Crossed products with amenable groups and functoriality properties

In the last lecture we defined, given a C*-dynamical System (A, G, α), a C*-Algebra $A \rtimes_\alpha G$, called the *crossed product* of the dynamical System. The construction of this algebra was similar to the construction of the group-C*-Algebra $C^*(G)$ of a locally compact group G. Especially we were also able to construct a reduced crossed product $A \rtimes_{\alpha,r} G$ similar to the reduced group-C*-Algebra $C_r^*(G)$. Considering this one would expect that some properties of group-C*-Algebras are also valid for the full and reduced crossed product. For example, if the group G is amenable we know that $C^*(G)$ and $C_r^*(G)$ are the same and that $C^*(G) = C_r^*(G)$ is nuclear. One of the main objectives of this lecture is to show that these two facts remain true for crossed products.

Furthermore, it turns out that the full and reduced crossed product are in many ways comparable to maximal and minimal tensor products of C*-Algebras. Specifically we want to see that the full and reduced crossed product define functors on suitable categories and just like in the tensor product case the functor for the full crossed product turns out to be exact, while the functor in the reduced case is not always exact.

These notes are based on the book [Wil07] by Dana P. Williams and the lecture notes [Phi] by N. Christopher Phillips.

We start by a short recap of the last lecture.

Preliminaries. Let (A, G, α) be a dynamical system. A *covariant representation* on a Hilbert space H is a pair (π, u), where $\pi : A \to B(H)$ is a *-homomorphism and $u : G \to U(H)$ is a unitary representation of G so that for each $s \in G, a \in A$ we have

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^*.$$

Given a covariant representation (π, u) we can define it's *integrated form* to be

$$\pi \rtimes u : C_c(G, A) \to B(H); f \mapsto \int_G \pi(f(s)) U_s \, ds.$$

This gives a *-homomorphism if we equip $C_c(G, A)$ with the multiplication

$$f * g(s) = \int_G f(r) \alpha_r(g(r^{-1}s)) \, dr$$

and involution

$$f^*(s) = \Delta(s^{-1}) \alpha_s(f(s^{-1})^*).$$

For $f \in C_c(G, A)$ we can now set

$$\|f\| = \sup\{\|\pi \rtimes u(f)\| \mid (\pi, u) \text{ covariant repr. of } (A, G, \alpha)\}.$$

This defines a C*-Norm on $C_c(G, A)$, called the *universal norm*. The universal norm remains unchanged if we restrict the definition to nondegenerate representations (π, u), where nondegenerate means that the representation π is nondegenerate.

The completion with respect to this norm is a C*-Algebra $A \rtimes_\alpha G$, called the *crossed product* of (A, G, α).

Now let $\pi : A \to B(H)$ be an arbitrary representation of A. Then we can define a representation

$$\widetilde{\pi} : A \to B(L^2(G, H)),$$

so that for $a \in A$, $\xi \in C_c(G, H)$ and $s \in G$ we have $\widetilde{\pi}(a)\xi(r) = \pi(\alpha_r^{-1}(a))(\xi(r))$. Further we define a unitary representation $V^\pi : G \to U(B(L^2(G, H)))$ by $V_s^\pi \xi(r) = \xi(s^{-1}r)$. The pair $(\widetilde{\pi}, V^\pi)$ is then a covariant representation of (A, G, α) and the integrated form $\widetilde{\pi} \rtimes V^\pi$ is called the *regular representation* associated to π and will be denoted by $\operatorname{Ind}(\pi)$. If we define for $f \in C_c(G, A)$

$$\|f\|_r = \sup\{\| \operatorname{Ind}(\pi)(f)\| \mid \pi \text{ repr. of } A\}$$

we get another norm on $C_c(G, A)$ called the *reduced norm*. The completion with respect to this norm will be called the *reduced crossed product* and will be denoted by $A \rtimes_{\alpha,r} G$.

If π is a faithful representation of A then $\operatorname{Ind}(\pi)$ is a faithful representation of $A \rtimes_{\alpha,r} G$.

By definition, we have for all $f \in C_c(G, A)$ that $\|f\|_r \leq \|f\|$. Thus, we get a canonical surjection

$$\kappa : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$$

extending the identity.

Our first objective is to show that this map κ is an isomorphism if G is amenable. To show this we first need to see that given a covariant representation (π, u) the representation $\operatorname{Ind}(\pi)$ is equivalent to a representation derived from (π, u). We first fix some notation.

Let H, K be Hilbert spaces. If $T \in B(H)$ and $S \in B(K)$ we can define an operator $T \otimes S \in B(H \otimes K)$ by $T \otimes S(x \otimes y) = T(x) \otimes S(y)$. If $\pi : A \to B(H)$ is a representation of A we can thus define a representation

$$1 \otimes \pi : A \to B(K \otimes H); a \mapsto \operatorname{id}_K \otimes \pi(a).$$

If $u : G \to U(K)$ and $v : G \to U(H)$ are unitary representations then

$$u \otimes v : G \to U(K \otimes H); s \mapsto u_s \otimes v_s$$

is a well defined unitary representation as well.

If $f \in C_c(G)$ and $h \in H$ then we can define an element $f\widetilde{\otimes}v$ in $C_c(G,H)$ by $f\widetilde{\otimes}v(s) = f(s)v$. This gives a unitary map $\Phi : L^2(G) \otimes H \mapsto L^2(G,H)$ which maps $f \otimes h$ to $f\widetilde{\otimes}h$ for $f \in C_c(G)$, $h \in H$.

Lemma 1. Let (A, G, α) be a dynamical system and (π, u) a covariant representation. Let Φ be as above and let λ be the left regular representation of G. Then the pair $(1 \otimes \pi, \lambda \otimes u)$ is a covariant representation that is unitarily equivalent to $(\widetilde{\pi}, V^\pi)$.

Proof. We first show that $(1 \otimes \pi, \lambda \otimes u)$ is covariant. Let $a \in A, s \in G, f \in C_c(G), v \in H$. Then we have

$$\begin{aligned} 1 \otimes \pi(\alpha_s(a))(f \otimes v) &= f \otimes \pi(\alpha_s(a))v = f \otimes (u_s\pi(a)u_s^*)v \\ &= \lambda_s \otimes u_s 1 \otimes \pi(a) \lambda_{s^{-1}} \otimes u_{s^{-1}}(f \otimes v). \end{aligned}$$

This proves the covariance.

We now define the unitary map that implements the equivalence. Let

$$z : C_c(G,H) \mapsto C_c(G,H); \xi \mapsto z(\xi),$$

where $z(\xi)(r) = u_r^{-1}(\xi(r))$. Then $z(\xi)$ is continuous since if $r_\lambda \to r$ we have

$$z(\xi)(r_\lambda) - z(\xi)(r) = \underbrace{u_{r_\lambda^{-1}}(\xi(r_\lambda)) - u_{r_\lambda^{-1}}(\xi(r))}_{\to 0 \text{ since } u_{r_\lambda} \text{ is unitary}} + \underbrace{u_{r_\lambda^{-1}}(\xi(r)) - u_{r^{-1}}(\xi(r))}_{\to 0} \to 0.$$

Since $\operatorname{supp}(z(\xi)) = \operatorname{supp}(\xi)$ the map z is well defined. An easy calculation shows that z is linear. For $z \in C_c(G,H)$ we have

$$\begin{aligned} \|z(\xi)\|_2^2 &= \int_G \langle z(\xi)(s), z(\xi)(s) \rangle \, ds = \int_G \langle u_s^{-1}(\xi(s))), u_s^{-1}(\xi(s))) \rangle \, ds \\ &= \int_G \langle \xi(s), \xi(s) \rangle = \|\xi\|_2^2. \end{aligned}$$

Thus z extends to an isometric linear map $L^2(G,H) \to L^2(G,H)$. If we set

$$w : C_c(G,H) \to C_c(G,H); \xi \mapsto w(\xi),$$

where $w(\xi)(r) = u_r(\xi(r))$, then by the same argument as aboze w extends to an isometric linear map $L^2(G,H) \to L^2(G,H)$. By definition w is the inverse of z on $C_c(G,H)$. By density if follows that w is the inverse of z on $L^2(G,H)$. It follows that z is a unitary map. Let $\Phi : L^2(G) \otimes H \to L^2(G,H)$ be the unitary map defined above. The equivalence is then implemented by the unitary map $z\Phi$.

To see this let $r \in G, f \in C_c(G), v \in H$. Then we have

$$z\Phi\lambda_r \otimes u_r\Phi^{-1}(f\widetilde{\otimes}v) = z(\lambda_r(f)\widetilde{\otimes}u_r(v)).$$

So for $s \in G$ we have

$$\begin{aligned} z\Phi\lambda_r \otimes u_r\Phi^{-1}(f\widetilde{\otimes}v)(s) &= u_s^{-1}(\lambda_r(f)\widetilde{\otimes}u_r(v)(s)) = u_s^{-1}(f(r^{-1}s)u_r(v)) \\ &= u_{(r^{-1}s)^{-1}}(f(r^{-1}s)v) = u_{(r^{-1}s)^{-1}}((f\widetilde{\otimes}v)(r^{-1}s)) \\ &= z(f\widetilde{\otimes}v)(r^{-1}s) = V_r^{\pi}(z(f\widetilde{\otimes}v))(s). \end{aligned}$$

This gives

$$z\Phi\lambda_r \otimes u_r\Phi^{-1}z^{-1} = V_r^{\pi}.$$

For $a \in A$ we further have

$$\begin{aligned} &z\Phi 1 \otimes \pi(a)\Phi^{-1}(f\widetilde{\otimes}v)(s) = z(f\widetilde{\otimes}\pi(a)v)(s) = u_s^{-1}(f(s)\pi(a)v) \\ &= u_{s^{-1}}\pi(a)f(s)v = \pi(\alpha_{s^{-1}}(a))(u_s^{-1}f(s)v) = \widetilde{\pi}(a)(z(f\widetilde{\otimes}v))(s). \end{aligned}$$

This gives

$$z\Phi 1 \otimes \pi(a)\Phi^{-1}z^{-1} = \widetilde{\pi}.$$

This proves the claim. □

We now give a short reminder on amenability.

Definition 2. Let G be a locally compact group. Then G is called *amenable* if given a compact subset C of G and $\epsilon > 0$, there exists a nonnegative function $f \in C_c(G)$ with $\|f\|_2 = 1$ such that for every $s \in C$ we have

$$\|\lambda_s(f) - f\|_2 < \epsilon,$$

where λ is the left regular representation of G.

