Duality for Fell bundles
–
Morita-Rieffel equivalence

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Content

I.

Smash product and restricted smash product

Smash product

Let G be a discrete group. $\ell^2(G)$, the space of functions $f : G \to \mathbb{C}$ with $(\sum_{g \in G} |f(g)|^2)^{\frac{1}{2}} < \infty$, is a Hilbert space and the space of compact operator on $\ell^2(G)$, $\mathscr{K}(\ell^2(G))$, is a nuclear C^*-algebra.

For every $g, h \in G$,

$$e_{g,h} : \ell^2(G) \to \ell^2(G) \ , \quad \xi \mapsto \langle \xi, e_h \rangle e_g$$

is a rank-one operator with $e_{g,h}(e_k) = \delta_{h,k} e_g$ for all $k \in G$, where $(e_g)_{g \in G}$ is the canonical basis of $\ell^2(G)$. We have

$$\mathscr{K}(\ell^2(G)) = [e_{g,h} : g, h \in G] \ ,$$

where $[\cdot]$ stands for the closed linear span.

Smash product

Let $\mathscr{B} = \{B_g\}_{g\in G}$ be a Fell bundle. Consider the set

$$\mathscr{B}_0^\sharp G := \sum_{g,h\in G} B_{g^{-1}h} \otimes e_{g,h} \qquad \subseteq \quad C^*(\mathscr{B}) \otimes \mathscr{K}(\ell^2(G)) \, .$$

- $\mathscr{B}_0^\sharp G$ is a *-subalgebra.

$$\begin{aligned} \to \quad (B_{g^{-1}h} \otimes e_{g,h})(B_{k^{-1}l} \otimes e_{k,l}) &= (B_{g^{-1}h} B_{k^{-1}l} \otimes e_{g,h} \circ e_{k,l}) \\ &\subseteq \delta_{h,k}(B_{g^{-1}h} B_{h^{-1}l} \otimes e_{g,l}) \\ &\subseteq (B_{g^{-1}l} \otimes e_{g,l}) \end{aligned}$$

Smash product

Definition

The *smash product* $\mathscr{B}\sharp G$ of the Fell bundle $\mathscr{B}$ by G is the closure of $\mathscr{B}_0^\sharp G$.

- The choice of $C^*(\mathscr{B})$ is arbitrary: The smash product is (up to isomorphism) independent of a C^*-algebra B with grading $\bigoplus_{g\in G} B_g \cong \mathscr{B}$!

  We have

$$\mathscr{B}\sharp G = \lim_{F\uparrow G} \sum_{g,h\in F} B_{g^{-1}h} \otimes e_{g,h} \qquad \text{with } F \subseteq G \text{ finite}$$

  and

$$\sum_{g,h\in F} B_{g^{-1}h} \otimes e_{g,h} \subseteq B \otimes \mathscr{K}(\ell^2(G))$$

  is a closed $*$-subalgebra for all G-graded C^*-algebras B with grading isomorphic to $\mathscr{B}$.

Subspaces and ideals in $\mathscr{B}\sharp G$

Let $J \subseteq \mathscr{B}\sharp G$ be a closed subspace.

QUESTION:
Let $w \in \mathscr{B}\sharp G$. Can we characterize $w \in J$?

Subspaces and ideals in $\mathscr{B}\sharp G$

Lemma

Let $g, h \in G$ be given. For every $w \in \mathscr{B}\sharp G$, there exists a unique $w_{g,h} \in B_{g^{-1}h}$ such that

$$(1 \otimes e_{g,g}) w (1 \otimes e_{h,h}) = w_{g,h} \otimes e_{g,h} \ .$$

Proof.

