IV

Reduced Cross-Sectional Algebras

&

Generalized Fourier Coefficients

CONTENT

I. Introduction

Definition: Let G be group. A C^*-grading for a C^*-algebra B is a collection $\{B_g\}_{g\in G}$ of closed subspaces $B_g \subset B$ such that

(i) $\bigoplus_{g\in G} B_g \underset{\text{dense}}{\subseteq} B$

(ii) $B_g B_h \subseteq B_{gh}$

(iii) $B_g^* \subseteq B_{g^{-1}}$ for all $g, h \in G$.

If B has a C^*-grading it is called G-graded C^*-algebra and each B_g a grading subspace.

Remark: Every C^*-grading $\{B_g\}_{g\in G}$ forms a Fell bundle with the norm, multiplication operation and adjoint operation of B.

$$
\begin{tikzcd}
B = \overline{\bigoplus_{g \in G} B_g}^{\|\cdot\|_B} \arrow[d] & G\text{-graded } C^*\text{-algebras} \arrow[d, shift right=2ex] \\
\{B_g\}_{g \in G} & \text{Fell bundles} \arrow[u, shift right=2ex, "?"']
\end{tikzcd}
$$

Question: Can every Fell bundle be obtained from a graded C^*-algebra?

$\longrightarrow$ YES!

$\longrightarrow$ In general, there is more than one way to complete a Fell bundle to get a graded C^*-algebra!

Fell bundle $\quad$ $*$-repr. $\quad$ associative $*$-algebra $\quad$ cross-sectional algebra $\leftarrow$ C^*-algebra

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, "\{j_g\}"] \arrow[rr, bend right, "\{\hat{j}_g\}"] \arrow[drr, "\{\pi_g\}"'] & C_c(\mathcal{B}) \arrow[r, "\kappa"] & C^*(\mathcal{B}) := \overline{C_c(\mathcal{B})/N}^{\|\cdot\|^{\sim}_{max}} \arrow[d, dashed, "\exists!\,\varphi"] \\
 & & A
\end{tikzcd}
$$

universal representation

<u>Problem</u>: So far, we only know that

$$\|y\|_{max} := \sup\{\, p(y) : p \text{ is a } C^*\text{-seminorm on } C_c(\mathcal{B})\}$$

is bounded, but it might be that

$$N = \{\, y \in C_c(\mathcal{B}) : \|y\|_{max} = 0\}$$

is all of $C_c(\mathcal{B})$ and $\|\cdot\|_{max}$ the zero seminorm.

$\rightsquigarrow$ Construction of a nontrivial representation $(\neq 0)$ of $\mathcal{B}$ with a nontrivial seminorm

$\quad\}$ completion process

$\Rightarrow$ REDUCED CROSS-SECTIONAL ALGEBRA $C^*_{red}(\mathcal{B})$

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\{j_g\}"] \arrow[rr, bend left, "\{\hat{j}_g\}"] \arrow[rrd, "\{\lambda_g\}"'] & C_c(\mathcal{B}) \arrow[r, tail, "\kappa"] \arrow[rd, tail] \arrow[rd, phantom, shift left=4, "\circlearrowright"] & C^*(\mathcal{B}) \arrow[d, dotted, two heads, "\Lambda"] \\
& & C^*_{red}(\mathcal{B})
\end{tikzcd}$$

isometry

graded C^*-algebra

$$C^*(\mathcal{B}) := \overline{C_c(\mathcal{B})}^{\|\cdot\|_{max}^{\sim}} \Big/ N$$

$$\mathcal{B} = \{B_g\}_{g \in G}$$

isometry

$$C^*_{red}(\mathcal{B}) \subset \mathcal{L}(\ell^2(\mathcal{B}))$$

graded C^*-algebra

II.1 HILBERT MODULES

Definition: Let A be a C^*-algebra.

A right pre-Hilbert A-module is a complex vector space M equipped with a right A-module structure and an inner product

$$\langle \cdot, \cdot \rangle : M \times M \to A$$

such that

(i) $\langle \xi, \lambda\eta + \eta' \rangle = \lambda \langle \xi, \eta \rangle + \langle \xi, \eta' \rangle$

(ii) $\langle \xi, \xi \rangle \geq 0$ and $\langle \xi, \xi \rangle = 0 \Rightarrow \xi = 0$

(iii) $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$

(iv) $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$ $\qquad \forall \xi, \eta, \eta' \in M,\ a \in A,\ \lambda \in \mathbb{C}$

Remark: (1) $\langle \cdot, \cdot \rangle$ is conjugate-linear in the first component.