Remark 3. This is only one of the many possible ways of defining amenability. We could for example replace the 2-norm in the above formulation by any p-norm for $1 \leq p < \infty$ and would get an equivalent definition of amenability.

There are many amenable groups. For example all abelian groups and all compact groups are amenable.

A good reference regarding amenability is [Pat88]. The equivalence of Definition 2 with more standard definitions of amenability is proved there, as are the facts mentioned above.

Theorem 4. Let (A, G, α) be a dynamical system and let G be amenable. Then the canonical map

$$\kappa : A \rtimes_{\alpha} G \to A \rtimes_{\alpha,r} G$$

is an isomorphism.

Proof. If suffices to show that κ is isometric on $C_c(G, A)$. So let $f \in C_c(G, A)$. Since $\|f\|_r \leq \|f\|$ it suffices to show that $\|f\| \leq \|f\|_r$. For this it suffices to show that given a covariant representation (π, u) of (A, G, α) we have $\|\pi \rtimes u(f)\| \leq \|f\|_r$.

So let (π, u) be a covariant representation of (A, G, α). We will show that $\|\pi \rtimes u(f)\| \leq \|\operatorname{Ind}(\pi)(f)\|$, which implies the claim. By Lemma 1 the covariant representation $(1 \otimes \pi, \lambda \otimes u)$ is unitarily equivalent to the covariant representation $(\widetilde{\pi}, V^\pi)$. But then the integrated forms are equivalent as well. So we have $\|\operatorname{Ind}(\pi)(f)\| = \|(1 \otimes \pi) \rtimes (\lambda \otimes u)(f)\|$ and it suffices to show that

$$\|\pi \rtimes u(f)\| \leq \|(1 \otimes \pi) \rtimes (\lambda \otimes u)(f)\|.$$

This is trivial if $\|\pi \rtimes u(f)\| = 0$. So let $\|\pi \rtimes u(f)\| > 0$. Let $0 < \epsilon < \|\pi \rtimes u(f)\|$. Then there exists $\xi_0 \in H$ with $\|\xi_0\| = 1$ and

$$\|\pi \rtimes u(f)\| - \frac{\epsilon}{2} < \|\pi \rtimes u(f)(\xi_0)\|.$$

Let $S = \operatorname{supp}(f)$, let $C > 0$ be a constant so that $\|f(s)\| \leq C$ for alle $s \in G$ and let $L > 0$ be a constant so that $\mu(S) < L$. Since S is compact and G is amenable we get a nonnegative function $g \in C_c(G)$ with $\|g\|_2 = 1$, such that for every $s \in S$ we have

$$\|\lambda_s(g) - g\|_2 < \frac{\epsilon}{2CL}.$$

Now consider $g \otimes \xi_0 \in L^2(G) \otimes H$. We have

$$\|g \otimes \xi_0\| = \|g\|_2 \|\xi_0\| = 1.$$

We further have

$$\begin{aligned}
&(1 \otimes \pi) \rtimes (\lambda \otimes u)(f)(g \otimes \xi_0) = \int_G \mathrm{id}_{L^2(G)} \otimes \pi(f(s))\lambda_s \otimes u_s(g \otimes \xi_0)\, ds \\
&= \int_G \lambda_s(g) \otimes \pi(f(s))u_s(\xi_0)\, ds \\
&= \int_G (\lambda_s(g) - g) \otimes \pi(f(s))u_s(\xi_0)\, ds + \int_G g \otimes \pi(f(s))u_s(\xi_0)\, ds.
\end{aligned}$$

We now have

$$\begin{aligned}
&\| \int_G (\lambda_s(g) - g) \otimes \pi(f(s))u_s(\xi_0)\, ds\| \leq \int_G \|(\lambda_s(g) - g) \otimes \pi(f(s))u_s(\xi_0)\|\, ds \\
&= \int_S \|(\lambda_s(g) - g) \otimes \pi(f(s))u_s(\xi_0)\|\, ds = \int_S \|(\lambda_s(g) - g)\| \|\pi(f(s))u_s(\xi_0)\|\, ds \\
&\leq \int_S \frac{\epsilon}{2CL} C\, ds = \mu(S)\frac{\epsilon}{2L} \leq \frac{\epsilon}{2}.
\end{aligned}$$

But this gives

$$
\begin{aligned}
&\|(1 \otimes \pi) \rtimes (\lambda \otimes u)(f)(g \otimes \xi_0)\| \\
&\geq \| \int_G g \otimes \pi(f(s)) u_s(\xi_0) \, ds\| - \| \int_G (\lambda_s(g) - g) \otimes \pi(f(s)) u_s(\xi_0) \, ds\| \\
&\geq \| \int_G g \otimes \pi(f(s)) u_s(\xi_0) \, ds\| - \frac{\epsilon}{2} = \|g \otimes \int_G \pi(f(s)) u_s(\xi_0) \, ds\| - \frac{\epsilon}{2} \\
&= \|g \otimes (\pi \rtimes u(f)(\xi_0))\| - \frac{\epsilon}{2} = \|g\|_2 \|\pi \rtimes u(f)(\xi_0)\| - \frac{\epsilon}{2} \\
&= \|\pi \rtimes u(f)(\xi_0)\| - \frac{\epsilon}{2} > \|\pi \rtimes u(f)\| - \epsilon.
\end{aligned}
$$

We thus have

$$\|(1 \otimes \pi) \rtimes (\lambda \otimes u)(f)\| \geq \|(1 \otimes \pi) \rtimes (\lambda \otimes u)(f)(g \otimes \xi_0)\| > \|\pi \rtimes u(f)\| - \epsilon.$$

Since $0 < \epsilon < \|\pi \rtimes u(f)\|$ was arbitrary we finally have

$$\|\pi \rtimes u(f)\| \leq \|(1 \otimes \pi) \rtimes (\lambda \otimes u)(f)\|.$$

This proves the claim. □

Remark 5. Let G be a locally compact group. It can be shown that if the natural map

$$C^*(G) \to C_r^*(G)$$

is an isomorphism, then G amenable. Thus, the natural map is an isomorphism if and only if the group is amenable, and we get yet another equivalent definition of amenability. A proof can be found in [Wil07, Theorem A.18].

This property of group-C^*-algebras does not generalize to arbitrary dynamical systems. To see this consider an arbitrary locally compact group G and the dynamical system $(C_0(G), G, \mathrm{lt})$ given by the action of G on itself by left translation. Using imprimitivity theorems it can be show that the natural map

$$C_0(G) \rtimes_{\mathrm{lt}} G \to C_0(G) \rtimes_{\mathrm{lt},r} G$$

is always an isomorphism. See for example [Wil07, Theorem 4.23], where it is shown that $C_0(G) \rtimes_{\mathrm{lt}} G$ is simple. This gives the claim.

We now turn to studying the functoriality properties of the crossed product.

Definition 6. Let (A, G, α) and (B, G, β) be dynamical systems. A homomorphism $\varphi : A \to B$ is called *equivariant* if for all $a \in A, s \in G$ we have

$$\varphi(\alpha_s(a)) = \beta_s(\varphi(a)).$$

We now show that every equivariant homomorphism induces a homomorphism between the corresponding crossed products.

Theorem 7. Suppose that (A, G, α) and (B, G, β) are dynamical systems and that $\varphi : A \to B$ is equivariant. Then the map

$$\Phi : C_c(G, A) \to C_c(G, B); f \mapsto \varphi \circ f$$

is a $*$-homomorphism. It extends to $*$-homomorphisms

$$\varphi \rtimes \mathrm{id} : A \rtimes_\alpha G \to B \rtimes_\beta G$$

and

$$\varphi \rtimes_r \mathrm{id} : A \rtimes_{\alpha,r} \to B \rtimes_{\beta,r} G.$$

Proof. We first show that Φ is a $*$-homomorphism. It is obvious that Φ is linear. Let $f, g \in C_c(G, A)$. For $s \in G$ we then have

$$\begin{aligned}\Phi(f)\Phi(g)(s) &= \int_G \Phi(f)(r)\beta_r(\Phi(g)(r^{-1}s))\,dr = \int_G \varphi(f(r))\beta_r(\varphi(g(r^{-1}s)))\,dr \\ &= \int_G \varphi(f(r)\alpha_r(g(r^{-1}s)))\,dr = \varphi(\int_G f(r)\alpha_r(g(r^{-1}s))\,dr) = \varphi(fg(s)).\end{aligned}$$

This gives $\Phi(f)\Phi(g) = \varphi \circ (fg) = \Phi(fg)$. We further have

$$\begin{aligned}\Phi(f)^*(s) &= \Delta(s^{-1})\beta_s(\Phi(f)(s^{-1})^*) = \Delta(s^{-1})\beta_s(\varphi(f(s^{-1})^*)) \\ &= \varphi(\Delta(s^{-1})\alpha_s(f(s^{-1})^*)) = \varphi(f^*(s)).\end{aligned}$$

Thus, we have $\Phi(f)^* = \Phi(f^*)$ and Φ is a $*$-homomorphism.

We now show that Φ is bounded with respect to the universal norms. For this let $f \in C_c(G, A)$ and let (π, u) be a covariant representation of (B, G, β). Then $(\pi\circ\varphi, u)$ is a covariant representation of (A, G, α) since for $a \in A, s \in G$ we have

$$\pi \circ \varphi(\alpha_s(a)) = \pi(\beta_s(\varphi(a))) = u_s\pi(\varphi(a))u_s^*.$$

This gives

$$\begin{aligned}\|\pi \rtimes u(\Phi(f))\| &= \| \int \pi(\Phi(f)(s))u_s\,ds\| = \| \int_G \pi \circ \varphi(f(s))u_s\,ds\| \\ &= \|(\pi \circ \varphi) \rtimes u(f)\| \leq \|f\|.\end{aligned}$$

Since (π, u) was an arbitrary covariant representation we get $\|\Phi(f)\| \leq \|f\|$. Thus Φ extends to a $*$-homomorphism $\varphi \rtimes \mathrm{id} : A \rtimes_\alpha G \to B \rtimes_\beta G$.

We now show that Φ is bounded with respect to the reduced norms. For this let π be a representation of B on a Hilbert space H. Then we get a covariant representation $(\widetilde{\pi}, V^\pi)$ of (B, G, β). But $\pi \circ \varphi$ is a representation of A on H and we get a covariant representation $(\widetilde{\pi \circ \varphi}, V^{\pi\circ\varphi})$ of (A, G, α) on $B(L^2(G, H))$. Since V^π depends only on the underlying Hilbert space of π we get $V^{\pi\circ\varphi} = V^\pi$.

