})^*)\alpha_r(\triangle((r^{-1}s)^{-1})\alpha_{r^{-1}s}(g((r^{-1}s)^{-1})^*)\,d\mu(r)$$

$$= f^* * g^*(s)$$

$\square$

The *-algebra $C_C(G, A)$

Proposition

We get a *-Algebra structure on the vector-space $C_C(G, A)$ by defining the convolution

$$f * g(s) = \int f(r)\alpha_r(g(r^{-1}s))d\,\mu(r)$$

and the involution

$$f^*(s) = \triangle(s^{-1})\alpha_s(f(s^{-1})^*).$$

Integrated forms

Proposition

If (π, U) is a covariant representation on H, then

$$\pi \rtimes U(f) = \int \pi(f(s))U_s \, d\mu(s)$$

defines a L^1-norm decreasing *-homomorphism of $C_C(G, A)$ on H called the integrated form of (π, U).

Proof.

$$\begin{aligned}\pi \rtimes U(f^*) &= \int \pi(\triangle(s^{-1})\alpha_s(f(s^{-1})^*)U_s \, d\mu(s) \\ &= \int U_s \pi(f(s^{-1})^*) \triangle (s^{-1}) \, d\mu(s) \\ &= \int U_{s^{-1}} \pi(f(s))^* \, d\mu(s) = \pi \rtimes U(f)^*\end{aligned}$$

$\square$

Integrated forms

Proposition

If (π, U) is a covariant representation on H, then

$$\pi \rtimes U(f) = \int \pi(f(s))U_s \, d\mu(s)$$

defines a L^1-norm decreasing *-homomorphism of $C_C(G, A)$ on H called the integrated form of (π, U).

Lemma

If (π, U) is a covariant representation with π nondegenerate, then $\pi \rtimes U$ is nondegenerate.

Proof.

Let $\xi \in H$, $\varepsilon > 0$. Choose

- $a \in A : \|\pi(a)\xi - \xi\| < \frac{\varepsilon}{2}$ and $\|a\| \leq 1$
- neighborhood V of e: $\|U_s\xi - \xi\| < \frac{\varepsilon}{2} \forall s \in V$
- $\varphi \in C_c^+(G) :$ $\operatorname{supp}\varphi \subseteq V$ and $\int \varphi(s)\, d\mu(s) = 1$

Now define $f = \varphi \otimes a \in C_c(G, A)$.

$$\begin{aligned}
\|\pi \rtimes U(f)\xi - \xi\| &= \|\int \pi(f(s))U_s\, d\mu(s)\xi - \xi \int \varphi(s)\, d\mu(s)\| \\
&= \|\int \varphi(s)(\pi(a)U_s\xi - \xi)\, d\mu(s)\| \\
&\leq \int \varphi(s)(\|\pi(a)(U_s\xi - \xi)\| + \|\pi(a)\xi - \xi\|)\, d\mu(s) < \varepsilon
\end{aligned}$$

$\square$

Lemma

Let π be a faithful representation and $(\widetilde{\pi}, U)$ the associated regular representation. Then $\widetilde{\pi} \rtimes U$ is faithful on $C_c(G, A)$.

For $r \in G$ define $\iota_G(r) : C_c(G, A) \to C_c(G, A)$ by

$$\iota_G(r)(f)(s) = \alpha_r(f(r^{-1}s)).$$

Note that for any covariant representation (π, U)

$$\begin{aligned} \pi \rtimes U(\iota_G(r)f) &= \int \pi(\alpha_r(f(r^{-1}s)))U_s \, d\mu(s) \\ &= \int U_r \pi(f(s))U_s \, d\mu(s) = U_r \pi \rtimes U(f). \end{aligned}$$

Proof.

Let $0 \neq f \in C_c(G, A)$, then WLOG $f(e) \neq 0$. Choose

- $\xi, \eta \in H: \ c := \langle \pi(f(e))\xi, \eta \rangle \neq 0$
- nbhd. V of e: $|\langle \pi(\alpha_{r^{-1}}(f(s)))\xi, \eta\rangle - c| < \frac{|c|}{2} \forall r, s \in V$
- symmetric nbhd. W of e such that $W^2 \subseteq V$
- $\varphi \in C_c^+(G)$: $\operatorname{supp}\varphi \subseteq W$, $\int\int \varphi(s^{-1}r)\varphi(r)\, d\mu(r)d\mu(s) = 1$

Define $\widetilde{\xi} = \varphi \otimes \xi, \widetilde{\eta} = \varphi \otimes \eta \in L^2(G, H)$.

$$\begin{aligned} |\left\langle \widetilde{\pi} \rtimes U(f)\widetilde{\xi}, \widetilde{\eta}\right\rangle - c| &= |\int\int \left\langle \widetilde{\pi}(f(s))U_s\widetilde{\xi}(r), \widetilde{\eta}(r)\right\rangle \, d\mu(r)d\mu(s) - c| \\ &\leq \int\int \varphi(s^{-1}r)\varphi(r)|\langle \pi(\alpha_r^{-1}(f(s)))\xi, \eta\rangle - c|\, d\mu(r)d\mu(s) \\ &< \frac{|c|}{2} \end{aligned}$$

$\square$

The universal norm

Definition

We can define the C^*-norm

$$\|f\| = \sup\{\|\pi \rtimes U(f)\| \mid (\pi, U) \text{ covariant representation of } (A, G, \alpha)\}$$

and call it the universal norm. The completion $A \rtimes_\alpha G$ of $C_C(G, A)$ by this norm is called the crossed product of A by G.

