INDUCED ALGEBRAS

1. Main Theorem

Assumption: All topological spaces are locally compact Hausdorff and all groups are locally compact unimodular, unless specified otherwise.

Definition 1.1. Let H act freely and properly on a space P, and let α be an action of H on a C^*-algebra A. The *induced C^*-algebra* is defined by

$$\mathrm{Ind}_H^P(A) = \{f \in C_b(P, A) : f(h \cdot p) = \alpha_h(f(p)), \text{ and } pH \mapsto ||f(p)|| \text{ is in } C_0(P/H)\}.$$

Comments: The map $Hp \mapsto ||f(p)||$ is well-defined. One can check that this is a closed $*$-subalgebra of $C_b(P, A)$, and therefore it is a C^*-algebra.

Remark 1.2. To get the *ordinary induced C^*-algebra*, we take $P = G$ and H to be a subgroup of G acting by right translation. Then

$$\mathrm{Ind}_H^G(A) = \{f \in C_b(G, A) : f(gh) = \alpha_h^{-1}(f(g)), \text{ and } Hg \mapsto ||f(g)|| \text{ is in } C_0(G/H)\}.$$

We have a well defined action of G on $\mathrm{Ind}_H^G A$, called the *induced action*, and given by $\mathrm{Ind}\alpha_g(f)(s) = f(g^{-1}s)$. We remark that $\mathrm{Ind}_H^G : (A, H, \alpha) \mapsto (\mathrm{Ind}_H^G(A), G, \mathrm{Ind}\alpha)$ is a fanctor between the correspondence categories.
Notice that if (A, G, β) is a C^*-dynamical system s.t. $\beta|_H = \alpha$, then the map $\varphi : C_b(G, A) \to C_b(G, A)$ given by $\varphi(f)(s) = \beta_s(f(s))$ defines an isomorphism of $\mathrm{Ind}_H^G(A)$ onto $C_0(G/H, A)$ mapping $\mathrm{Ind}\alpha$ to $lt \otimes \beta$.

Remark 1.3. $\mathrm{Ind}_H^G(A)$ is the C^*-analogue of the usual construction of the induced G-space $G \times_H Y$ of a topological H-space Y. $G \times_H Y$ is defined as the quotient of $G \times Y$ by the H-action $h \cdot (g, y) = (gh^{-1}, hy)$, and which is equipped with the G-action $k \cdot (g, y) = (kg, y)$. Indeed, if Y is locally compact, one checks that $\mathrm{Ind}_H^G(C_0(Y)) \cong C_0(G \times_H Y)$, with the induced action mapped to the mentioned G-action.

Proposition 1.4. Let (A, G, α) be a C^*-dynamical system and H be a closed subgroup of G acting by right translation. Equip $G \times \mathrm{Prim}(A)$ with the right H action $(g, P)h \mapsto (gh, h^{-1} \cdot P)$. Then

$$\mathrm{Prim}(\mathrm{Ind}_H^G(A)) \cong (G \times \mathrm{Prim}(A))/H$$

Proof. We sketch the argument. Let $s \in G$ and $\pi \in \hat{A}$ be given. Then one can show that $M(s, \pi)(f) = \pi(f(s))$ gives an irreducible representation of $\mathrm{Ind}_H^G(A)$. The idea is showing that $\mathrm{Ind}_H^G(A)$ is non-trivial, namely for any $a \in A$ and $s \in G$, there exists $f \in \mathrm{Ind}_H^G(A)$ s.t. $f(s) = a$. Therefore, $M(s, \pi)$ is irreducible if π is. The map $(s, \pi) \mapsto \ker M(s, \pi)$ gives the desired isomorphism. $\square$

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Theorem 1.5. Let (B, G, β) be a system and let H be a closed subgroup of G. Then (B, G, β) is isomorphic to an induced system $(\mathrm{Ind}_H^G(A), G, \mathrm{Ind}\alpha)$ if and only if there exsists a continuuos G-equivariant map $\varphi : \mathrm{Prim}(B) \to G/H$, where G acts on $\mathrm{Prim}(B)$ via $s \cdot P = \beta_s(P)$.