For $a \in A, \xi \in C_c(G, H)$ and $s \in G$ we have

$$\begin{aligned}\widetilde{\pi}(\varphi(a))(\xi)(s) &= \pi(\beta_s^{-1}(\varphi(a)))(\xi(s)) = \pi(\varphi(\alpha_s^{-1}(a)))(\xi(s)) \\ &= \widetilde{\pi \circ \varphi}(a)(\xi)(s).\end{aligned}$$

This gives $\widetilde{\pi} \circ \varphi = \widetilde{\pi \circ \varphi}$. We get

$$\begin{aligned}\mathrm{Ind}(\pi)(\Phi(f)) &= \widetilde{\pi} \rtimes V^\pi(\Phi(f)) = (\widetilde{\pi} \circ \varphi) \rtimes V^\pi(f) \\ &= \widetilde{\pi \circ \varphi} \rtimes V^{\pi\circ\varphi}(f) = \mathrm{Ind}(\pi \circ \varphi)(f),\end{aligned}$$

where the second equality follows from the calculation for the universal norm. Thus, we have

$$\|\mathrm{Ind}(\pi)(\Phi(f))\| = \|\mathrm{Ind}(\pi \circ \varphi)(f)\| \leq \|f\|_r.$$

This π was arbitrary it follows that $\|\Phi\|_r \leq \|f\|_r$. Thus, Φ extends to a $*$-homomorphism $\varphi \rtimes_r \mathrm{id} : A \rtimes_{\alpha,r} G \to B \rtimes_{\beta,r} G$. □

Definition 8. Let G be a locally compact group. A dynamical system (A, G, α) is called a *G-algebra*. The G-algebras form a category with equivariant homomorphisms as morphisms. The assignments

$$(A, G, \alpha) \mapsto A \rtimes_\alpha G, \qquad \varphi \mapsto \varphi \rtimes \mathrm{id}$$

and

$$(A, G, \alpha) \mapsto A \rtimes_{\alpha,r} G, \qquad \varphi \mapsto \varphi \rtimes_r \mathrm{id}$$

define covariant functors from the category of G-algebras to the category of C^*-algebras.

Proof. Let (A, G, α), (B, G, β) and (C, G, γ) be dynamical systems and $\varphi : A \to B$, $\psi : B \to C$ equivariant. Then for $a \in A$ and $s \in G$ we have

$$\psi \circ \varphi(\alpha_s(a)) = \psi(\beta_s(\varphi(a)) = \gamma_s(\psi \circ \varphi(a)),$$

so $\psi \circ \varphi$ is covariant as well. Since

$$\mathrm{id}_A(\alpha_s(a)) = \alpha_s(a) = \alpha_s(\mathrm{id}_A(a))$$

the identity is covariant as well. Thus, the G-algebras form a category.

For $f \in C_c(G, A)$ one has

$$(\psi \rtimes \mathrm{id}) \circ (\varphi \rtimes \mathrm{id})(f) = \psi \rtimes \mathrm{id}(\varphi \circ f) = \psi \circ \varphi \circ f = (\psi \circ \varphi) \rtimes \mathrm{id}(f).$$

By density one thus has $(\psi \rtimes \mathrm{id}) \circ (\varphi \rtimes \mathrm{id}) = (\psi \circ \varphi) \rtimes \mathrm{id}$. We further have

$$\mathrm{id}_A \rtimes \mathrm{id}(f) = \mathrm{id}_A \circ f = f = \mathrm{id}_{A \rtimes_\alpha G}(f).$$

We thus have $\mathrm{id}_A \rtimes \mathrm{id} = \mathrm{id}_{A\rtimes_\alpha G}$. Thus, the crossed product defines a functor. The same calculations show that the reduced crossed product defines a functor as well. $\square$

Corollary 9. Let (A, G, α) and (B, G, β) be dynamical systems and let $\varphi : A \to B$ be an equivariant isomorphism. Then we have

$$A \rtimes_\alpha G \xrightarrow[\varphi\rtimes\mathrm{id}]{\cong} B \rtimes_\beta G$$

and

$$A \rtimes_{\alpha,r} G \xrightarrow[\varphi\rtimes_r\mathrm{id}]{\cong} B \rtimes_{\beta,r} G.$$

Proof. If φ is equivariant, then we have for $b \in B$ and $s \in G$

$$\varphi^{-1}(\beta_s(b)) = \varphi^{-1}(\beta_s(\varphi(\varphi^{-1}(a)))) = \alpha_s(\varphi^{-1}(a)),$$

so φ^{-1} is equivariant as well. By the functoriality properties we thus have the equalities $(\varphi \rtimes \mathrm{id})^{-1} = \varphi^{-1} \rtimes \mathrm{id}$ and $(\varphi \rtimes_r \mathrm{id})^{-1} = \varphi^{-1} \rtimes_r \mathrm{id}$, which proves the claim. $\square$

Our next objective is to investigate the maps $\varphi \rtimes \mathrm{id}$ and $\varphi \rtimes_r \mathrm{id}$. For this we need a fact from set theoretic topology. The proof of the following theorem is based on [Echb, Satz 8.9].

Theorem 10 (Partitions of unity)**.** Let X be a locally compact space and let $K \subset X$ be compact. Let $U_1, ..., U_n \subset X$ be open sets with $K \subset \cup_{i=1}^n U_i$. Then there exist continuous functions $\varphi_1, ..., \varphi_n : X \to [0, 1]$ such that

- $\mathrm{supp}(\varphi_i)$ is compact and contained in U_i for $1 \leq i \leq n$
- $\sum_{i=1}^n \varphi_i(x) = 1$ for $x \in K$ and $\sum_{i=1}^n \varphi(x) \leq 1$ for all $x \in X$

Proof. We first show that we can find $V_1, ..., V_n$ open with $\overline{V_i}$ compact, $\overline{V_i} \subset U_i$ for $1 \leq i \leq n$ and $K \subset \cup_{i=1}^n V_i$. For this consider $x \in K$. Let U_x be the intersection of all the U_i, $1 \leq i \leq n$, that contain x. This is an open neighborhood of x, and since X is locally compact there is an open set V_x with $\overline{V_x}$ compact, $\overline{V_x} \subset U_x$ and $x \in V_x$. Then $(V_x)_{x\in K}$ is an open cover of the compact set K, and we can find a finite subcover $V_{x_1}, ..., V_{x_l}$. For $1 \leq i \leq n$ let V_i be the union over all the V_{x_j} with $x_j \in U_i$. Then V_i is an open set, and since V_i is the union of finitely many sets whose closures are contained in U_i, we also have that $\overline{V_i}$ is compact with $\overline{V_i} \subset U_i$. If we have $x \in K$ then x

is contained in some V_{x_j}. Now x_j is contained in some U_i. But then we have $x \in V_{x_j} \subset V_i$. Thus, we have $K \subset \cup_{i=1}^n V_j$.

By Urysohn's lemma we can now find functions $\psi_i : X \to [0,1]$ with $\psi_i|_{\overline{V_i}} = 1$ and $\operatorname{supp}(\psi_i) \subset U_i$. Then for all $x \in \cup_{i=1}^n V_i$ we have $\sum_{i=1}^n \psi_i(x) \geq 1$. Now again by Urysohn's lemma there is a function $g : X \to [0,1]$ with $g|_K = 1$ and $\operatorname{supp}(g) \subset \cup_{i=1}^n V_i$. Now set for $1 \leq i \leq n$

$$\varphi_i : X \to [0,1]; x \mapsto \begin{cases} \frac{g(x)\psi_i(x)}{\sum_{k=1}^n \psi_k(x)} & \text{if } x \in \cup_{k=1}^n V_k \\ 0 & \text{if } x \notin \operatorname{supp}(g) \end{cases}.$$

Since $\operatorname{supp}(g) \subset \cup_{k=1}^n V_k$ the function is defined for all $x \in X$. If we have $x \in (X \setminus \operatorname{supp}(g)) \cap \cup_{k=1}^n V_k$ it follows that

$$\frac{g(x)\psi_i(x)}{\sum_{k=1}^n \psi_k(x)} = 0$$

and thus φ_i is well defined. Since $X \setminus \operatorname{supp}(g)$ and $\cup_{k=1}^n V_k$ are open and the restriction of φ_i to these open sets is continuous it follows that φ_i is continuous as well. Since $\operatorname{supp}(\varphi_i) \subset \operatorname{supp}(g) \subset \cup_{k=1}^n \overline{V_i}$ the support of φ_i is compact and since $\operatorname{supp}(\varphi_i) \subset \operatorname{supp}(\psi_i) \subset U_i$ the first part of the claim follows.

For $x \in K$ we further have $\sum_{i=1}^n \varphi_i(x) = g(x) = 1$ and for $x \in X$ arbitrary we have $\sum_{i=1}^n \varphi_i(x) \leq 1$. This proves the theorem.

□

Lemma 11. Let A, B be C^*-algebras and let $\varphi : A \to B$ be a surjective $*$-homomorphism. Let X be a locally compact space. Then the maps

$$C_0(X, A) \to C_0(X, B); f \mapsto \varphi \circ f$$

and

$$C_c(X, A) \to C_c(X, B); f \mapsto \varphi \circ f$$

are surjective.

Proof. The map $C_0(X, A) \to C_0(X, B); f \mapsto \varphi \circ f$ is a $*$-homomorphism between C^*-algebras and hence has closed range. To prove the first claim it thus suffices to show that this homomorphism has dense range.

By Urysohn's lemma we have that $C_c(X, B) \subset C_0(X, B)$ is dense. We now show that $\operatorname{span}\{f \widetilde{\otimes} b | f \in C_c(X), b \in B\}$ is dense in $C_c(X, B)$ and thus in $C_0(X, B)$.

For this let $g \in C_c(X, B)$ and $\epsilon > 0$. Let $K = \operatorname{supp}(g)$. Now since g is continuous there exists for every $x \in K$ an open neighborhood V_x of x so that for $y \in V_x$ we have $\|g(x) - g(y)\| < \epsilon$. Then $(V_x)_{x \in K}$ is an open cover of K, and we can find a finite subcover $V_{x_1}, ..., V_{x_n}$. By Theorem 10 we can find $\varphi_1, ..., \varphi_n : X \to [0,1]$ with $\text{supp}(\varphi_i) \subset V_{x_i}$, $\text{supp}(\varphi_i)$ compact, $\sum_{i=1}^n \varphi \leq 1$ and for $x \in K$ we have $\sum_{i=1}^n \varphi_i(x) = 1$. Now set $f = \sum_{i=1}^n \varphi_i \widetilde{\otimes} g(x_i)$.