(2) $\langle \xi a, \eta \rangle = a^* \langle \xi, \eta \rangle \qquad \forall a \in A,\ \xi, \eta \in M$

$\ulcorner$ $\langle \xi a, \eta \rangle = (\langle \eta, \xi a \rangle)^* \overset{(iii)}{=} (\langle \eta, \xi \rangle a)^* \overset{*\text{ in }A}{=} a^* \langle \eta, \xi \rangle^* \overset{iv}{=} a^* \langle \xi, \eta \rangle$ $\lrcorner$

(3) A left pre-Hilbert A-module M can be defined analogously using (i') $\langle \lambda\xi + \xi', \eta \rangle = \lambda \langle \xi, \eta \rangle + \langle \xi', \eta \rangle$

(iii') $\langle a\xi, \eta \rangle = a \langle \xi, \eta \rangle$ $\qquad \forall \xi, \xi', \eta \in M,\ a \in A,\ \lambda \in \mathbb{C}$

Proposition / Definition:

Let M be a right pre-Hilbert A-module. Then

$$\|\cdot\|_2 : M \to \mathbb{C}, \quad \|\xi\|_2 := \|\langle \xi, \xi \rangle\|_A^{1/2}$$

defines a norm on M.

If M is complete relative to $\|\cdot\|_2$, it is a (right) Hilbert A-module.

Proof: For all states f on A

$$(\xi, \eta) \longmapsto f(\langle \xi, \eta \rangle) \qquad \forall \xi, \eta \in M$$

is an semi inner product and $\xi \longmapsto f(\langle \xi, \xi \rangle)^{1/2}$ a seminorm on M. We have $\|\langle \xi, \xi \rangle\|_A^{1/2} = \sup\{ f(\langle \xi, \xi \rangle)^{1/2} : f \in S(A)\}$ and hence, $\|\xi\|_2 = \|\langle \xi, \xi \rangle\|_A^{1/2}$ as a pointwise supremum of seminorms on M is also a seminorm and by (ii) of the previous definition, it is a norm. $\square$

Proposition / Definition

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Examples: (i) Every C^*-algebra A is a Hilbert module over itself with inner product $\langle a, b \rangle := a^* b \quad \forall a, b \in A$.

(ii) Every (closed) ideal $J \trianglelefteq A$ is a sub-Hilbert module of A.

Remark: The completion of a left pre-Hilbert A-module via $\|\cdot\|_2$ is a (left) Hilbert A-module.

<u>Definition</u>: Let M, N be right Hilbert A-modules and $T: M \to N$ a bounded A-linear operator.

An adjoint of T is an operator $T^*: N \to M$, such that

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle \qquad \forall \xi \in M, \eta \in N.$$

An operator T is called adjointable operator, if it has an adjoint.

<u>Note</u>: Not every operator T has an adjoint!

$\rightsquigarrow$ A unital, $J \trianglelefteq A$ not unital, $J \xhookrightarrow{\iota} A$.

$\Rightarrow \iota$ is a bounded A-linear operator

Suppose ι^* exists:

$$\rightsquigarrow \quad x^* = x^* 1_A = \langle \iota(x), 1_A \rangle = \langle x, \iota^*(1_A) \rangle = x^* \iota^*(1_A) \quad \forall x \in J$$

$$\Rightarrow \iota^*(1_A) \text{ is a unit for } J \quad \lightning \quad \Rightarrow \iota^* \text{ does not exist!}$$

$\mathcal{L}(M)$ denotes the set of all adjointable operators on the Hilbert module M. It is a C^*-algebra!

Lemma: Let A be a C^*-algebra and M a right (left) Hilbert A-module. For an approximate unit $\{v_i\}_{i\in I} \subset A$

$$\xi = \lim_{i\in I} \xi v_i \quad \left(\xi = \lim_{i\in I} v_i \xi \right) \qquad \forall \xi \in M.$$

Proof:
$$\begin{aligned} \|\xi - \xi v_i\|_2^2 &= \|\langle \xi - \xi v_i, \xi - \xi v_i \rangle\|_A \\ &= \|\langle \xi,\xi\rangle - \langle \xi,\xi\rangle v_i - v_i^*\langle \xi,\xi\rangle + v_i^*\langle \xi,\xi\rangle v_i\|_A \\ &\leq \|\langle \xi,\xi\rangle - \langle \xi,\xi\rangle v_i\|_A + \|v_i^*\|_A \cdot \|\langle \xi,\xi\rangle v_i - \langle \xi,\xi\rangle\|_A \\ &\xrightarrow{i\to\infty} 0. \end{aligned}$$

$\square$

II.