Let

$$w = \sum_{k,l \in G} b_{k^{-1}l} \otimes e_{k,l} \in \mathscr{B}_0^\sharp G$$

with $b_{k^{-1}l} \in B_{k^{-1}l}$ for all $k, l \in G$. Then we have

$$\begin{aligned}(1 \otimes e_{g,g}) w (1 \otimes e_{h,h}) &= \sum_{k,l \in G} (1 \otimes e_{g,g})(b_{k^{-1}l} \otimes e_{k,l})(1 \otimes e_{h,h}) \\ &= \sum_{k,l \in G} \delta_{g,k}\delta_{h,l}(b_{k^{-1}l} \otimes e_{k,l}) = b_{g^{-1}h} \otimes e_{g,h}\end{aligned}$$

Set $w_{g,h} := b_{g^{-1}h}$. Since $\mathscr{B}_0^\sharp G$ is dense in $\mathscr{B}\sharp G$, the uniqueness of $w_{g,h}$ for an arbitrary $w \in \mathscr{B}\sharp G$ follows. □

Subspaces and ideals in $\mathscr{B}\sharp G$

Proposition

Let $J \subseteq \mathscr{B}\sharp G$ be a closed subspace and $w \in \mathscr{B}\sharp G$.

(i) If $w_{g,h} \otimes e_{g,h} \in J$ for all $g, h \in G$, then $w \in J$.

(ii) If $w \in J$ and $(1 \otimes \mathscr{K})J(1 \otimes \mathscr{K}) \subseteq J$, then $w_{g,h} \otimes e_{g,h} \in J$ for all $g, h \in G$.

Proof sketch.

For every finite subset $F \subseteq G$ set $P_F = \sum_{g \in F} 1 \otimes e_{g,g}$. Then

$$w = \lim_{F \uparrow G} P_F w P_F$$

for $w \in \mathscr{B}\sharp G$. From the previous proposition, we have

$$P_F w P_F = \sum_{g,h \in F} w_{g,h} \otimes e_{g,h} .$$

$\Rightarrow$ (i) follows, since J is closed. (ii) follows directly. □

The restricted smash product $\mathscr{B}\flat G$

Definition

The *restricted smash product* $\mathscr{B}\flat G$ of the Fell bundle $\mathscr{B}$ by G is defined as

$$\mathscr{B}\flat G := \overline{\sum_{g,h\in G} [B_{g^{-1}}B_h] \otimes e_{g,h}} \qquad \subseteq C^*(\mathscr{B}) \otimes \mathscr{K}(\ell^2(G)) \ .$$

REMARKS:

(i) Since $[B_{g^{-1}}B_h] \subseteq B_{g^{-1}h}$, it holds that $\mathscr{B}\flat G \subseteq \mathscr{B}\sharp G$.

(ii) If $\mathscr{B}$ is saturated, i.e., $[B_g B_h] = B_{gh}$ for all $g, h \in G$, we also have $\mathscr{B}\flat G = \mathscr{B}\sharp G$.

(iii) $\mathscr{B}\flat G$ is a closed two-sided ideal of $\mathscr{B}\sharp G$.

(iv) For every $w \in \mathscr{B}\flat G$ it holds: $w_{g,h} \in [B_{g^{-1}}B_h]$ for all $g, h \in G \Leftrightarrow w \in \mathscr{B}\flat G$.

II.

Dual partial actions

Dual global and partial actions

Let G be a discrete group. The left regular representation

$$\lambda_g^G : \ell^2(G) \to \ell^2(G)\ e_h \to e_{gh}$$

(for all $g \in G$) has the properties

$$\lambda_g^G \circ e_{h,k} = \lambda_g^G \langle \cdot, e_k \rangle e_h = \langle \cdot, e_k \rangle e_{gh} = e_{gh,k}$$

and

$$e_{h,k} \circ \lambda_g^G = \langle \lambda_g^G(\cdot), e_k \rangle e_h = \langle \circ, e_{g^{-1}k} \rangle = e_{h,g^{-1}k}$$

for all $g \in G$, $e_{h,k} \in \ell^2$ and hence,

$$\lambda_g^G \circ e_{h,k} \circ \lambda_{g^{-1}}^G = e_{gh,gk}\ .$$

Therefore,

$$(1 \otimes \lambda_g^G)(B_{h^{-1}k} \otimes e_{h,k})(1 \otimes \lambda_{g^{-1}}^G) = B_{(gh)^{-1}(gk)} \otimes e_{gh,gk} \in \mathscr{B}\sharp G$$

$\Longrightarrow \mathscr{B}\sharp G$ is invariant under this conjugation!