Proof. $(\Rightarrow)$: It is easy to see that the action $\mathrm{Ind}\alpha$ of G on $\mathrm{Prim}(\mathrm{Ind}_H^G(A))$ maps via the isomorphism in Proposition 1.4 to the action $g \cdot [s, P] = [gs, P]$. Therefore $[s, p] \mapsto sH$ is a continuous G-equivariant map of $\mathrm{Prim}(\mathrm{Ind}_H^G(A))$ onto G/H.
$(\Leftarrow)$: If $\varphi : \mathrm{Prim}(B) \to G/H$ is given, define $A = B/I$, where

$$I = \cap\{P \in \mathrm{Prim}(B) : \varphi(P) = eH\}.$$

Since I is H-invariant, $\beta|_H$ induces an action α of H on A and (B, G, β) is isomorphic to $(\mathrm{Ind}_H^G(A), G, \mathrm{Ind}(\alpha))$ via $b \mapsto f_b$, where $f_b(s) = \beta_{s^{-1}}(b) + I$. □

Corollary 1.6. Let X be a locally compact G-space and let H be a closed subgroup of G. Then X is G-homeomorphic to $G \times_H Y$ for some H-space Y if and only if there exists a G-equivariant continuous map $\varphi : X \to G/H$. If such a map is given, then Y can be chosen as $\varphi^{-1}(eH)$ and the homeomorphism $G \times_H Y \cong X$ is given by $[g, y] \mapsto gy$.

Remark 1.7. If $G \curvearrowright P$ is proper and free, then the orbit space P/G is Hausdorff. Moreover, $\mathrm{Ind}(A)$ is a $C_0(P/G)$-algebra with fibers all isomorphic to A.

Theorem 1.8. Let H, K be groups, A be a C^*-algebra and P be a topological space. Assume that $\sigma : K \to \mathrm{Aut}(A)$ and $\tau : H \to \mathrm{Aut}(A)$ are two commuting actions and moreover, K and H admit commuting, free, proper actions on a topological space P. We define actions

$$\alpha := \mathrm{Ind}\tau : K \to \mathrm{Aut}(\mathrm{Ind}_H^P(A_\tau)) \text{ and } \beta := \mathrm{Ind}\sigma : H \to \mathrm{Aut}(\mathrm{Ind}_K^P(A_\sigma))$$

by

$$\alpha_k(f)(p) = \sigma_k(f(k^{-1} \cdot p)) \text{ and } \beta_h(f)(p) = \tau_h(f(h^{-1} \cdot p)).$$

Then

$$\mathrm{Ind}A_\tau \rtimes_\alpha K \sim_M \mathrm{Ind}A_\sigma \rtimes_\beta H.$$

The result also holds for the reduced crossed products.

Proof. One has to check that α, β are well-defined strongly continuous actions. For example,

$$\alpha_k(f)(hp) = \sigma_k(f(k^{-1}hp)) = \sigma_k(f(hk^{-1}p)) = \sigma_k(\tau_h(f(k^{-1}p))) = \tau_h(\alpha_k(f(p))),$$

so $\alpha_k(f) \in \mathrm{Ind}A_\tau$. Notice that α, β are the diagonal actions on $C_b(P, A)$, restricted to the induced algebras. For the Morita equivalence, our imprimitivity bimodule will be a completion of an $B = C_c(K, \mathrm{Ind}A_\tau)$-$C = C_c(H, \mathrm{Ind}A_\sigma)$ module $X_0 = C_c(P, A)$ w.r.t. the following actions and inner products:

(1) $b \cdot x(p) = \int_K b(t, p)\sigma_t(x(t^{-1}p))dt$;
(2) $x \cdot c(p) = \int_H \tau_s^{-1}(x(sp)c(s, sp))ds$;
(3) ${}_B\langle x, y\rangle(k, p) = \int_H \tau_s(x(s^{-1}p)\sigma_k(y(k^{-1}s^{-1}p)^*))ds$;
(4) $\langle x, y\rangle_C(h, p) = \int_K \sigma_t(x(t^{-1}p)^*\tau_h(y(t^{-1}h^{-1}p)))dt$.