REGULAR REPRESENTATION

OF

A FELL BUNDLE

Proposition:

Let $\mathcal{B} = \{B_g\}_{g\in G}$ be a Fell bundle.

(i) By $j_1 : B_1 \to C_c(\mathcal{B})$, $C_c(\mathcal{B})$ is a right B_1-module.

(ii) $\langle \cdot, \cdot \rangle : C_c(\mathcal{B}) \times C_c(\mathcal{B}) \longrightarrow B_1$,

$$\langle y, z \rangle := \sum_{g\in G} (y_g)^* z_g \qquad \forall y, z \in C_c(\mathcal{B})$$

is a B_1-valued inner-product of $C_c(\mathcal{B})$, which makes $C_c(\mathcal{B})$ a right pre-Hilbert module. $\|\cdot\|_2$ denotes the induces norm.

(iii) $\ell^2(\mathcal{B}) := \overline{C_c(\mathcal{B})}^{\langle\cdot,\cdot\rangle}$ is a right Hilbert B_1-module.

(iv) $j_g : B_g \to \ell^2(\mathcal{B})$ is an isometry for all $g \in G$.

Proof: (iv) $\forall b \in B_g$: $\|j_g(b)\|_2^2 = \|\langle j_g(b), j_g(b)\rangle\| = \|b^* b\| = \|b\|^2$.

STRATEGY

$$\mathcal{B} \xrightarrow{\lambda} \begin{array}{c} C_c(\mathcal{B}) \\ \big\downarrow \\ C_c(\mathcal{B}) \end{array}$$

$$\rightsquigarrow \text{ completion}$$

$$\mathcal{B} \longrightarrow \begin{array}{c} \ell^2(\mathcal{B}) \\ \big\downarrow \\ \ell^2(\mathcal{B}) \end{array} \in \mathcal{L}(\ell^2(\mathcal{B}))$$

Lemma: Let $g, h \in G$, $b \in \mathcal{B}_g$, $c \in \mathcal{B}_h$. Then

$$\underbrace{c^* b^* b c}_{\in \mathcal{B}_1} \leq \|b\|^2 \underbrace{c^* c}_{\in \mathcal{B}_1} .$$

Proof: Recall: $h \leq \|h\|$ for $h \in A_+$, i.e. $v^*(\|h\| - h)v \geq 0 \quad \forall v \in A$

(alternatives: 1. $A \rightsquigarrow A^\sim$
2. $\|h\| v^* v - v^* h v \geq 0$

Take an approximate $\{v_i\}_{i \in I} \subset A$. Since $b^* b \in (\mathcal{B}_1)_+$,

$$\Rightarrow \quad v_i^* (\|b\|^2 - b^* b) v_i \geq 0$$

$$\Rightarrow \quad \exists a_i \in \mathcal{B}_1 : \quad v_i^* (\|b\|^2 - b^* b) v_i = a_i^* a_i$$

$$\Rightarrow \quad 0 \leq (\underbrace{a_i c}_{\in \mathcal{B}_h})^* (a_i c) = c^* a_i^* a_i c = c^* v_i^* (\|b\|^2 - b^* b) v_i c$$

$$\xrightarrow{i \to \infty} \quad c^* (\|b\|^2 - b^* b) c = \|b\|^2 c^* c - c^* b^* b c$$

$$\Leftrightarrow \quad c^* b^* b c \leq \|b\|^2 c^* c \qquad \square$$

Proposition: Let $b \in B_g$. Then

$$\lambda_g(b): \begin{array}{ccc} y & \longmapsto & j_g(b) * y \\ \cap & & \cap \\ C_c(\mathcal{B}) & & C_c(\mathcal{B}) \end{array}$$

is continuous relative to $\|\cdot\|_2$ on $C_c(\mathcal{B})$, can be therefore extended to $\ell^2(\mathcal{B})$ s.t. $\|\lambda_g(b)\|_2 = \|b\|$ and $\lambda_g(b)(j_h(c)) = j_{gh}(bc)$ $\forall h \in G$, $c \in B_h$.