The universal norm

Definition

We can define the C^*-norm

$$\begin{aligned}\|f\| &= \sup\{\|\pi \rtimes U(f)\| \mid (\pi, U) \text{ covariant representation of } (A, G, \alpha)\} \\ &= \sup\{\|\pi \rtimes U(f)\| \mid (\pi, U) \text{ nondeg. cov. rep. of } (A, G, \alpha)\}\end{aligned}$$

and call it the universal norm. The completion $A \rtimes_\alpha G$ of $C_C(G, A)$ by this norm is called the crossed product of A by G.

Examples

- the crossed product of $(A, \{e\}, \mathrm{id})$ is A
- the crossed product of $(\mathbb{C}, G, \mathrm{id})$ is $C^*(G)$
- Take $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ with $\alpha_n(f)(z) = f(e^{2\pi i\theta n}z)$.Then the crossed product for θ irrational is A_θ
- If we have G finite with $|G| = n$, then the crossed product of $(C_0(G), G, lt)$ is M_n

Proof.

Remember the covariant representation (M, λ). Now $\{\delta_s\}_{s\in G}$ is an ONB for the Hilbert space $L^2(G)$, thus $B(L^2(G)) \cong M_n$. For $f \in C_c(G, C_0(G))$ we calculate

$$M \rtimes \lambda(f)(h)(s) = \sum_{r\in G} f(r,s)h(r^{-1}s) = \sum_{r\in G} f(sr^{-1}, s)h(s),$$

so $M \rtimes \lambda(f)$ is given by the matrix $(f(sr^{-1}, s))_{(s,r)}$. Thus $M \rtimes \lambda$ is bijective.

$\square$

Representations of the crossed product

Proposition

There is a homomorphism $\iota_A : A \to M(A \rtimes_\alpha G)$ and a strictly continuous unitary valued homomorphism $\iota_G : G \to \mathcal{U}M(A \rtimes_\alpha G)$ such that for $f \in C_C(G, A)$

$$\iota_G(r)(f)(s) = \alpha_r(f(r^{-1}s)) \text{ and } \iota_A(a)(f)(s) = af(s).$$

(ι_A, ι_G) is covariant in that

$$\iota_A(\alpha_r(a)) = \iota_G(r)\iota_A(a)\iota_G(r)^*.$$

ι_A is nondegenerate, both are faithful and if (π, U) is a nondegenerate covariant representation, then

$$(\pi \rtimes U)^{-}(\iota_A(a)) = \pi(a) \text{ and } (\pi \rtimes U)^{-}(\iota_G(s)) = U_s.$$

Proof.

Remember that

$$\pi \rtimes U(\iota_G(r)f) = U_r \pi \rtimes U(f).$$

We also have

$$\pi \rtimes U(\iota_A(a)f) = \int \pi(af(s))U_s \, d\mu(s) = \pi(a)\pi \rtimes U(f),$$

so we can extend the maps onto $A \rtimes_\alpha G$. Furthermore they are adjointable since

$$(\iota_G(r)f)^* * g = f^* * \iota_G(r^{-1})g,$$
$$(\iota_A(a)f)^* * g = f^* * \iota_A(a^*)g.$$

Now

$$\iota_A(a)(h \otimes b) = h \otimes ab.$$

$\square$

Lemma

There is a homomorphism $\widetilde{\iota}_G : C^*(G) \to M(A \rtimes_\alpha G)$ such that for $h \in C_C(G)$

$$\widetilde{\iota}_G(h) = \int h(s)\iota_G(s)\, d\mu(s).$$

Furthermore $\iota_A(a)\widetilde{\iota}_G(h) = h \otimes a$.

Proof.

$$\iota_A(a)\iota_G(h) = \int \iota_A(a)h(s)\iota_G(s)\, d\mu(s) = \int \iota_A(h\otimes a(s))\iota_G(s)\, d\mu(s).$$

Now if (π, U) is a nondegenerate covariant representation, then

$$(\pi \rtimes U)^{-}(\iota_A(a)\iota_G(h)) = \int \pi(h \otimes a(s))U_s\, d\mu(s) = \pi \rtimes U(h \otimes a).$$

$\square$

Proposition

If ρ is a nondegenerate representation of $A \rtimes_{\alpha} G$ on H, then

$$U_s = \overline{\rho}(\iota_G(s)) \text{ and } \pi(a) = \overline{\rho}(\iota_A(a))$$

define a nondegenerate covariant representation (π, U), such that

$$\rho = \pi \rtimes U.$$

Proof.

We have

$$\pi(A)\rho(A \rtimes_\alpha G)H = \rho(\iota_A(A)A \rtimes_\alpha G)H,$$

so π is nondegenerate. Furthermore

$$\begin{aligned}\pi \rtimes U(\iota_A(a)\iota_G(h)) &= \pi(a)\int h(s)U_s\,d\mu(s)\\ &= \bar{\rho}(\iota_A(a))\int h(s)\bar{\rho}(\iota_G(s))\,d\mu(s) = \rho(\iota_A(a)\iota_G(h)).\end{aligned}$$

$\square$

The reduced crossed product

Definition

We can define the C^*-norm

$$\|f\|_r = \sup\{\|\widetilde{\pi} \rtimes U(f)\| \mid (\widetilde{\pi}, U) \text{ regular representation of } (A, G, \alpha)\}$$

and the completion $A \rtimes_{\alpha,r} G$ of $C_C(G, A)$ by this norm is called the reduced crossed product of A by G.

Proposition

If π is a faithful representation of A and $(\widetilde{\pi}, U)$ is the associated regular representation, then $\widetilde{\pi} \rtimes U$ is faithful on $A \rtimes_{\alpha,r} G$.