We skip the technical checking.

□

2. Applications

Theorem 2.1. (Diagonal Actions). Let H be a group acting properly and freely on a space P, and τ be an action of H on a C^*-algebra A. Then

$$\mathrm{Ind}A_\tau \sim_M C_0(P, A) \rtimes_{lt\otimes\tau} H.$$

Proof. Theorem 1.8 with $K = \{e\}$. Then $\mathrm{Ind}A_\sigma = C_0(P, A)$ and we are done. □

Theorem 2.2. (Green's Symmetric Impremitivity Theorem). Let H, K act freely and properly on a space P. If the actions commute, then

$$C_0(P/H) \rtimes_{lt} K \sim_M C_0(P/K) \rtimes_{lt} H.$$

Proof. Theorem 1.8 with $A = \mathbb{C}$, as $\mathrm{Ind}A_\tau = C_0(P/H)$ and $\mathrm{Ind}A_\sigma = C_0(P/K)$. □

Examples 2.3.

(1) Take $H = \{e\}$ in Theorem 2.2. We get that if K is acting freely and properly on a space P, then the crossed product $C_0(P) \rtimes_{lt} K$ is Morita equivalent to the commutative C^*-algebra $C_0(P/K)$.

(2) Let K and H be closed subgroups of a group G. Let K act by left multiplication on G and H act by right multiplication. Then the actions commute, so immediately from Theorem 2.2 we get

$$C_0(G/H) \rtimes_{lt} K \sim_M C_0(K \setminus G) \rtimes_{rt} H.$$

(3) Let $G = \mathbb{R}$ and $K = \mathbb{Z}$, $H = \alpha\mathbb{Z}$, for some $\alpha \in \mathbb{R}$. By the previous example

$$C_0(\mathbb{R}/\mathbb{Z}) \rtimes \alpha\mathbb{Z} \sim_M C_0(\mathbb{R}/\alpha\mathbb{Z}) \rtimes \mathbb{Z}.$$

That is,

$$A_\alpha \sim_M A_{\alpha^{-1}}$$

Theorem 2.4. (Crossed Products of Ordinary Induced C^*-Algebras). Let H a closed subgroup of K acting by right translation. Let $\tau : H \to \mathrm{Aut}(A)$ be an action of H on a C^*-algebra A. Then $\mathrm{Ind}A_\tau$ is the ordinary induced C^*-algebra, $\mathrm{Ind}_H^K(A)$, with induced action $\alpha : K \to \mathrm{Aut}(\mathrm{Ind}A_\tau)$ given by $\alpha_k(f)(k') = f(k^{-1}k')$ ($\alpha = \mathrm{Ind}(\tau)$). Then

$$\mathrm{Ind}A_\tau \rtimes_\alpha K \sim_M A \rtimes_\tau H.$$

Proof. Apply Theorem 1.8 with $P = K$ acting on itself by left translation, and $\sigma : K \to \mathrm{Aut}(A)$ the trivial action. It is left to show that in this case $\mathrm{Ind}A_\sigma \cong A$ and the isomorphism carries the action $\beta : H \to \mathrm{Aut}(\mathrm{Ind}A_\sigma)$, $\beta_h(f)(k) = \tau_h(f(kh))$, to the action τ. Indeed, $f \mapsto f(e)$ defines an isomorphism of $\mathrm{Ind}A_\sigma$ onto A. Indeed, for $a = f(e)$,

$$\beta_h(a) = \beta_h(f(e)) = \tau_h(f(h)) = \tau_h(f(e)) = \tau_h(a).$$

□

Remark 2.5. Let K be a scond countable group and H a closed subgroup of K acting on a C^*-algebra A. Phil Green's theorem gives the following decomposition

$$\mathrm{Ind}_H^K A \rtimes_{\mathrm{Ind}\tau} K \cong (A \rtimes_\tau H) \otimes \mathcal{K}(L^2(K/H)),$$

where the L^2 space is taken with respect to some quasi-invariant measure on K/H.