Proof: • $\forall y \in C_c(\mathcal{B})$, $h \in G$:

$$\lambda_g(b)y|_h = j_g(b) * y|_h = \sum_{\tilde{g} \in G} j_g(b)|_{\tilde{g}}\, y_{\tilde{g}^{-1}h} = b\, y_{g^{-1}h}$$

$\rightarrow$ $c \in B_h$: $\forall k \in G$

$$\lambda_g(b)(j_h(c))|_k = b\, j_h(c)|_{g^{-1}k} = \delta_{h,g^{-1}k}\, bc = \delta_{gh,k}\, bc = j_{gh}(bc)|_k .$$

• $\forall y \in C_c(\mathcal{B})$:

$$\langle \lambda_g(b)y, \lambda_g(b)y\rangle = \sum_{h \in G} (y_{g^{-1}h})^* b^* b\, y_{g^{-1}h} = \sum_{h \in G} (y_h)^* b^* b\, y_h \overset{\text{Lemma}}{\leq} \|b\|^2 \sum_{h \in G} y_h^* y_h = \|b\|^2 \langle y, y\rangle$$

$$\Longrightarrow \|\lambda_g(b)y\|_2 \leq \|b\|\,\|y\|_2 \Rightarrow \lambda_g(b) \text{ is bounded with } \|\lambda_g(b)\| \leq \|b\|$$

• $\xi \in \ell^2(\mathcal{B})$ with $j_{g^{-1}}(b^*) = \xi \xrightarrow{j_{g^{-1}} \text{ isometry}} \|\xi\|_2 = \|b\|$

$$\rightarrow \lambda_g(b)\xi = \lambda_g(b) j_{g^{-1}}(b^*) = j_g(b) * j_{g^{-1}}(b^*) = j_1(bb^*)$$

$$\Rightarrow \|b\|^2 = \|b^*b\| = \|j_1(b^*b)\|_2 = \|\lambda_g(b)\xi\|_2 \leq \|\lambda_g(b)\|\,\|\xi\|_2 = \|\lambda_g(b)\| \cdot \|b\| \Rightarrow \|\lambda_g(b)\| \geq \|b\|.$$

$\square$

<u>Proposition / Definition</u>: The collection of maps $\lambda = \{\lambda_g\}_{g\in G}$ is a representation of $\mathcal{B}$ in $\mathcal{L}(\ell^2(\mathcal{B}))$. It is called the regular representation of $\mathcal{B}$.

<u>Proof</u>: (i) $\lambda_g(b)^* = \lambda_{g^{-1}}(b^*) \quad \forall b \in \mathcal{B}_g$ (i.e. λ_g is adjointable)

$\rightarrow \quad \forall y,z \in C_c(\mathcal{B})$, $b\in\mathcal{B}_g$: $\langle \lambda_g(b)y, z\rangle = \sum_{h\in G} (y_{g^{-1}h})^* b^* z = \sum_{h\in G} (y_h)^* b^* z_{gh}$

$$= \sum_{h\in G} (y_h)^* \Big(\sum_{\ell\in G} j_{g^{-1}}(b^*)\big|_\ell z_{\ell^{-1}h} \Big)$$

$$= \sum_{h\in G} (y_h)^* \big(j_{g^{-1}}(b^*) * z\big)\big|_h = \langle y, \lambda_{g^{-1}}(b^*) z\rangle$$

$C_c(\mathcal{B})$ is dense in $\ell^2(\mathcal{B})$ $\Rightarrow \langle \lambda_g(b)\xi, \eta\rangle = \langle \xi, \lambda_{g^{-1}}(b^*)\eta\rangle \quad \forall \xi,\eta \in \ell^2(\mathcal{B})$

$\Rightarrow \lambda$ is adjointable with $\lambda_g(b)^* = \lambda_{g^{-1}}(b^*)$

(ii) (*) $\lambda_g(b_g)\lambda_h(b_h) = \lambda_{gh}(b_g b_h)$: $\forall b_g \in \mathcal{B}_g,\ b_h \in \mathcal{B}_h,\ y \in C_c(\mathcal{B})$:

$$\lambda_g(b_g)(\lambda_h(b_h)y) = j_g(b_g) * j_h(b_h) * y = j_{gh}(b_g b_h) * y = \lambda_{gh}(b_g b_h) y$$