Now we have for $x \in X$ that $g(x) = \sum_{i=1}^n \varphi_i(x) g(x)$. Furthermore, we have for $1 \leq i \leq n$

$$\varphi_i(x)\|g(x_i) - g(x)\| \leq \varphi_i(x)\epsilon.$$

We thus have for $x \in X$

$$\begin{aligned}\|f(x) - g(x)\| &= \|\sum_{i=1}^n \varphi_i(x)(g(x_i) - g(x))\| \leq \sum_{i=1}^n \varphi_i(x)\|g(x_i) - g(x)\| \\ &\leq \sum_{i=1}^n \varphi_i(x)\epsilon = \epsilon.\end{aligned}$$

We thus have $\|f - g\|_\infty \leq \epsilon$. This proves the density. To prove the first claim it thus suffices to show that for $f \in C_c(X)$ and $b \in B$ we have that $f \widetilde{\otimes} b$ is in the range of the above map. Since φ is surjective there is $a \in A$ with $b = \varphi(a)$. Then $f \widetilde{\otimes} a \in C_c(G, A)$ and we have for $x \in X$

$$\varphi \circ f \widetilde{\otimes} a(x) = \varphi(f(x)a) = f(x)\varphi(a) = f(x)b = f \widetilde{\otimes} b(x).$$

This proves the first claim.

To prove the second claim let $g \in C_c(X, B)$. Let $K = \text{supp}(g)$. Let U be an open set with compact closure and $K \subset U$. Then $g|_U \in C_0(U, B)$. By the first part of the proof there is $f_0 \in C_0(U, A)$ so that $\varphi \circ f_0 = g|_U$. Then let

$$f = \begin{cases} f_0(x) & \text{if } x \in U \\ 0 & \text{else} \end{cases}.$$

Then $f \in C_c(X, A)$ and $\varphi \circ f = g$. □

Theorem 12. Let (A, G, α) and (B, G, β) be dynamical systems and let $\varphi : A \to B$ be equivariant.

(1) If φ is injective, then so is $\varphi \rtimes_r \text{id}$

(2) If φ is surjective, then so are $\varphi \rtimes_r \text{id}$ and $\varphi \rtimes \text{id}$

(3) If $\varphi(A) \subset B$ is an ideal, then $\varphi \rtimes_r \text{id}(A \rtimes_{\alpha,r} G) \subset B \rtimes_{\beta,r} G$ and $\varphi \rtimes \text{id}(A \rtimes_\alpha G) \subset B \rtimes_\beta G$ are ideals as well.

Proof. (1): Let $\pi : B \to B(H)$ be a faithful representation of B. Then $\text{Ind}(\pi) : B \rtimes_{\beta,r} G \to B(L^2(G, H))$ is faithful as well. Furthermore, $\pi \circ \varphi$ is a faithful representation of A, so that $\operatorname{Ind}(\pi \circ \varphi) : A \rtimes_{\alpha,r} G \to B(L^2(G,H))$ is also faithful. The calculation in Theorem 7 shows that

$$\operatorname{Ind}(\pi)(\varphi \rtimes_r \operatorname{id}(f)) = \operatorname{Ind}(\pi)(\varphi \circ f) = \operatorname{Ind}(\pi \circ \varphi)(f)$$

which gives

$$\|\varphi \rtimes_r \operatorname{id}(f)\|_r = \|\operatorname{Ind}(\pi)(\varphi \rtimes_r \operatorname{id}(f))\| = \|\operatorname{Ind}(\pi \circ \varphi)(f)\| = \|f\|_r.$$

Thus $\varphi \rtimes_r \operatorname{id}$ is isometric on the dense subspace $C_c(G,A)$ and hence is an isometric map.

(2): Let φ be surjective. It suffices to show that $\varphi \rtimes_r \operatorname{id}$ and $\varphi \rtimes \operatorname{id}$ have dense range. But if $g \in C_c(G,B)$, then by Theorem 11 there is a $f \in C_c(G,A)$ with

$$g = \varphi \circ f = \varphi \rtimes_r \operatorname{id}(f) = \varphi \rtimes \operatorname{id}(f).$$

(3): Since $\varphi \rtimes \operatorname{id}(A \rtimes_\alpha G) = \overline{\varphi \rtimes \operatorname{id}(C_c(G,A))}^{\|\cdot\|}$ and $\varphi \rtimes_r \operatorname{id}(A \rtimes_{\alpha,r} G) = \overline{\varphi \rtimes_r \operatorname{id}(C_c(G,A))}^{\|\cdot\|_r}$, it suffices to show that for $f \in C_c(G,A)$ and $g \in C_c(G,B)$ there are $h_1, h_2 \in C_c(G,A)$ with $g(\varphi \circ f) = \varphi \circ h_1$ and $(\varphi \circ f)g = \varphi \circ h_2$.

For $s \in G$ we have

$$g(\varphi \circ f)(s) = \int_G g(r)\beta_r(\varphi(f(r^{-1}s)))\,dr = \int_G \underbrace{g(r)\varphi(\alpha_r(f(r^{-1}s)))}_{\in \varphi(A)}\,dr \in \varphi(A).$$

Thus we have $g(\varphi \circ f) \in C_c(G, \varphi(A))$. But since $\varphi : A \to \varphi(A)$ is surjective there exists $h_1 \in C_c(G,A)$ with $g(\varphi \circ f) = \varphi \circ h_1$ by Theorem 11.

Similarly, we have

$$(\varphi \circ f)g(s) = \int_G \varphi(f(r))\beta_r(g(r^{-1}s))\,dr \in \varphi(A).$$

This gives $(\varphi \circ f)g \in C_c(G, \varphi(A))$, so again there is $h_2 \in C_c(G,A)$ with $(\varphi \circ f)g = \varphi \circ h_2$. This proves the claim. □

Now let (A, G, α) be a dynamical system and let $I \subset A$ be an ideal. We call I *α-invariant* if for every $a \in I$ and $s \in G$ we have $\alpha_s(I) \subset I$. It then follows that $\alpha_s(I) = I$, since for $a \in I$ we have $a = \alpha_s(\underbrace{\alpha_{s^{-1}}(a)}_{\in I}) \in \alpha_s(I)$. It follows that $\alpha_s|_I \in \operatorname{Aut}(I)$ and we get a dynamical system (I, G, α) by restricting the automorphisms from the action α. If I is an α-invariant ideal we can further set for $s \in G$

$$\alpha_s^I : {}^{A}\!/_{I} \longrightarrow {}^{A}\!/_{I}; a + I \mapsto \alpha_s(a) + I.$$

This is well defined since if $a, b \in A$ with $a - b \in I$, we have $\alpha_s(a) - \alpha_s(b) = \alpha_s(a-b) \in I$. Furthermore, α_s^I is a $*$-homomorphism with inverse $\alpha_{s^{-1}}^I$, so $\alpha_s^I \in \mathrm{Aut}(A/I)$. The map $s \mapsto \alpha_s^I$ is a strongly continuous group homomorphism, so we get a dynamical system $(A/I, G, \alpha^I)$.

The maps $\iota : I \to A; a \mapsto a$ and $q : A \to A/I; a \mapsto a + I$ are equivariant since for $s \in G$, $a \in I$ and $b \in A$ we have

$$\iota(\alpha_s(a)) = \alpha_s(a) = \alpha_s(\iota(a))$$

and

$$q(\alpha_s(b)) = \alpha_s(b) + I = \alpha_s^I(a + I) = \alpha_s^I(q(a)).$$

Thus we get maps

$$I \rtimes_\alpha G \xrightarrow{\iota \rtimes \mathrm{id}} A \rtimes_\alpha G \xrightarrow{q \rtimes \mathrm{id}} A/I \rtimes_{\alpha^I} G$$

and

$$I \rtimes_{\alpha,r} G \xrightarrow{\iota \rtimes_r \mathrm{id}} A \rtimes_{\alpha,r} G \xrightarrow{q \rtimes_r \mathrm{id}} A/I \rtimes_{\alpha^I,r} G.$$

We now want to investigate whether these maps give short exact sequences of C^*-algebras. For this we need a lemma.

Lemma 13. Let G be a locally compact group. Then there is an bounded self-adjoint approximate unit for $C^*(G)$ in $C_c(G)$.

Proof. The set of neighborhoods of the neutral element e of G becomes a directed set if we set $U \leq V \iff V \subset U$.

If V is a neighborhood of the neutral element there is a symmetric neighborhood U of the identity with $U \subset V$. By Urysohn's lemma there is a nonnegative function $f \in C_c(G)$ with $f(e) = 1$ and $\mathrm{supp}(f) \subset U$. By scaling if necessary we can assume that $\int_G f(s)\,ds = 1$. Set $u_V = \frac{f+f^*}{2}$. This gives a net $(u_V)_V$ in $C_c(G)$ with u_V nonnegative, integral one, $\mathrm{supp}(u_V) \subset V$ and $u_V^* = u_V$. Since $(u_V)_V$ is bounded in the L^1-norm the net is also bounded in the universal norm on $C^*(G)$ by 1.

Now let $g \in C_c(G)$ be arbitrary. We show that $u_V * g$ converges to g in the inductive limit topology.

Let $\epsilon > 0$. Now g is uniformly continuous and thus there exists a neighborhood W of the neutral element so that for $s, g \in G$ with $sg^{-1} \in W$ we have $|g(s) - g(r)| < \epsilon$. We can assume W to be compact. Now for all neighborhoods of the identity $V \subset W$ we have for $s \notin W\,\mathrm{supp}(g)$

$$u_V * g(s) = \int_G u_V(r) g(r^{-1}s)\,dr = \int_V u_V(r)\,\underbrace{g(r^{-1}s)}_{=0}\,dr = 0.$$

We thus have $\operatorname{supp}(u_V * g) \subset W \operatorname{supp}(g)$. Thus, the support of $u_V * g$ is eventually contained in the compact set $W \operatorname{supp}(g)$. We further have for $s \in G$

$$\begin{aligned}|u_V * g(s) - g(s)| &= |\int_G u_V(r)g(r^{-1}s) - g(s)\,dr| \leq \int_G u_V(r)|g(r^{-1}s) - g(s)|\,dr \\ &= \int_W u_V(r)|g(r^{-1}s) - g(s)|\,dr \leq \epsilon \int_W u_V(r)\,dr = \epsilon.\end{aligned}$$

Thus, for all neighborhoods V of the identity with $V \geq W$ we have that $\|u_V * g - g\|_\infty \leq \epsilon$. Thus, $(u_V * g)_V$ converges to g in the inductive limit topology and then also in the norm on $C^*(G)$.