Corollary 2.6. (Green's Imrimitivity Theorem). Suppose H is a closed subgroup of K acting by right translation. Let $\sigma : K \to \mathrm{Aut}(A)$ be an action of K on a C^*-algebra A. Then

$$C_0(K/H, A) \rtimes_{lt\otimes\sigma} K \sim_M A \rtimes_{\sigma|_H} H.$$

Proof. Apply Theorem 2.4 with $\tau = \sigma|_H$. Then $\mathrm{Ind}(A_\tau)$ is the ordinary induced C^*-algebra equipped with the action α of K by left translation. However, since σ extends to an action of K, the discussion in Remark 1.2 implies that

$$C_0(K/H, A) \rtimes_{lt\otimes\sigma} K \cong \mathrm{Ind}(A_\tau) \rtimes_\alpha K \sim_M A \rtimes_{\sigma|_H} H.$$

□

Remark 2.7. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on a C^*-algebra A. By taking $H = \{e\}$ in Corollary 2.6, we get

$$C_0(G, A) \rtimes_{lt\otimes\alpha} G \sim_M A.$$

However, actually more than that is true:

$$C_0(G, A) \rtimes_{lt\otimes\alpha} G \cong A \otimes K$$

Indeed, in this case we obtain a unitary isomorphism between Green's bimodule X and the Hilbert-A-module $L^2(G, A)$ via the transformation

$$U : X \to L^2(G, A) \text{ given by } U(x)(s) = x(s)$$

Thus

$$C_0(G, A) \rtimes_{lt\otimes\alpha} G \cong K_A(X) = K_A(L^2(G, A)) \cong A \otimes K(L^2(G)).$$

In particular, it follows that $C_0(G) \rtimes_{lt} G \cong K(L^2(G))$.

Corollary 2.8. Let H be a closed subgroup of K, (A, K, α) be a C^*-dynamical system, and $\varphi : \mathrm{Prim}(A) \to K/H$ be a K-equivariant continuous map. Then

$$I = \cap\{P \in \mathrm{Prim}(A) : \varphi(P) = eH\}.$$

is an H invariant ideal, and

$$A \rtimes_\alpha K \sim_M A/I \rtimes_{\alpha^I} H$$

Proof. By Theorem 1.5 $(A, K, \alpha) \cong (\mathrm{Ind}(A/I), K, \mathrm{Ind}(\alpha^I))$, therefore

$$A \rtimes_\alpha K \cong \mathrm{Ind}(A/I) \rtimes_{\alpha^I} K$$

and the last is Morita equivalent to $A/I \rtimes_{\alpha^I} H$ by Theorem 2.4. □

Examples 2.9.

(1) Let P be a locally compact G space and H a closed subgroup of G. Suppose that $\varphi : P \to G/H$ is a G-equivariant continuous map. Let $Y = \varphi^{-1}(eH)$. Then Y is a H-space, and $C_0(P) \rtimes_{lt} G \sim_M C_0(Y) \rtimes_{lt} H$.

(2) Let H be a closed subgroup of G acting on a space Y. Then

$$C_0(G \times_H Y) \rtimes_{lt} G \sim_M C_0(Y) \rtimes_{lt} H.$$

(3) Let $\mathbb{R}$ act on the two-torus $\mathbb{T}^2$ by an irrational flow, i.e. there exists an irrational number $\theta \in (0, 1)$ s.t. $t \cdot (z_1, z_2) = (e^{2\pi it} z_1, e^{2\pi i\theta t} z_2)$. Then $\mathbb{T}^2 \cong \mathbb{R} \times_{\mathbb{Z}} \mathbb{T}$ equivariantly, where $\mathbb{Z}$ acts on $\mathbb{T}$ by irrational rotation given by θ. Thus

$$C(\mathbb{T}^2) \rtimes_\theta \mathbb{R} \sim_M A_\theta$$