$C_c(\mathcal{B})$ is dense in $\ell^2(\mathcal{B})$ $\Longrightarrow$ (*). $\square$

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\lambda_g"] \arrow[dr, "\hat{j}_g"'] & \mathcal{L}(\ell^2(\mathcal{B})) \\
& C^*(\mathcal{B}) \arrow[u, dashed, "\exists!\ \Lambda"']
\end{tikzcd}$$

$$\Lambda \circ \hat{j}_g = \lambda_g \quad \forall\, g \in G$$

<u>Definition</u>: Λ is the regular representation of $C^*(\mathcal{B})$ and $C^*_{red}(\mathcal{B}) := \Lambda(C^*(\mathcal{B}))$ is called reduced cross sectional C^*-algebra of $\mathcal{B}$.

$$\rightarrow \quad C^*_{red}(\mathcal{B}) \cong C^*(\mathcal{B}) / \ker \Lambda$$

<u>Example</u>: $\mathcal{B} = \mathbb{C} \times G \;\longrightarrow\; C^*_{red}(G)$ is the reduced group C^*-algebra

Lemma / Definition: For each $g \in G$ there exists a contractive linear map $E_g : C^*_{red}(\mathcal{B}) \to B_g$ defined by

$$E_g(\lambda_h(b)) = \begin{cases} b & g = h \\ 0 & g \neq h \end{cases} \qquad \forall h \in G,\ b \in B_g$$

For all $z \in C^*_{red}(\mathcal{B})$, $E_g(z)$ is called g^{th} Fourier coefficient of z.

Proof: $\forall g \in G$: $P_g : C_c(\mathcal{B}) \to B_g$, $y \longmapsto y_g$ is continuous with respect to the Hilbert module norm

$$\left[\underbrace{\|P_g y\|^2}_{\|P_g j_g(y_g)\| = \|y_g\| = \|j_g(y_g)\|} = \|y_g\|^2 = \|y_g^* y_g\| \overset{?}{\leq} \Big\|\sum_{h \in G} y_h^* y_h\Big\| = \|\langle y, y\rangle\| = \|y\|_2^2 \qquad \forall y \in C_c(\mathcal{B}) \right.$$

$\Rightarrow \|P_g\| = 1 \ \forall g \in G$

$\to P_g$ extends to a continuous map from $\ell^2(\mathcal{B}) \to B_g$.

- $\{v\}_i$ approx. unit in B_1

$$\begin{array}{ccccccc} B_1 & \xrightarrow{j_1} & \ell^2(\mathcal{B}) & \xrightarrow{z} & \ell^2(\mathcal{B}) & \xrightarrow{P_g} & B_g \\ v_i & \longmapsto & j_1(v_i) & \longmapsto & z(j_1(v_1)) & \longmapsto & P_g(z(j_1(v_1))) \end{array} \qquad \forall z \in C^*_{red}(\mathcal{B})$$

$\lim_i P_g(z(j_1(v_i)))$ exists:

- $z \in \lambda_h(b)$, $h \in G$, $b \in B_h$:

$$P_g(z(j_1(v_i))) = P_g(\lambda_h(b)(j_1(v_i))) = P_g(j_h(b) * j_1(v_i)) = P_g(j_h(bv_i)) = j_h(bv_i)\big|_g = \delta_{g,h}\, bv_i \xrightarrow{i \to \infty} \delta_{g,h}\, b$$

- $z \in \sum_{h \in G} \lambda_h(B_h) = \Lambda(\kappa(C_c(\mathcal{B})))$ $\leftarrow$ dense in $C^*_{red}(\mathcal{B})$

$$\begin{tikzcd} \bigoplus_{g \in G} B_g \arrow[r, "\{j_g\}", "="'] \arrow[rr, bend left, "\{\hat{j}_g\}"] \arrow[rrr, bend right, "\{\lambda_g\}"'] & C_c(\mathcal{B}) \arrow[r, "\kappa"] & C^*(\mathcal{B}) \arrow[r, "\Lambda"] & C^*_{red}(\mathcal{B}) \end{tikzcd}$$

- P_g, z, j_1 are uniformly bounded $\Rightarrow \lim_i P_g(z(j_1(v_i)))$ exists for all $z \in C^*_{red}(\mathcal{B})$