If $x \in C^*(G)$ and $\epsilon > 0$ there is an $g \in C_c(G)$ with $\|x - g\| \leq \frac{\epsilon}{3}$ and we get

$$\|u_V x - x\| \leq \underbrace{\|u_V x - u_V g\|}_{\leq \|x-g\|} + \underbrace{\|u_V g - g\|}_{\to 0} + \|g - x\| \leq \epsilon$$

eventually. Thus, we have $u_V x \to x$. Since $(u_V)_V$ is self-adjoint we also have $x u_V \to x$. So $(u_V)_V$ is an approximate unit for $C^*(G)$. □

Theorem 14. Let (A, G, α) be a dynamical system and let $I \subset A$ be an α-invariant ideal. Let $\iota : I \to A$ be the inclusion and let $q : A \to A/I$ be the quotient map. Then

$$0 \longrightarrow I \rtimes_\alpha G \xrightarrow{\iota \rtimes \mathrm{id}} A \rtimes_\alpha G \xrightarrow{q \rtimes \mathrm{id}} A/I \rtimes_{\alpha^I} G \longrightarrow 0$$

is a short exact sequence of C^*-algebras.

Proof. By Theorem 12(2) the map $q \rtimes \mathrm{id}$ is surjective. We now show that $\iota \rtimes \mathrm{id}$ is injective. To see this it suffices to show that $\iota \rtimes \mathrm{id}$ is isometric on $C_c(G, I)$. Let $f \in C_c(G, I)$. Since $\iota \rtimes \mathrm{id}$ is continuous we know that $\|\iota \rtimes \mathrm{id}(f)\|_A = \|f\|_A \leq \|f\|_I$. Now let (π, u) be a nondegenerate covariant representation of (I, G, α) on a Hilbert space H. Then π can be uniquely extended to a representation $\overline{\pi} : A \to B(H)$ so that $\overline{\pi}|_I = \pi$. Then $(\overline{\pi}, u)$ is a covariant representation of (A, G, α), since for $a \in A, s \in G$ and $b \in I, \xi \in H$ we have

$$\begin{aligned}\overline{\pi}(\alpha_s(a))\pi(b)\xi &= \pi(\alpha_s(a)b)\xi = \pi(\alpha_s(a\alpha_{s^{-1}}(b)))\xi = u_s\pi(a\alpha_{s^{-1}}(b))u_s^*\xi \\ &= u_s\overline{\pi}(a)u_{s^{-1}}\pi(b)u_{s^{-1}}^*u_s^*\xi = u_s\overline{\pi}(a)u_s^*\pi(b)\xi.\end{aligned}$$

Since π is nondegenerate this implies that $\overline{\pi}(\alpha_s(a)) = u_s\overline{\pi}(a)u_s^*$, which shows that $(\overline{\pi}, u)$ is covariant. This gives

$$\|\pi \rtimes u(f)\| = \|\int_G \pi(f(s))u_s\,ds\| = \|\int_G \overline{\pi}(f(s))u_s\,ds\| = \|\overline{\pi} \rtimes u(f)\| \leq \|f\|_A.$$

Since (π, u) was an arbitrary covariant representation of (I, G, α) it follows that $\|f\|_I \leq \|f\|_A$. Thus, we get $\|\iota \rtimes \mathrm{id}(f)\|_A = \|f\|_A = \|f\|_I$, which shows that $\iota \rtimes \mathrm{id}$ is isometric on $C_c(G, I)$ and is thus an injective map.

It remains to show that $\ker(q \rtimes \mathrm{id}) = \mathrm{im}(\iota \rtimes \mathrm{id})$. If $f \in C_c(G, I)$ we have

$$q \rtimes \mathrm{id}(\iota \rtimes \mathrm{id}(f)) = q \circ \iota \circ f = 0.$$

By density this gives $\mathrm{im}(\iota \rtimes \mathrm{id}) \subset \ker(q \rtimes \mathrm{id})$.

By Theorem 12(3) it follows that $\mathrm{im}(\iota \rtimes \mathrm{id}) \subset A \rtimes_\alpha G$ is an ideal. Thus, there is a nondegenerate representation $\rho : A \rtimes_\alpha G \to B(H)$ so that $\ker(\rho) = \mathrm{im}(\iota \rtimes \mathrm{id})$. But then there is a nondegenerate covariant representation (π, u) of (A, G, α) on H so that $\pi \rtimes u = \rho$. Let $J = \ker(\pi)$. Then we have $I \subset J$. Assume otherwise. Then there is a $a \in I$ with $\pi(a) \neq 0$. Let $(f_i)_i$ be an approximate unit for $C^*(G) \in C_c(G)$. Then $(f_i \widetilde{\otimes} a)_i$ is a net in $C_c(G, I)$ and we have

$$\pi \rtimes u(f_i \widetilde{\otimes} a) = \int_G \pi(f_i(s)a)u_s \, ds = \pi(a) \int_G f_i(s)u_s \, ds = \pi(a)u(f_i).$$

Now the $*$-representation $u : C^*(G) \to B(H)$ is nondegenerate. Thus, we have $u(f_i) \to \mathrm{id}_H$ in the strong operator topology. Now let $\xi \in H$ with $\pi(a)\xi \neq 0$. We then have

$$0 \neq \pi(a)\xi = \lim_i \pi(a)u(f_i)\xi = \lim_i \pi \rtimes u(f_i \widetilde{\otimes} a)\xi$$

Thus there exists i_0 with $\pi \rtimes u(f_{i_0} \widetilde{\otimes} a) \neq 0$, which gives $f_{i_0} \widetilde{\otimes} a \notin \ker(\pi \rtimes u) = \ker(\rho) = \mathrm{im}(\iota \rtimes \mathrm{id})$. But $f_{i_0} \widetilde{\otimes} a = \iota \rtimes \mathrm{id}(f_{i_0} \widetilde{\otimes} a) \in \mathrm{im}(\iota \rtimes \mathrm{id})$, which gives a contradiction. Thus we have $I \subset J$.

Since $I \subset J = \ker(\pi)$ there is a $*$-homomorphism $\pi' : A/I \to B(H)$ so that

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "q"'] & B(H) \\ A/I \arrow[ur, "\pi'"'] & \end{tikzcd}$$

commutes. But then (π', u) is a covariant representation for $(A/I, G, \alpha^I)$, since for $a + I \in A/I$ and $s \in G$ we have

$$\pi'(\alpha_s^I(a + I)) = \pi'(\alpha_s(a) + I) = \pi(\alpha_s(a)) = u_s\pi(a)u_s^* = u_s\pi'(a + I)u_s^*.$$

For $f \in C_c(G, A)$ we then have

$$\begin{aligned} \pi' \rtimes u(q \rtimes \mathrm{id}(f)) = \pi' \rtimes (q \circ f) &= \int_G \pi'(q \circ f(s))u_s \, ds \\ &= \int_G \pi(f(s))u_s \, ds = \pi \rtimes u(f). \end{aligned}$$

We thus have $(\pi' \rtimes u) \circ (q \rtimes \mathrm{id}) = \pi \rtimes u$. But this gives

$$\ker(q \rtimes \mathrm{id}) \subset \ker(\pi \rtimes u) = \mathrm{im}(\iota \rtimes \mathrm{id}).$$

This shows $\mathrm{im}(\iota \rtimes \mathrm{id}) = \ker(q \rtimes \mathrm{id})$. We thus have exactness in the middle term and have proven the exactness of the sequence. $\square$

Given Theorem 14 it is natural to ask whether a similar statement also holds for the reduced crossed product. We have the following.

Theorem 15. Let (A, G, α) be a dynamical system and $I \subset A$ an α-invariant ideal. Let $\iota : I \to A$ be the inclusion and let $q : A \to A/I$ be the quotient map. Then we get a sequence

$$0 \longrightarrow I \rtimes_{\alpha,r} G \xrightarrow{\iota \rtimes_r \mathrm{id}} A \rtimes_{\alpha,r} G \xrightarrow{q \rtimes_r \mathrm{id}} A/I \rtimes_{\alpha^I,r} G \longrightarrow 0$$

where $\iota \rtimes_r \mathrm{id}$ is injective, $q \rtimes_r \mathrm{id}$ is surjective, and we have the inclusion $\mathrm{im}(\iota \rtimes_r \mathrm{id}) \subset \ker(q \rtimes_r \mathrm{id})$.

Proof. The injectivity of $\iota \rtimes_r \mathrm{id}$ follows from Theorem 12(1), the surjectivity of $q \rtimes_r \mathrm{id}$ follows from Theorem 12(2) and the inclusion follows exactly as in the proof of Theorem 14. $\square$

Remark 16. It can be shown that there is a C^*-dynamical system (A, G, α) with an α-invariant ideal $I \subset A$ so that the sequence of Theorem 15 is *not exact*. A locally compact group so that this sequence is always exact is called an *exact group*. A concrete example of a non exact group can be found in [Osa18].

If G is an amenable group we have $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$ by Theorem 4 and the induced maps from an equivariant homomorphism also agree. By Theorem 14 is follows that every amenable group is exact.

Our final objective for this talk is to show that the crossed product of a nuclear C^*-algebra with an amenable group is again nuclear. For this we need to consider tensor products of dynamical systems.

Remark 17. Let (A, G, α) and (B, G, β) be dynamical systems.If $\varphi \in \mathrm{Aut}(A)$ and $\psi \in \mathrm{Aut}(B)$ we get automorphisms

$$\varphi \otimes_{\max} \psi : A \otimes_{\max} B \to A \otimes_{\max} B$$

and

$$\varphi \otimes_{\min} \psi : A \otimes_{\min} B \to A \otimes_{\min} B$$

defined by $a \otimes b \mapsto \varphi(a) \otimes \psi(b)$ with $(\varphi \otimes \psi)^{-1} = \varphi^{-1} \otimes \psi^{-1}$. Then we get strongly continuous group homomorphisms

$$\alpha \otimes_{\max} \beta : G \to \mathrm{Aut}(A \otimes_{\max} B); s \mapsto \alpha_s \otimes_{\max} \beta_s$$

and

$$\alpha \otimes_{\min} \beta : G \to \mathrm{Aut}(A \otimes_{\min} B); s \mapsto \alpha_s \otimes_{\min} \beta_s.$$

This gives C^*-dynamical systems $(A \otimes_{\max} B, G, \alpha \otimes_{\max} \beta)$ and $(A \otimes_{\min} B, G, \alpha \otimes_{\min} \beta)$. If it is clear whether we are talking about the maximal or minimal tensor product or if it does not make a difference we will sometimes drop the subscripts from the notation.