Dual global and partial actions

We also have

$$(1 \otimes \lambda_g^G)(B_{h^{-1}}B_k \otimes e_{h,k})(1 \otimes \lambda_{g^{-1}}^G) = B_{h^{-1}}B_k \otimes e_{gh,gk} \in \mathscr{B}\flat G$$

and in general

$$B_{h^{-1}}B_k \subsetneq [B_{(gh)^{-1}}B_{gk}]$$

$\Longrightarrow \mathscr{B}\flat G$ is NOT invariant under this conjugation!

Dual global and partial action

Set

$$\Gamma_g : \mathscr{B}\sharp G \to \mathscr{B}\sharp G\ , \quad b \mapsto (1 \otimes \lambda_g^G) b (1 \otimes \lambda_{g^{-1}}^G)$$

for all $g \in G$.

Definition

Let $\mathscr{B}$ be a Fell bundle.

(i) $\Gamma = \{\Gamma_g\}$ is called the *dual global action* for $\mathscr{B}$.

(ii) Δ, the restriction of Γ to $\mathscr{B}\flat G$, is called the *dual partial action* for $\mathscr{B}$.

The dual partial action

The spaces $E_g := \Gamma_g(\mathscr{B}\flat G) \cap \mathscr{B}\flat G$, which are domains and targets of Δ can be characterized in the following way:

Proposition

Set $D_g := [B_g B_{g^{-1}}]$ for every $g \in G$. Then

$$E_g = \overline{\sum_{h,k\in G} [B_{h^{-1}} D_g B_k] \otimes e_{h,k}} \ .$$

Proof.

"$\subseteq$" An element $w \in E_g$ clearly fulfills $w \in \mathscr{B}\flat G$ and this is equivalent to the fact, that $w_{h,k} \in [B_{h^{-1}} B_k]$ for all $h, k \in G$. For the inclusion "$\subseteq$" it suffices thus to show, that $w_{h,k} \in [B_{h^{-1}} D_g B_k]$ for all $h, k \in G$. By the definition of E_g, we additionally have $w \in \Gamma_g(\mathscr{B}\flat G)$ and therefore we can set $y := \Gamma_{g^{-1}}(w) \in \mathscr{B}\flat G$, which implies $w_{h,k} = y_{g^{-1}h, g^{-1}k} \in [B_{h^{-1}g} B_{g^{-1}k}]$. Since $[B_{h^{-1}g} B_{g^{-1}k}]$ is a left $[B_{h^{-1}} B_h]$- and a right $[B_{k^{-1}} B_k]$-ideal, we choose two approximate identities $\{e_\lambda\}_{\lambda\in\Lambda} \subset [B_{h^{-1}} B_h]$ and $\{e_{\lambda'}\}_{\lambda'\in\Lambda'} \subset [B_{k^{-1}} B_k]$ and we obtain

$$w_{h,k} = \lim_{\lambda,\lambda'} e_\lambda w_{h,k} e_{\lambda'} \in [B_{h^{-1}} B_h B_{h^{-1}g} B_{g^{-1}k} B_{k^{-1}} B_k] \subseteq [B_{h^{-1}} B_g B_{g^{-1}} B_k] \ ,$$

which means $w_{h,k} \in [B_{h^{-1}} D_g B_k]$ for all $h, k \in G$.

The dual partial action

Proof.