$\rightarrow \quad E_g(z) := \lim_i P_g(z(j_1(v_i))) \quad \forall z \in C^*_{red}(\mathcal{B})$

$\Rightarrow \quad E_g(\lambda_h(b)) = \delta_{g,h}\, b$

- $\|E_g(z)\| = \lim_i \|P_g(z(j_1(v_1)))\| \overset{\|v_i\| \leq 1,\ \|P_g\| = 1}{\leq} \|z\|$ $\quad \square$

$$\begin{tikzcd}
\mathcal{B}_g \arrow[r, "\lambda_g"] \arrow[d, "j_g"'] \arrow[dr, dashed, "\hat{j}_g"] & C^*_{red}(\mathcal{B}) \subset \mathcal{L}(\ell^2(\mathcal{B})) \\
C_c(\mathcal{B}) \arrow[r, "\kappa"] & C^*(\mathcal{B}) \arrow[u, dashed, two heads, "\Lambda"']
\end{tikzcd}$$

<u>Proposition</u>: Let $\mathcal{B}$ be a Fell bundle.

(i) κ is injective
(ii) $\Lambda \circ \kappa$ is injective
(iii) $\exists$ C^*-seminorm on $C_c(\mathcal{B})$ that is a norm
(iv) $\forall g \in G$: $\hat{j}_g$ is an isometry
(v) $\forall g \in G$: λ_g is an isometry
(vi) $C^*(\mathcal{B})$ is a graded C^*-algebra with grading subspaces $\hat{j}_g(\mathcal{B}_g)$
(vii) $C^*_{red}(\mathcal{B})$ is a graded C^*-algebra with grading subspaces $\lambda_g(\mathcal{B}_g)$

<u>Proof</u>: • (ii) $y \in C_c(\mathcal{B})$ s.t. $\Lambda(\kappa(y)) = 0$. Write $y = \sum_{h \in G} j_h(y_h)$.

$$\Rightarrow \forall g \in G: \quad 0 = E_g(\Lambda(\kappa(y))) = \sum_{h \in G} E_g(\Lambda(\kappa(j_h(y_h))))$$
$$= \sum_{h \in G} E_g(\lambda_h(y_h)) = \sum_{h \in G} \delta_{g,h} y_h = y_g \quad \Rightarrow y = 0$$

• (i) follows from (ii)

• (iii) $\|y\|_{max} = \|\kappa(y)\|$ is a norm, since κ is injective

• (iv) $b \in \mathcal{B}_g$:
$$\left.\begin{aligned} \|\hat{j}_g(b)\| &= \|\kappa(j_g(b))\| = \|j_g(b)\|_{max} \overset{(*)}{\leq} \|b\| \\ \|\hat{j}_g(b)\| &\geq \|\Lambda(\hat{j}_g(b))\| = \|\lambda_g(b)\| = \|b\| \end{aligned}\right\} \Rightarrow \|\hat{j}_g\| = 1$$

• (vii) λ_g isometry $\Rightarrow$ $\lambda_g(\mathcal{B}_g) \underset{\text{closed}}{\subset} C^*_{red}(\mathcal{B})$

$\lambda_g(B_g)$ are linearly independent:

Suppose $\sum_{g\in G} z_g = 0$ for $z_g \in \lambda_g(B_g)$, only finitely many nonzero terms

$z_g = \lambda_g(y_g)$ for some $y_g \in B_g$ $\quad y = \sum_{g\in G} j_g(y_g) \in C_c(\mathcal{B})$

$$\Lambda(\varkappa(y)) = \sum_{g\in G} \Lambda(\varkappa(j_g(y_g))) = \sum_{g\in G} \lambda_g(y_g) = \sum_{g\in G} z_g = 0 \overset{(ii)}{\Rightarrow} y = 0$$

$\Rightarrow z_g = \lambda_g(y_g) = 0 \quad \forall g \in G$

$$\bigoplus_{g\in G} \lambda_g(B_g) = \Lambda(\varkappa(C_c(\mathcal{B}))) \underset{\text{dense}}{\subseteq} C^*_{red}(\mathcal{B})$$

$$\lambda_g(B_g)\lambda_h(B_h) \overset{\lambda_g \text{ repr.}}{=} \lambda_{gh}(B_{gh}) \quad , \quad \lambda_g(B_g)^* = \lambda_{g^{-1}}(B_g^*) = \lambda_{g^{-1}}(B_{g^{-1}})$$

(vi) similar to (vii). $\square$

<u>Definition</u>: The reduced crossed product $A \rtimes_{red} G$ of a C^*-algebra A by a partial action θ of a group G is the reduced cross sectional algebra of the corresponding semi-direct product bundle.