The key step in proving the nuclearity of the crossed product of a nuclear C^*-algebra with an amenable group is calculating two particular crossed products involving tensor products. We give a short reminder on the needed properties of maximal and minimal tensor products. Proofs of these facts can be found in [RW98, Appendix B].

Let A, B be C^*-algebras. We can describe the norm on the minimal tensor product in the following way. If $\rho_A : A \to B(H_A)$ and $\rho_B : B \to B(H_B)$ are faithful representations of A and B respectively, we get a faithful representation

$$\rho_A \otimes \rho_B : A \otimes_{\min} B \to B(H_A \otimes H_B)$$

given by $\rho_A \otimes \rho_B(a \otimes b)(\xi_A \otimes \xi_B) = \rho_A(a)\xi_A \otimes \rho_B(b)\xi_B$.

The maximal crossed product on the other hand enjoys the following universal property.

Theorem 18. Let A, B be C^*-algebras.

(1) If $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ are nondegenerate representations with commuting ranges, there exists a nondegenerate representation $\pi_A \otimes_{\max} \pi_B : A \otimes_{\max} B \to B(H)$ so that

$$\pi_A \otimes_{\max} \pi_B(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

(2) If $\pi : A \otimes_{\max} B \to B(H)$ is a nondegenerate representation, there exists nondegenerate representations $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ with commuting ranges so that

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a).$$

If B is an arbitrary C^*-algebra and G is a locally compact group we always have a trivial dynamical system $\mathrm{id} : G \to \mathrm{Aut}(B); s \mapsto \mathrm{id}_B$.

Theorem 19. Let (A, G, α) be a dynamical system and let B be a C^*-algebra. Then there is a $*$-homomorphism

$$\begin{aligned} \Phi : C_c(G, A) \odot B &\to C_c(G, A \odot B) \\ f \otimes b &\mapsto (s \mapsto f(s) \otimes b). \end{aligned}$$

This map extends to isomorphisms

$$(A \rtimes_\alpha G) \otimes_{\max} B \to (A \otimes_{\max} B) \rtimes_{\alpha\otimes \mathrm{id}} G$$

and

$$(A \rtimes_{\alpha,r} G) \otimes_{\min} B \to (A \otimes_{\min} B) \rtimes_{\alpha\otimes \mathrm{id},r} G.$$

Especially for $f \in C_c(G), a \in A, b \in B$ we have $(f\widetilde{\otimes}a) \otimes b \mapsto f\widetilde{\otimes}(a \otimes b)$.

Remark 20. When writing $C_c(G, A \odot B)$ we take either the minimal norm or the maximal norm on $A \odot B$. Thus, we take $C_c(G, A \odot B)$ as a subspace of either $C_c(G, A \otimes_{\min} B)$ or $C_c(G, A \otimes_{\max} B)$.

The following argument is an elaboration of [Echa, Lemma 4.1].

Proof. We first show that Φ is a $*$-homomorphism. Since the map $(f, b) \mapsto (s \mapsto f(s) \otimes b)$ is a bilinear map of $C_c(G, A) \times B$ to $C_c(G, A \odot B)$ because every C^*-norm on $A \odot B$ is subcross, it follows that Φ is a well defined linear map.

If $f, g \in C_c(G, A)$ and $b, c \in B$ we have for $s \in G$

$$\begin{aligned}\Phi(f \otimes b)\Phi(g \otimes c)(s) &= \int_G \Phi(f \otimes b)(r)(\alpha \otimes \mathrm{id})_r(\Phi(g \otimes c)(r^{-1}s))\,dr \\ = \int_G f(r) \otimes b \cdot \alpha_r(g(r^{-1}s)) \otimes c\,dr &= \int_G f(r)\alpha_r(g(r^{-1}s)) \otimes bc\,dr \\ = \int_G f(r)\alpha_r(g(r^{-1}s))\,dr \otimes bc &= fg(s) \otimes bc = \Phi(fg \otimes bc)(s).\end{aligned}$$

This gives $\Phi(f \otimes b)\Phi(g \otimes c) = \Phi((f \otimes b)(g \otimes c))$. Since every element of $C_c(G, A) \odot B$ is a finite linear combination of elements of the form $f \otimes b$ with $f \in C_c(G, A), b \in B$, this shows that Φ is multiplicative on $C_c(G, A) \odot B$.

We further have

$$\begin{aligned}\Phi(f \otimes b)^*(s) &= \Delta(s^{-1})(\alpha \otimes \mathrm{id})_s(\Phi(f \otimes b)(s^{-1})^*) \\ = \Delta(s^{-1})\alpha_s \otimes \mathrm{id}_B(f(s^{-1})^* \otimes b^*) &= (\Delta(s^{-1})\alpha_s(f(s^{-1})^*)) \otimes b^* \\ = f^*(s) \otimes b^* &= \Phi(f^* \otimes b^*)(s) = \Phi((f \otimes b)^*)(s).\end{aligned}$$

This shows $\Phi(f \otimes b)^* = \Phi((f \otimes b)^*)$. It again follows that for every $x \in C_c(G, A)\odot B$ we have $\Phi(x)^* = \Phi(x^*)$. This shows that Φ is a $*$-homomorphism.

We show that Φ is isometric with respect to the norm of $(A \otimes_{\max} B) \rtimes_{\alpha\otimes\mathrm{id}} G$ on $C_c(G, A \odot B)$ and the norm of $(A \rtimes_\alpha G) \otimes_{\max} B$ on $C_c(G, A) \odot B$.

Let $x \in C_c(G, A) \odot B$. Let (π, u) be an arbitrary nondegenerate covariant representation of $(A \otimes_{\max} B, G, \alpha \otimes \mathrm{id})$ on a Hilbert space H. By the universal property of the maximal tensor product there are nondegenerate representation π_A, π_B of A and B so that $\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$.

We now proof that (π_A, u) is a covariant representation of (A, G, α). Firstly for $c \in A, d \in B$ we have

$$\pi_A(a)\pi(c \otimes d) = \pi_A(a)\pi_A(c)\pi_B(d) = \pi_A(ac)\pi_B(d) = \pi(ac \otimes d)$$

and similarly $\pi_B(b)\pi(c \otimes d) = \pi(c \otimes bd)$. This gives for $s \in G$

$$\begin{aligned} u_s\pi_A(a)\pi(c \otimes d) &= u_s\pi(ac \otimes d) = \pi(\alpha_s(ac) \otimes d)u_s \\ &= \pi_A(\alpha_s(a))\pi(\alpha_s(c) \otimes d)u_s = \pi_A(\alpha_s(a))u_s\pi(c \otimes d). \end{aligned}$$

Since π is nondegenerate this gives $u_s\pi_A(a)u_s^* = \pi_A(\alpha_s(a))$. Thus, (π_A, u) is covariant and thus gives a representation $\pi_A \rtimes u : A \rtimes_\alpha G \to B(H)$.

We now show that $\pi_A \rtimes u$ and π_B have commuting ranges. For this, let $a \in A, b \in B, f \in C_c(G)$ and $c \in A, d \in B$. We then have

$$\pi_A \rtimes u(f\widetilde{\otimes}a) = \int_G \pi_A(f(s)a)u_s\,ds = \pi_A(a)\int_G f(s)u_s\,ds = \pi_A(a)u(f).$$

and thus

$$\begin{aligned} \pi_A \rtimes u(f\widetilde{\otimes}a)\pi_B(b)\pi(c \otimes d) &= \pi_A(a)\int_G f(s)u_s\pi(c \otimes bd)\,ds \\ &= \pi_A(a)\int_G f(s)\pi(\alpha_s(c) \otimes bd)u_s\,ds \\ &= \pi_A(a)\pi_B(b)\int_G f(s)\pi(\alpha_s(c) \otimes d)u_s\,ds \\ &= \pi_B(b)\pi_A(a)\int_G f(s)u_s\pi(c \otimes d)\,ds = \pi_B(b)\pi_A \rtimes u(f\widetilde{\otimes}a)\pi(c \otimes d). \end{aligned}$$

By the nondegenerary of π it again follows that

$$\pi_A \rtimes u(f\widetilde{\otimes}a)\pi_B(b) = \pi_B(b)\pi_A \rtimes u(f\widetilde{\otimes}a).$$

Since $\operatorname{span}\{f\widetilde{\otimes}a | f \in C_c(G), a \in A\}$ is dense in $A \rtimes_\alpha G$ it follows that $\pi_A \rtimes u$ and π_B have commuting ranges. Thus, we get a representation

$$(\pi_A \rtimes u) \otimes_{\max} \pi_B : (A \rtimes_\alpha G) \otimes_{\max} B \to B(H).$$

If $f \in C_c(G, A)$ and $b \in B$ we now have

$$\begin{aligned} \pi \rtimes u(\Phi(f \otimes b)) &= \int_G \pi(f(s) \otimes b)u_s\,ds = \pi_B(b)\int_G \pi_A(f(s))u_s\,ds \\ &= \pi_B(b)\pi_A \rtimes u(f) = (\pi_A \rtimes u) \otimes_{\max} \pi_B(f \otimes b). \end{aligned}$$

Thus we get for $x \in C_c(G,A) \odot B$ that $\pi \rtimes u(\Phi(x)) = (\pi_A \rtimes u) \otimes_{\max} \pi_B(x)$. This gives $\|\pi \rtimes u(\Phi(x))\| \leq \|x\|_{\max}$. Since (π, u) was an arbitrary nondegenerate covariant representation we get $\|\Phi(x)\| \leq \|x\|$.

Now let $\pi : (A \rtimes_\alpha G) \otimes_{\max} B \to B(H)$ be a faithful nondegenerate representation. By the universal property of the maximal tensor product there are nondegenerate representations $\pi_\rtimes$ of $A \rtimes_\alpha G$ on H and π_B of B on H so that for $z \in A \rtimes_\alpha G$ and $b \in B$ we have $\pi(z \otimes b) = \pi_\rtimes(z)\pi_B(b) = \pi_B(b)\pi_\rtimes(z)$. Then by the construction of the crossed product there is a nondegenerate covariant representation (π_A, u) of (A, G, α) so that $\pi_A \rtimes u = \pi_\rtimes$.

Now π_B and $\pi_\rtimes$ are nondegenerate representations. Thus, they extend uniquely to representations $\overline{\pi_B}$ and $\overline{\pi_\rtimes}$ of the associated multiplier algebras. We show that the extended representations still have commuting ranges.