"⊇" We have to prove that $B_{h^{-1}} D_g B_k \otimes e_{h,k} \subseteq E_g$ for all $h, k \in G$. Firstly, $D_g B_k \subseteq B_k$ for all $g, k \in G$, because $D_g \subseteq B_1$, which implies $B_{h^{-1}} D_g B_k \otimes e_{h,k} \subseteq B_{h^{-1}} B_k \otimes e_{h,k} \subseteq \mathscr{B}\flat G$. Since

$$[B_{h^{-1}} D_g B_k] = [B_{h^{-1}} B_g B_{g^{-1}} B_k] \subseteq [B_{h^{-1}g} B_{g^{-1}k}] \ ,$$

we also obtain

$$B_{h^{-1}} D_g B_k \otimes e_{g^{-1}h, g^{-1}k} \in \mathscr{B}\flat G \ .$$

Therefore,

$$B_{h^{-1}} D_g B_k \otimes e_{h,k} = (1 \otimes \lambda_g^G)(B_{h^{-1}} D_g B_k \otimes e_{g^{-1}h, g^{-1}k})(1 \otimes \lambda_{g^{-1}}^G) \subseteq \Gamma_g(\mathscr{B}\flat G)$$

and hence $B_{h^{-1}} D_g B_k \otimes e_{h,k} \subseteq \mathscr{B}\flat G \cap \Gamma_g(\mathscr{B}\flat G) = E_g$ for all $h, k \in G$.

□

Globalization

Proposition

The dual global action for a Fell bundle $\mathscr{B} = \{B_g\}_{g \in G}$ is a globalization of the dual partial action for $\mathscr{B}$. Hence, the dual partial action of a Fell bundle $\mathscr{B}$ is globalizable.

Proof.

By the definition of globalization, we have to show that $\sum_{g \in G} \Gamma_g(\mathscr{B} \flat G)$ is dense in $\mathscr{B} \sharp G$. Let $g, h \in G$. Clearly, $B_{g^{-1}h} \otimes e_{1,g^{-1}h} \in \mathscr{B} \flat G$. Since $\Gamma_g(B_{g^{-1}h} \otimes e_{1,g^{-1}h}) = B_{g^{-1}h} \otimes e_{g,h}$, every element of $\mathscr{B}_0^\sharp G$ is in the orbit of Γ_g. Hence, the statement follows, because $\mathscr{B} \sharp G$ is the closure of $\mathscr{B}_0^\sharp G$. □

III.

Morita-Rieffel equivalence

Hilbert bimodule

Definition

Let A, B be C^*-algebras. A left Hilbert A-module and right Hilbert B-module M is called *Hilbert A-B-bimodule*, if

(i) $(a\xi)b = a(\xi b)$ and

(ii) ${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$ for all $\xi, \eta, \zeta \in M$, $a \in A$ and $b \in B$.

Remark:

We have $\|{}_A\langle \xi, \xi \rangle\|_A = \|\langle \xi, \xi \rangle_B\|_B$ for all $\xi \in M$, i.e. the induced norms agree.

Morita-Rieffel equivalence for C^*-algebras

Definition

Let A, B be C^*-algebras. A Hilbert A-B-bimodule is called *left* (resp. *right*) *full*, if $\langle M, M\rangle_A$ (resp. $\langle M, M\rangle_B$) is dense in A (resp. B). If M is left and right full, it is a *imprimitivity bimodule*.

Definition

Let A, B be C^*-algebras. A and B are *Morita-Rieffel equivalent*, if there exists a imprimitivity bimodule A-B-bimodule.

Morita-Rieffel equivalence for C^*-algebras

Example

Let B be a C^*-algebra and A a C^*-subalgebra of B. Set $M := [AB]$ and define

$$\langle \xi, \eta \rangle_A := \eta \xi^* \quad \text{and} \quad \langle \xi, \eta \rangle_B := \eta^* \xi \quad \text{for all } \eta, \xi \in M\ .$$

This is a well-defined Hilbert A-B-bimodule, if $ABA \subseteq A$, i.e. if A is a *hereditary subalgebra*.

- M is left full, since $A \subseteq M$.
- If A is a *full subalgebra*, i.e., if $[BAB] = B$, M is also right full.

Hence, M is an imprimitivity module and therefore, A and B are Morita-Rieffel equivalent.