<u>Corollary</u>:

$$\begin{tikzcd}
\mathcal{B}_g \arrow[r, "\lambda_g"] \arrow[d, "j_g"'] \arrow[dr, dashed, "j_g"] & C^*_{red}(\mathcal{B}) \\
C_c(\mathcal{B}) \arrow[r, "\kappa"] & C^*(\mathcal{B}) \arrow[u, dashed, two heads, "\Lambda"']
\end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd}
\{b\delta_g : b \in D_g\} = \mathcal{B}_g \arrow[r, "\lambda_g"] \arrow[d] \arrow[dr, dashed] & A \rtimes_{red} G \\
A \rtimes_{alg} G \arrow[r, "\kappa"] & A \rtimes G \arrow[u, dashed, two heads, "\Lambda"']
\end{tikzcd}$$

III.

GENERALIZED FOURIER COEFFICIENTS & PARSEVAL'S IDENTITY

Proposition: Let $z \in C^*_{red}(\mathcal{B})$. Then

$$\langle j_g(b), z j_h(c) \rangle = b^* E_{gh^{-1}}(z) c \qquad \forall\, g,h \in G,\ b \in \mathcal{B}_g,\ c \in \mathcal{B}_h$$

Proof: $\bigoplus_g \lambda_g \mathcal{B}_g$ dense in $C^*_{red}(\mathcal{B}) \Rightarrow$ enough to show this for $z = \sum_{k \in G} \lambda_k(y_k)$ with $y_k \in \mathcal{B}_k$

$$\Rightarrow \langle j_g(b), z j_h(c) \rangle = \sum_{k \in G} \langle j_g(b), \lambda_k(y_k) j_h(c) \rangle$$

$$= \sum_{k \in G} \langle j_g(b), j_k(y_k) * j_h(c) \rangle$$

$$= \sum_{k \in G} \langle j_g(b), j_{kh}(y_k c) \rangle$$

$$= \sum_{k \in G} \Big(\sum_{l \in G} \big(j_g(b)\big|_l\big)^* j_{kh}(y_k c)\big|_l \Big)$$

$$= \sum_{k \in G} \Big(b^* j_{kh}(y_k c)\big|_g \Big)$$

$$= b^* y_{gh^{-1}} c = b^* E_{gh^{-1}}(z) c \qquad \square$$

Theorem: For $f \in C^1(\mathbb{T})$: $f = 0 \iff \hat{f}(n) = 0 \quad \forall n \in \mathbb{Z}$.

Theorem: Let $z \in C^*_{red}(\mathcal{B})$. Then the following are equivalent

(i) $E_1(z^*z) = 0$

(ii) $E_g(z) = 0 \quad \forall g \in G$

(iii) $z = 0$

Proof: (iii) $\Rightarrow$ (i) : ✓ , (iii) $\Rightarrow$ (ii) : ✓

(i) $\Rightarrow$ (iii) : For $b \in B_g$

$$\langle z j_g(b), z j_g(b) \rangle = \langle j_g(b), z^*z\, j_g(b) \rangle = b^* E_1(z^*z) b = 0 \quad \Rightarrow z j_g(b) = 0$$

$\bigoplus_{g \in G} j_g(B_g)$ dense in $\ell^2(\mathcal{B}) \quad \Rightarrow \quad z = 0$

(ii) $\Rightarrow$ (iii) : For $b \in B_g$, $c \in B_h$:

$$\langle j_g(b), z\, j_h(c) \rangle = b^* E_{gh^{-1}}(z)\, c = 0$$

span$\{j_g(b)\}$ dense $\Rightarrow$ $z j_h(c) = 0 \xRightarrow{\text{span}(j_h(c)) \text{ dense}} z = 0$. $\square$

$$\hat{f}(0) := \int_{\mathbb{T}} f(x)\, e^{-2\pi i 0 x}\, dx = \int_{\mathbb{T}} f(x)\, dx \qquad f \in C(\mathbb{T})$$

Bessel's inequality : $\sum_{k=-n}^{n} |\hat{f}(n)|^2 \leq \|f\|_2^2$

Parseval's identity : $\int_{\mathbb{T}} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 \iff \|f\|_{L^2}^2 = \|\underbrace{\mathcal{F} f}_{(\hat{f}(n))_{n\in\mathbb{Z}}}\|_{\ell^2}^2$

Theorem : Let $z \in C^*_{red}(\mathcal{B})$.