For this let $v \in M(A \rtimes G), y \in M(B)$. Then we have for $z \in A \rtimes_\alpha G, b \in B$ and $\eta \in H$

$$\begin{aligned}
\overline{\pi_\rtimes}(v)\overline{\pi_B}(y)\pi_B(b)\pi_\rtimes(z)\eta &= \overline{\pi_\rtimes}(v)\pi_B(yb)\pi_\rtimes(z)\eta = \overline{\pi_\rtimes}(v))\pi_\rtimes(z)\pi_B(yb)\eta \\
&= \pi_\rtimes(vz)\pi_B(yb)\eta = \pi_B(yb)\pi_\rtimes(vz)\eta = \overline{\pi_B}(y)\pi_B(b)\pi_\rtimes(vz)\eta \\
&= \overline{\pi_B}(y)\pi_\rtimes(vz)\pi_B(b)\eta = \overline{\pi_B}(y)\overline{\pi_\rtimes}(v)\pi_B(b)\pi_\rtimes(z)\eta.
\end{aligned}$$

Since $\pi_\rtimes$ and π_B are nondegenerate it follows that $\overline{\pi_\rtimes}$ and $\overline{\pi_B}$ have commuting ranges.

Now let $i_A : A \to M(A \rtimes_\alpha G)$ and $i_G : G \to M(A \rtimes_\alpha G)$ be the canonical maps. We know that $\pi_A = \overline{\pi_\rtimes} \circ i_A$ and $u = \overline{\pi_\rtimes} \circ i_G$. This shows that π_B and π_A have commuting ranges. Thus, we get a representation $\pi_A \otimes_{\max} \pi_B : A \otimes_{\max} B \to B(H)$ given by $\pi_A(a)\pi_B(b) = \pi_A \otimes_{\max} \pi_B(a \otimes b)$. It also follows that π_B and u have commuting ranges.

We now show that $(\pi_A \otimes_{\max} \pi_B, u)$ is a covariant representation of $(A \otimes_{\max} B, G, \alpha \otimes \mathrm{id})$. For $a \in A, b \in B, s \in G$ we have

$$\begin{aligned}
\pi_A \otimes_{\max} \pi_B((\alpha \otimes \mathrm{id})_s(a \otimes b)) &= \pi_A(\alpha_s(a))\pi_B(b) = u_s\pi_A(a)u_s^*\pi_B(b) \\
&= u_s\pi_A \otimes_{\max} \pi_B(a \otimes b)u_s^*.
\end{aligned}$$

This shows the covariance. Thus, we get a representation

$$(\pi_A \otimes_{\max} \pi_B) \rtimes u : (A \otimes_{\max} B) \rtimes_{\alpha \otimes \mathrm{id}} G \to B(H).$$

For $f \in C_c(G, A)$ and $b \in B$ we then have

$$\begin{aligned}
(\pi_A \otimes_{\max} \pi_B) \rtimes u(\Phi(f \otimes b)) &= \int_G \pi_A \otimes_{\max} \pi_B(f(s) \otimes b)u_s \, ds \\
= \int_G \pi_A(f(s))u_s \, ds \, \pi_B(b) &= \pi_A \rtimes u(f)\pi_B(b) = \pi_\rtimes(f)\pi_B(b) = \pi(f \otimes b).
\end{aligned}$$

Thus we have for $x \in C_c(G,A) \odot B$ that $(\pi_A \otimes_{\max} \pi_B) \rtimes u(\Phi(x)) = \pi(x)$. This gives

$$\|x\| = \|\pi(x)\| = \|(\pi_A \otimes_{\max} \pi_B) \rtimes u(\Phi(x))\| \leq \|\Phi(x)\|.$$

This gives $\|\Phi(x)\| = \|x\|$. Thus, $\Phi : C_c(G,A) \odot B \to C_c(G, A \odot B)$ is isometric. Since $C_c(G,A) \odot B \subset (A \rtimes_\alpha G) \otimes_{\max} B$ is dense it follows that Φ extends to an isometric $*$-homomorphism $(A \rtimes_\alpha G) \otimes_{\max} B \to (A \otimes_{\max} B) \rtimes_{\alpha \otimes \mathrm{id}} G$. Since for $f \in C_c(G), a \in A, b \in B$ we have that $f \widetilde{\otimes} (a \otimes b) = \Phi((f \widetilde{\otimes} a) \otimes b)$ is in its image this homomorphism has dense range and is thus surjective.

Thus, we get an isomorphism

$$(A \rtimes_\alpha G) \otimes_{\max} B \xrightarrow{\cong} (A \otimes_{\max} B) \rtimes_{\alpha \otimes \mathrm{id}} G$$

that maps $f \otimes b$ to $(s \mapsto f(s) \otimes b)$ for $f \in C_c(G,A)$ and $b \in B$.

We now show that Φ is isometric if we consider the norm from $(A \otimes_{\min} B) \rtimes_{\alpha \otimes \mathrm{id}, r} G$ on $C_c(G, A \odot B)$ and the norm from $(A \rtimes_{\alpha, r} G) \otimes_{\min} B$ on $C_c(G,A) \odot B$.

To prove this we first need to show that for two Hilbert spaces H, K the map

$$U : C_c(G,H) \odot K \to C_c(G, H \otimes K); f \otimes \xi \mapsto (s \mapsto f(s) \otimes \xi)$$

extends to a unitary map $L^2(G,H) \otimes K \to L^2(G, H \otimes K)$.

Since $(f, \xi) \mapsto (s \mapsto f(s) \otimes \xi)$ is a bilinear map from $C_c(G,H) \times K$ to $C_c(G, H \otimes K)$ the map U is well defined and linear. If $x = \sum_{i=1}^n f_i \otimes \xi_i, f_i \in C_c(G,H), x_i \in K$, we have

$$\begin{aligned}
\|U(x)\|_2^2 &= \int_G \langle U(x)(s), U(x)(s) \rangle \, ds = \sum_{i,j=1}^n \int_G \langle U(f_i \otimes \xi_i)(s), U(f_j \otimes \xi_j)(s) \rangle \, ds \\
&= \sum_{i,j=1}^n \int_G \langle f_i(s), f_j(s) \rangle \langle \xi_i, \xi_j \rangle \, ds = \sum_{i,j=1}^n \langle f_i, f_j \rangle \langle \xi_i, \xi_j \rangle \\
&= \sum_{i,j=1}^n \langle f_i \otimes \xi_i, f_j \otimes \xi_j \rangle = \|x\|^2.
\end{aligned}$$

Thus U is an isometric linear map on $C_c(G,H) \odot K$. Since $C_c(G,H) \odot K \subset L^2(G,H) \otimes K$ is dense it follows that U extends to an isometric linear map $L^2(G,H) \otimes K \to L^2(G, H \otimes K)$. Since every element of the form $f \widetilde{\otimes} (\xi_H \otimes \xi_K) = U((f \widetilde{\otimes} \xi_H) \otimes \xi_K)$, $\xi_H \in H, \xi_K \in K$, is in the range of U it follows that U has dense range and is thus surjective. We thus get a unitary map

$$U : L^2(G,H) \otimes K \xrightarrow{\cong} L^2(G, H \otimes K)$$

so that for $f \in C_c(G,H), \xi \in K$ we have $U(f \otimes \xi) = (s \mapsto f(s) \otimes \xi)$.

Now let $\rho_A : A \to B(H_A)$ and $\rho_B : B \to B(H_B)$ be faithful nondegenerate representations. Then $\rho_A \otimes \rho_B : A \otimes_{\min} B \to B(H_A \otimes H_B)$ is faithful as well. Thus,

$$\mathrm{Ind}(\rho_A \otimes \rho_B) : (A \otimes_{\min} B) \rtimes_{\alpha \otimes \mathrm{id}, r} G \to B(L^2(G, H_A \otimes H_B))$$

is faithful as well.

On the other hand $\mathrm{Ind}(\rho_A) : A \rtimes_{\alpha,r} G \to B(L^2(G,H))$ is also faithful. But then

$$\mathrm{Ind}(\rho_A) \otimes \rho_B : (A \rtimes_{\alpha,r} G) \otimes_{\min} B \to B(L^2(G,H) \otimes K)$$

is faithful as well. We now show that for $x \in C_c(G,A) \odot B$ we have

$$U^* \,\mathrm{Ind}(\rho_A \otimes \rho_B)(\Phi(x))U = \mathrm{Ind}(\rho_A) \otimes \rho_B(x).$$

For this let $f \in C_c(G,A), b \in B$. Then for $g \in C_c(G), \xi_A \in H_A, \xi_B \in H_B$ and $r \in G$ we have

$$\begin{aligned}
&\mathrm{Ind}(\rho_A \otimes \rho_B)(\Phi(f \otimes b))(U((g \widetilde{\otimes} \xi_A) \otimes \xi_B))(r) \\
&= \widetilde{\rho_A \otimes \rho_B} \rtimes V^{\rho_A \otimes \rho_B}(\Phi(f \otimes b))(g \widetilde{\otimes} (\xi_A \otimes \xi_B))(r) \\
&= \int_G \widetilde{\rho_A \otimes \rho_B}(f(s) \otimes b) V_s^{\rho_A \otimes \rho_B}(g \widetilde{\otimes} (\xi_A \otimes \xi_B))(r)\, ds \\
&= \int_G \rho_A \otimes \rho_B(\alpha_r \otimes \mathrm{id}(f(s) \otimes b))(V_s^{\rho_A \otimes \rho_B}(g \widetilde{\otimes} (\xi_A \otimes \xi_B))(r))\, ds \\
&= \int_G \rho_A \otimes \rho_B(\alpha_r(f(s)) \otimes b)(g(s^{-1}r)\xi_A \otimes \xi_B)\, ds \\
&= \int_G \rho_A(\alpha_r(f(s))) g(s^{-1}r)\xi_A \otimes \rho_B(b)\xi_B\, ds \\
&= \int_G \rho_A(\alpha_r(f(s))) g(s^{-1}r)\xi_A\, ds \otimes \rho_B(b)\xi_B \\
&= \int_G \rho_A(\alpha_r(f(s)))(V_s^{\rho_A}(g \widetilde{\otimes} \xi_A)(r))\, ds \otimes \rho_B(b)\xi_B \\
&= \int_G \widetilde{\rho_A}(f(s)) V_s^{\rho_A}(g \widetilde{\otimes} \xi_A)(r)\, ds \otimes \rho_B(b)\xi_B \\
&= \widetilde{\rho_A} \rtimes V^{\rho_A}(f)(g \widetilde{\otimes} \xi_A)(r) \otimes \rho_B(b)\xi_B \\
&= \mathrm{Ind}(\rho_A)(f)(g \widetilde{\otimes} \xi_A)(r) \otimes \rho_B(b)\xi_B \\
&= U(\mathrm{Ind}(\rho_A)(f) g \widetilde{\otimes} \xi_A \otimes \rho_B(b)\xi_B)(r) \\
&= U(\mathrm{Ind}(\rho_A) \otimes \rho_B(f \otimes b)((g \widetilde{\otimes} \xi_A) \otimes \xi_B))(r).
\end{aligned}$$