Morita-Rieffel equivalence for C^*-algebras

Theorem

Let A, B to separable C^*-algebras. A and B are Morita-Rieffel equivalent if and only if they are *stably isomorphic*, i.e.,

$$\mathscr{K} \otimes A \simeq \mathscr{K} \otimes B$$

with the algebra $\mathscr{K}$ of compact operators on a a separable, infinite-dimensional Hilbert space.

Definition

Let

$$\theta^k = (A^k, G, \{A_g^k\}_{g \in G}, \{\theta^k\}_{g \in G})$$

be C^*-algebraic partial dynamical systems with $k = 1, 2$. θ^1 and θ^2 are *Morita-Rieffel equivalent*, if there exists a Hilbert A^1-A^2-bimodule M and a (set-theoretical) partial action $\gamma = (\{M_g\}_{g \in G}, \{\gamma_g\}_{g \in G})$ such that

(i) M_g is a norm-closed, sub-A^1-A^2-bimodule of M for all $g \in G$

(ii) $A_g^k = [\langle M_g, M_g \rangle_{A^k}]$ for $k = 1, 2$ and all $g \in G$

(iii) $\gamma_g : M_{g^{-1}} \to M_g$ is a $\mathbb{C}$-linear map for all $g \in G$

(iv) $\langle \gamma_g(\xi), \gamma_g(\eta) \rangle_{A^k} = \theta_g^k (\langle \xi, \eta \rangle_{A^k})$ for $k = 1, 2$, all $\xi, \eta \in M_{g^{-1}}$ and all $g \in G$.

The partial dynamical system

$$\gamma = (M, G, \{M_g\}_{g \in G}, \{\gamma_g\}_{g \in G})$$

is then called an *imprimitivity system* for θ^1 and θ^2.

MR equivalence for C^*-algebraic partial dynamical systems

REMARKS:

- (iv) is well-defined, since $\langle \xi, \eta \rangle_{A^k} \in A^k_{g^{-1}}$.
- By (ii), M_g is left and right full as a Hilbert A^1_g-A^2_g-bimodule, and hence A^1_g and A^2_g are Morita-Rieffel equivalent for all $g \in G$.
- In particular ($g = 1$), A^1 and A^2 are Morita-Rieffel equivalent.
- In general, γ_g are no Hilbert-(bi)module homomorphisms! We have

$$\gamma_g(a\xi) = \theta^1_g(a)\gamma_g(\xi) \qquad \text{and} \qquad \gamma_g(\xi b) = \gamma_g(\xi)\theta^2_g(b)$$

  for all $a \in A^1_{g^-}, b \in A^2_{g^{-1}}, \xi \in M_{g^{-1}}$ and $g \in G$.

Morita-Rieffel equivalence

Theorem

If two C^*-algebraic partial dynamical systems $\theta^k = (A^k, G, \{A_g^k\}_{g\in G}, \{\theta^k\}_{g\in G})$ with $k = 1, 2$ are Morita-Rieffel equivalent, then $A^1 \rtimes G$ and $A^2 \rtimes G$ are Morita-Rieffel equivalent as C^*-algebras.

Theorem

Every C^*-algebraic partial action $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is Morita-Rieffel equivalent to the dual partial action Δ on $\mathscr{B}\flat G$, where $\mathscr{B}$ is the semi-direct product bundle.

- Every C^*-algebraic partial action is Morita-Rieffel equivalent to a partial action which admits a globalization.

$\longrightarrow$ The proofs of these theorems can be found in notes of the subsequent talk!