(i) For every finite $K \subset G$: $\sum_{g \in K} E_g(z)^* E_g(z) \leq E_1(z^* z)$ (BESSEL)

(ii) If $z = \sum_{g \in G} \lambda_g(y_g)$ for $y \in C_c(\mathcal{B})$, then $\sum_{g \in G} E_g(z)^* E_g(z) = E_1(z^* z)$

(iii) $\sum_{g \in G} E_g(z)^* E_g(z) = E_1(z^* z)$ where the series converges unconditionally.

$\lim_{K \subseteq G} \{\sum_{g \in K} E_g(z)^* E_g(z)\}_{K \subseteq G} = E_1(z^* z)$ (PARSEVAL)

Proof : (ii) By Prop. $E_g(z) = y_g \Rightarrow$ the sum is finite

$$\to E_1(z^* z) = E_1\Big(\sum_{g,h \in G} \underbrace{\lambda_g(y_g)^*}_{= \lambda_{g^{-1}}(y_g^*)} \lambda_h(y_h)\Big) = E_1\Big(\sum_{g,h \in G} \lambda_{g^{-1}h}(y_g^* y_h)\Big) \overset{\text{Def } E_1}{=} \sum_{g \in G} y_g^* y_g = \sum_{g \in G} E_g(z)^* E_g(z)$$

(i) Let $z = \sum_{g \in G} \lambda_g(y_g)$ for $y \in C_c(\mathcal{B})$: $E_1(z^* z) \overset{(ii)}{=} \sum_{g \in G} E_g(z)^* E_g(z) \geq \sum_{g \in K} E_g(z)^* E_g(z)$

for all $K \subseteq G$ finite. By this density argument and the continuity of E_g $\forall g \in G$, (i) holds for all $z \in C^*_{red}(\mathcal{B})$.

(iii) Let $K = \{g_1, g_2, \dots, g_n\} \subset G$ be finite. Define

$$E_K : C^*_{red}(\mathcal{B}) \longrightarrow M_n(C^*_{red}(\mathcal{B}))$$

$$z \longmapsto \begin{pmatrix} E_{g_1}(z) & 0 & \cdots & 0 \\ E_{g_2}(z) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{g_n}(z) & 0 & \cdots & 0 \end{pmatrix}$$

$[\, B_g \hookrightarrow C^*_{red}(\mathcal{B}) \,]$

- E_K is linear ✓ and contractive

$$\|E_K(z)\|^2 = \|E_K(z)^* E_K(z)\| = \|\sum_{g\in K} E_g(z)^* E_g(z)\| \overset{(i)}{\leq} \|E_1(z^*z)\| \leq \|z^*z\| = \|z\|^2$$

- (∗) $\left(E_K(z)^* E_K(z)\right)_{1,1} = \sum_{g\in K} E_g(z)^* E_g(z) \quad \forall z \in C^*_{red}(\mathcal{B}),\ K \subseteq G$

- $\forall z, z_0 \in C^*_{red}(\mathcal{B})$:

$$\|E_K(z)^* E_K(z) - E_K(z_0)^* E_K(z_0)\| = \|(E_K(z) + E_K(z_0))^* (E_K(z) - E_K(z_0))\|$$

$$= \|E_K(z+z_0)^* E_K(z-z_0)\|$$

$$\leq \|z\|\cdot\|z-z_0\| + \|z_0\|\cdot\|z-z_0\|$$

$\Rightarrow \{E_K(\cdot)^* E_K(\cdot) : K \subseteq G,\ |K| < \infty\}$ is equicontinuous

$\overset{(ii)}{\Longrightarrow}$ (∗) $\to E_1(z^*z)$ for $K \uparrow G$ $\xrightarrow[\text{+ equic.}]{\text{density}}$ (∗) $\to E_1(z^*z)\ \forall z \in C^*_{red}(\mathcal{B})$.

□