Thus we have proven

$$U^* \,\mathrm{Ind}(\rho_A \otimes \rho_B)(\Phi(f \otimes b))U((g \widetilde{\otimes} \xi_A) \otimes \xi_B) = \mathrm{Ind}(\rho_A) \otimes \rho_B(f \otimes b)((g \widetilde{\otimes} \xi_A) \otimes \xi_B).$$

By density this gives

$$U^* \operatorname{Ind}(\rho_A \otimes \rho_B)(\Phi(f \otimes b))U = \operatorname{Ind}(\rho_A) \otimes \rho_B(f \otimes b).$$

By linearity we thus get for all $x \in C_c(G, A) \odot B$

$$U^* \operatorname{Ind}(\rho_A \otimes \rho_B)(\Phi(x))U = \operatorname{Ind}(\rho_A) \otimes \rho_B(x).$$

This gives

$$\begin{aligned} \|\Phi(x)\|_r &= \| \operatorname{Ind}(\rho_A \otimes \rho_B)(\Phi(x))\| = \|U \operatorname{Ind}(\rho_A \otimes \rho_B)(\Phi(x))U^*\| \\ &= \| \operatorname{Ind}(\rho_A) \otimes \rho_B(x)\| = \|x\|_{\min}. \end{aligned}$$

Thus Φ is isometric in the specified norms. Since $C_c(G, A) \odot B \subset (A \rtimes_{\alpha,r} G) \otimes_{\min} B$ is a dense linear subspace we get an isometric extension of Φ to a $*$-homomorphism $(A \rtimes_{\alpha,r} G) \otimes_{\min} B \to (A \otimes_{\min} B) \rtimes_{\alpha\otimes\mathrm{id},r} G$. Since again all elements of the form $f\widetilde{\otimes}(a \otimes b), f \in C_c(G), a \in A, b \in B$ are in its range this extension is also surjective since it has dense range.

Thus, we get an isomorphism

$$(A \rtimes_{\alpha,r} G) \otimes_{\min} B \xrightarrow{\cong} (A \otimes_{\min} B) \rtimes_{\alpha\otimes\mathrm{id},r} G$$

which for $f \in C_c(G, A), b \in B$ maps $f \otimes b$ to $(s \mapsto f(s) \otimes b)$. $\square$

Remark 21. Let (A, G, α) and (B, G, β) be dynamical systems. Then we have $A \otimes_{\max} B \cong B \otimes_{\max} A$ and $A \otimes_{\min} B \cong B \otimes_{\min} A$ via $a \otimes b \mapsto b \otimes a$. Let $\varphi_{\max}$ and $\varphi_{\min}$ be these isomorphisms. These are equivariant since for $a \in A, b \in B, s \in G$ we have

$$\begin{aligned} \varphi_{\max}(\alpha_s \otimes \beta_s(a \otimes b)) &= \varphi_{\max}(\alpha_s(a) \otimes \beta_s(b)) = \beta_s(b) \otimes \alpha_s(a) \\ &= \beta_s \otimes \alpha_s(\varphi_{\max}(a \otimes b)). \end{aligned}$$

By Corollary 9 we get isomorphisms

$$\varphi_{\max} \rtimes \mathrm{id} : (A \otimes_{\max} B) \rtimes_{\alpha\otimes\beta} G \xrightarrow{\cong} (B \otimes_{\max} A) \rtimes_{\beta\otimes\alpha} G.$$

and

$$\varphi_{\max} \rtimes_r \mathrm{id} : (A \otimes_{\max} B) \rtimes_{\alpha\otimes\beta,r} G \xrightarrow{\cong} (B \otimes_{\max} A) \rtimes_{\beta\otimes\alpha,r} G.$$

Similarly we get isomorphisms

$$\varphi_{\min} \rtimes \mathrm{id} : (A \otimes_{\min} B) \rtimes_{\alpha\otimes\beta} G \xrightarrow{\cong} (B \otimes_{\min} A) \rtimes_{\beta\otimes\alpha} G.$$

and

$$\varphi_{\min} \rtimes_r \mathrm{id} : (A \otimes_{\min} B) \rtimes_{\alpha\otimes\beta,r} G \xrightarrow{\cong} (B \otimes_{\min} A) \rtimes_{\beta\otimes\alpha,r} G.$$

Especially for the trivial dynamical system (A, G, id) Theorem 19 gives

$$\begin{tikzcd}
A \otimes_{\max} (B \rtimes_\beta G) \arrow[r, "\cong"] \arrow[d, dashed, "\cong"', "a\otimes(f\widetilde{\otimes}b) \mapsto f\widetilde{\otimes}(a\otimes b)"] & (B \rtimes_\beta G) \otimes_{\max} A \arrow[d, "\cong", "(f\widetilde{\otimes}b)\otimes a \mapsto f\widetilde{\otimes}(b\otimes a)"'] \\
(A \otimes_{\max} B) \rtimes_{\mathrm{id}\otimes\beta} G & (B \otimes_{\max} A) \rtimes_{\beta\otimes\mathrm{id}} G \arrow[l, "\cong"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes_{\min} (B \rtimes_{\beta,r} G) \arrow[r, "\cong"] \arrow[d, dashed, "\cong"', "a\otimes(f\widetilde{\otimes}b) \mapsto f\widetilde{\otimes}(a\otimes b)"] & (B \rtimes_{\beta,r} G) \otimes_{\min} A \arrow[d, "\cong", "(f\widetilde{\otimes}b)\otimes a \mapsto f\widetilde{\otimes}(b\otimes a)"'] \\
(A \otimes_{\min} B) \rtimes_{\mathrm{id}\otimes\beta,r} G & (B \otimes_{\min} A) \rtimes_{\beta\otimes\mathrm{id},r} G \arrow[l, "\cong"']
\end{tikzcd}$$

This enables us to calculate the crossed product of the trivial dynamical system (A, G, id). We remind that for a locally compact group G we have $\mathbb{C} \rtimes_{\mathrm{id}} G = C^*(G)$ and $\mathbb{C} \rtimes_{\mathrm{id},r} G = C^*_r(G)$.

Corollary 22. Let A be a C^*-algebra and let G be a locally compact group. Then we have

(1) $A \rtimes_{\mathrm{id}} G \cong A \otimes_{\max} C^*(G)$

(2) $A \rtimes_{\mathrm{id},r} G \cong A \otimes_{\min} C^*_r(G)$

These isomorphisms are given by $f\widetilde{\otimes}a \mapsto a \otimes f$ for $f \in C_c(G), a \in A$.

Proof. By Theorem 19 and Corollary 21 we have

$$\begin{aligned}
A \rtimes_{\mathrm{id}} G &\cong (A \rtimes_{\mathrm{id}} G) \otimes_{\max} \mathbb{C} \cong (A \otimes_{\max} \mathbb{C}) \rtimes_{\mathrm{id}\otimes\mathrm{id}} G \\
&\cong A \otimes_{\max} (\mathbb{C} \rtimes_{\mathrm{id}} G) = A \otimes_{\max} C^*(G)
\end{aligned}$$

and

$$\begin{aligned}
A \rtimes_{\mathrm{id},r} G &\cong (A \rtimes_{\mathrm{id},r} G) \otimes_{\min} \mathbb{C} \cong (A \otimes_{\min} \mathbb{C}) \rtimes_{\mathrm{id}\otimes\mathrm{id},r} G \\
&\cong A \otimes_{\min} (\mathbb{C} \rtimes_{\mathrm{id},r} G) = A \otimes_{\min} C^*_r(G).
\end{aligned}$$

Applying the individual isomorphisms given above shows that $f\widetilde{\otimes}a$ maps to $a \otimes f$ for $f \in C_c(G)$ and $a \in A$. □

Theorem 23. Let (A, G, α) be a dynamical system with A nuclear and G amenable. Then $A \rtimes_\alpha G$ is nuclear.

Proof. Let B be a C^*-algebra.

Since A is nuclear the natural map $A \otimes_{\max} B \to A \otimes_{\min} B$ is an isomorphism. It is trivial that this isomorphism is equivariant. Thus, we get an isomorphism

$$(A \otimes_{\max} B) \rtimes_{\alpha\otimes\mathrm{id}} G \xrightarrow{\cong} (A \otimes_{\min} B) \rtimes_{\alpha\otimes\mathrm{id}} G$$

such that for every $f \in C_c(G), a \in A, b \in B$ the element $f\widetilde{\otimes}(a \otimes b)$ is mapped to $f\widetilde{\otimes}(a \otimes b)$.

Then by Theorem 19 and Theorem 4 we have a diagram

$$\begin{tikzcd}
(A \rtimes_\alpha G) \otimes_{\max} B \arrow[r, "\cong", "\text{Thm. 17}"'] \arrow[dddd, dashed, "\varphi"', "\cong"] & (A \otimes_{\max} B) \rtimes_{\alpha\otimes \mathrm{id}} G \arrow[d, "\cong"', "A \text{ nuclear}"] \\
 & (A \otimes_{\min} B) \rtimes_{\alpha\otimes \mathrm{id}} G \arrow[d, "\cong"', "G \text{ amenable}"] \\
 & (A \otimes_{\min} B) \rtimes_{\alpha\otimes \mathrm{id}, r} G \arrow[dd, "\cong"', "\text{Thm. 17}"] \\
 & \\
(A \rtimes_\alpha G) \otimes_{\min} B & (A \rtimes_{\alpha, r} G) \otimes_{\min} B \arrow[l, "\cong"', "G \text{ amenable}"]
\end{tikzcd}$$

Thus we get an isometric $*$-isomorphism $\varphi : (A \rtimes_\alpha G) \otimes_{\max} B \to (A \rtimes_\alpha G) \otimes_{\min} B$. One easily checks that for $f \in C_c(G), a \in A, b \in B$ we have $\varphi((f\widetilde{\otimes}a) \otimes b) = (f\widetilde{\otimes}a) \otimes b$. By density this gives that $\varphi|_{(A\rtimes_\alpha G)\odot B} = \mathrm{id}$. But since φ is isometric this gives that the maximal norm and the minimal norm on $(A \rtimes_\alpha G) \odot B$ agree.

Since B was arbitrary it follows that $A \rtimes_\alpha G$ is nuclear. $\square$

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