Adjoint Hilbert bimodule

Definition

Let A, B be two C^*-algebras and M a Hilbert A-B-bimodule. The *adjoint Hilbert bimodule* to M is a set M^*, such that there is a bijection $\xi \in M \mapsto \xi^* \in M^*$, with a vector space, B-left module and A-right module structure, defined by

$$\xi^* + \lambda\eta^* := (\xi + \overline{\lambda}\eta)^* \qquad b\xi^* := (\xi b^*)^* \qquad \xi^* a := (a^*\xi)^*$$

for all $a \in A$, $b \in B$, $\xi, \eta \in M$ and $\lambda \in \mathbb{C}$ and with an A-valued and B-valued inner product

$$\langle \xi^*, \eta^* \rangle_A := \langle \xi, \eta \rangle_A \qquad \langle \xi^*, \eta^* \rangle_B := \langle \xi, \eta \rangle_B$$

for all $\xi, \eta \in M$. Then M^* is a Hilbert B-A-bimodule.

Construction of the linking algebra

Let A, B be two C^*-algebras, M a Hilbert A-B-bimodule and M^* its adjoint. The complex vector space $A \times M \times M^* \times B$ written as

$$L = \begin{pmatrix} A & M \\ M^* & B \end{pmatrix}$$

is a C^*-algebra with the multiplication

$$\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} := \begin{pmatrix} a_1 a_2 + \langle \xi_1, \eta_2 \rangle_A & a_1 \xi_2 + \xi_1 b_2 \\ \eta_1^* a_2 + b_1 \eta_2^* & \langle \eta_1, \xi_2 \rangle_B + b_1 b_2 \end{pmatrix},$$

and the involution

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* := \begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix}$$

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $\xi, \xi_1, \xi_2, \eta, \eta_1, \eta_2 \in M$.

Construction of the linking algebra

Taking the columns of the multiplication in L, there are representations

$$\pi_B : L \to \mathscr{L}(M \oplus B) \ , \quad \begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \mapsto \left(\begin{pmatrix} \xi_2 \\ b_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1\xi_2 + \xi_1 b_2 \\ \langle \eta_1, \xi_2 \rangle_B + b_1 b_2 \end{pmatrix} \right)$$

and

$$\pi_A : L \to \mathscr{L}(A \oplus M^*) \ , \quad \begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \mapsto \left(\begin{pmatrix} a_2 \\ \eta_2^* \end{pmatrix} \mapsto \begin{pmatrix} a_1 a_2 + \langle \xi_1, \eta_2 \rangle_A \\ \eta_1^* a_2 + b_1 \eta_2^* \end{pmatrix} \right)$$

of L, where $M \oplus B$ is a right Hilbert B-module, $A \oplus M^*$ a left Hilbert A-module and $\mathscr{L}(M \oplus B)$ and $\mathscr{L}(A \oplus M^*)$ are the spaces of adjointable operators. Then

$$\| \cdot \| : L \to \mathbb{R}_{\geq 0} \ , \quad c \mapsto \max\{\pi_A(c), \pi_B(c)\}$$

defines a norm on L, whereby it becomes a C^*-algebra, the so called *linking algebra*.

Morita-Rieffel equivalence and the linking algebra

Proposition

Let $\alpha = (A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $\beta = (B, G, \{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ be two C^*-algebraic partial dynamical systems that are Morita-Rieffel equivalence with the imprimitivity system $\gamma = (M, G, \{M_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$ and the linking algebra L of M.

(i) $L_g := \begin{pmatrix} A_g & M_g \\ M_g^* & B_g \end{pmatrix}$ is a closed two-sided ideal for every $g \in G$

(ii) $\lambda_g : L_{g^{-1}} \to L_g$, $\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} \mapsto \begin{pmatrix} \alpha_g(a) & \gamma_g(\xi) \\ \gamma_g(\eta)^* & \beta_g(b) \end{pmatrix}$ is a *-isomorphism for every $g \in G$

(iii) $\lambda = (\{L_g\}_{g\in G}, \{\lambda_g\}_{g\in G})$ is a C^*-algebraic partial action of G on L.

Proof Sketch.

(i) $LL_gL \subseteq L_g$ for all $g \in G$:
By the definition of the multiplication, if suffices to show that

(a) $a_1 a \in A_g$ for all $a_1 \in A_g$ and $a \in A$

(b) $\langle \xi_1, \eta \rangle_A \in A_g$ for all $\xi_1 \in M_g$ and $\eta \in M$

(c) $a_1 \xi \in M_g$ for all $a_1 \in A_g$ and $\xi \in M$.

Morita-Rieffel equivalence and the linking algebra

Proof Sketch.

Since by the same reasoning (as in the following proofs) and by taking adjoints, it follows from

(a) that $aa_1 \in A_g, b_1 b \in B_g$ and $bb_1 \in B_g$ for all $a_1 \in A_g, a \in A, b_1 \in B_g$ and $b \in B$;
(b) that $\langle \xi, \eta_1 \rangle_A \in A_g, \langle \xi_1, \eta \rangle_B \in B_g$ and $\langle \xi, \eta_1 \rangle_B \in B_g$ for all $\xi_1, \eta_1 \in M_g$ and $\xi, \eta \in M$
(c) that $a\xi_1, \xi_1 b, \xi b_1 \in M_g, \eta_1^* a, \eta^* a_1, b_1 \eta^*, b\eta_1^* \in M_g^*$ for all $a_1 \in A_g, a \in A, b_1 \in B_g$, $b \in B, \xi_1 \in M_g, \xi \in M, \eta_1^* \in M_g^*, \eta^* \in M^*$.

Taking summads of these elements, we get $LL_g \subseteq L_g$ as well as $LL_g \subseteq L_g$, which imply that L_g is a two-sided ideal.

Proof of (a): This is immediately clear, since $A_g \subset A$ is a two-sided ideal for all $g \in G$.

Proof of (b): M_g is a left Hilbert A_g-module. Let $\xi, \eta \in M_g$. There exists an approximate unit $\{e_\lambda\}_{\lambda \in \Lambda} \subset A_g$ such that $\xi = \lim_{\lambda \in \Lambda} e_\lambda \xi$. Then we obtain
$\langle \xi, \eta \rangle_A = \langle \lim_{\lambda \in \Lambda} e_\lambda \xi, \eta \rangle_A = \lim_{\lambda \in \Lambda} e_\lambda \langle \xi, \eta \rangle_A \in A_g$.

Proof of (c): Since $A_g = [\langle M_g, M_g \rangle_A]$, there are $\chi, \zeta \in M_g$ such that $a_1 = \langle \chi, \zeta \rangle_A$. Therefore, we obtain $a_1 \xi = \langle \chi, \zeta \rangle_A \xi = \chi \langle \zeta, \xi \rangle_B \in M_g$, since M is a Hilbert A-B-bimodule.

Since the norm topology of L and the product topology for $L = A \times M \times M^* \times B$ set up the same topology on L and because $A_g \subset A, B_g \subset B, M_g \subset M$ and $M_g^* \subset M$ are all closed, $L_g \subset L$ is also a closed ideal for all $g \in G$.

Morita-Rieffel equivalence and the linking algebra

Proof Sketch.

(ii) Using the definition of the involution for L and the properties, that α_g, β_g are *-isomorphisms and that γ_g is bijective, we have

$$\lambda_g\left(\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^*\right) = \lambda_g\left(\begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix}\right) = \begin{pmatrix} \alpha_g(a^*) & \gamma_g(\eta) \\ \gamma_g(\xi)^* & \beta_g(b^*) \end{pmatrix} = \begin{pmatrix} \alpha_g(a)^* & \gamma_g(\eta) \\ \gamma_g(\xi)^* & \beta_g(b)^* \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_g(a) & \gamma_g(\xi) \\ \gamma_g(\eta)^* & \beta_g(b) \end{pmatrix}^* = \lambda_g\left(\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}\right)^*$$

for $a \in A_{g^{-1}}, b \in B_{g^{-1}}$ and $\xi, \eta \in M_{g^{-1}}$. Hence, it is a *-isomorphism.

(iii) A direct sum of C^*-partial actions is again a C^*-algebraic partial action and λ_g is the direct sum of four partial actions.

□

Thank you for your attention!