Fell's absorption principle and graded C^*-algebras

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Disclaimer on notation

Convention

We make no notational distinction between a representation

$$\pi : \mathcal{B} \to \mathcal{L}(H)$$

of a Fell bundle $\mathcal{B}$ and its integrated form

$$\pi : C^*(\mathcal{B}) \to \mathcal{L}(H).$$

Convention

Denote by

$$\lambda^G : C^*(G) \to \mathcal{L}(\ell^2 G), \quad (\lambda^G(u_g)f)(h) := f(g^{-1}h)$$

the left regular representation. We have $C_r^*(G) = \lambda^G(C^*(G))$.

Tensor products of C^*-algebras

Let A and B be C^*-algebras and denote by $A \odot B$ their algebraic tensor product (as $\mathbb{C}$-vector spaces). We turn $A \odot B$ into a $*$-algebra via

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1 a_2) \otimes (b_1 b_2), \quad (a_1 \otimes b_1)^* := a_1^* \otimes a_2^*$$

We want to complete $A \odot B$ in order to get a C^*-algebra.

Theorem (Takesaki?)

There is a minimal and a maximal C^*-norm on $A \odot B$.

Definition

The completions w.r.t. these norms are the *minimal tensor product* $A \otimes B$ and the *maximal tensor product* $A \otimes_{\max} B$.

The minimal tensor product

Proposition

Let $A \subseteq \mathcal{L}(H)$ and $B \subseteq \mathcal{L}(K)$ be C^*-algebras. Let $H \otimes K$ be the completion of $H \odot K$ w.r.t. the inner product

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle.$$

We get a representation

$$A \odot B \subseteq \mathcal{L}(H \otimes K), \quad (a \otimes b)(\xi \otimes \eta) = (a\xi) \otimes (b\eta).$$

Then

$$A \otimes B \cong \overline{A \odot B}^{\|\cdot\|} \subseteq \mathcal{L}(H \otimes K).$$

The maximal tensor product

Proposition

Let A and B be C^*-algebras. Then $A \otimes_{\max} B$ satisfyies the following universal property:

For all $*$-homomorphisms $\pi : A \to C$ and $\rho : B \to C$ satisfying

$$\pi(a)\rho(b) = \rho(b)\pi(a), \quad \forall a \in A, b \in B,$$

there is a (unique) $*$-homomorphisms

$$\pi \times \rho : A \otimes_{\max} B \to C$$

satisfying

$$\pi \times \rho(a \otimes b) = \pi(a)\rho(b)$$

Fell's absorption principle

Fell's absorption principle for unitary representations

Theorem (Fell's absorption principle)

Let G be a discrete group and $\pi : C^*(G) \to \mathcal{L}(H)$ a (non-degenerate) representation. Then there is a $*$-homomorphism

$$\psi : C_r^*(G) \to \mathcal{L}(H \otimes \ell^2 G)$$

such that the following diagram commutes.

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\pi \otimes \lambda^G} & \mathcal{L}(H \otimes \ell^2 G) \\ \downarrow \lambda^G & \overset{\psi}{\nearrow} & \\ C_r^*(G) & & \end{array}$$

Moreover, ψ is faithful.

Slogan: λ^G *absorbs* π.

Fell's absorption principle for Fell bundles

Theorem (Fell's absorption principle)

Let $\mathcal{B}$ be a Fell bundle and let $\pi : C^*(\mathcal{B}) \to \mathcal{L}(H)$ be a representation. Then the representation

$$\pi \otimes \lambda^G : C^*(\mathcal{B}) \to \mathcal{L}(H \otimes \ell^2 G), \quad \mathcal{B}_g \ni b \mapsto \pi(b) \otimes \lambda^G(g)$$

factors through $C_r^*(\mathcal{B})$, i.e. we have a commutative diagram

$$\begin{array}{ccc} C^*(\mathcal{B}) & \xrightarrow{\pi\otimes\lambda^G} & \mathcal{L}(H \otimes \ell^2 G) \\ \big\downarrow \Lambda & \overset{\psi}{\nearrow} & \\ C_r^*(\mathcal{B}) & & \end{array}$$

If π_1 is faithful, then so is ψ.

Proof #1: Existence of ψ

We have to show

$$\ker \Lambda \subseteq \ker \pi \otimes \lambda^G.$$

Consider the canonical faithful trace (= 1st Fourier coefficient)

$$\tau = E_1 : C_r^*(G) \to \mathbb{C}, \quad \lambda^G(g) \mapsto \begin{cases} 1, & g = 1 \\ 0, & g \neq 1 \end{cases}$$

Check that the following diagram commutes!

$$\begin{array}{ccccc} C^*(\mathcal{B}) & \xrightarrow{\pi \otimes \lambda^G} & \mathcal{L}(H) \otimes C_r^*(G) & \xrightarrow{\mathrm{id} \otimes \tau} & \mathcal{L}(H) \\ \downarrow \Lambda & & & \nearrow \pi_1 & \\ C_r^*(\mathcal{B}) & \xrightarrow{\quad E_1 \quad} & B_1 & & \end{array}$$

For $x^*x \in \ker \Lambda$, we get

$$\Lambda(x^*x) = 0 \quad \Rightarrow \quad (\mathrm{id} \otimes \tau) \circ (\pi \otimes \lambda^G)(x^*x) = 0$$

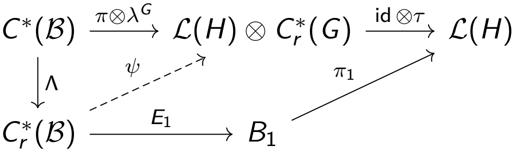
$$\overset{\tau \text{ faithful}}{\Longrightarrow} \quad (\pi \otimes \lambda^G)(x^*x) = 0$$

Proof #2: Faithfulness of ψ

Now suppose that $\pi_1 : B_1 \to \mathcal{L}(H)$ is faithful. We have to check that

$$\psi : C_r^*(\mathcal{B}) \to \mathcal{L}(H \otimes \ell^2 G)$$

is faithful. Again, consider the diagram



For $x^*x \in \ker(\psi)$, we get

$$\pi_1 \circ E_1(x^*x) = 0 \quad \overset{\pi_1 \text{ faithful}}{\Longrightarrow} \quad E_1(x^*x) = 0 \quad \overset{E_1 \text{ faithful}}{\Longrightarrow} \quad x^*x = 0.$$

Coaction maps

Corollary

We have canonical inclusions (into spatial tensor products)

$$\mathrm{id}_{C^*(\mathcal{B})} \otimes \lambda^G : C^*_r(\mathcal{B}) \hookrightarrow C^*(\mathcal{B}) \otimes C^*_r(G)$$

$$\Lambda \otimes \lambda^G : C^*_r(\mathcal{B}) \hookrightarrow C^*_r(\mathcal{B}) \otimes C^*_r(G).$$

both given by

$$B_g \ni b \mapsto b \otimes \lambda^G(g).$$

A coaction map for $\otimes_{\max}$

Theorem

Let $\mathcal{B}$ be a Fell bundle. Then the representation

$$\mathcal{S} : C^*(\mathcal{B}) \to C^*_r(\mathcal{B}) \otimes_{\max} C^*_r(G), \quad B_g \ni b \mapsto b \otimes \lambda^G(g)$$

is faithful.

Caveat

The analogous map

$$C^*(\mathcal{B}) \to C^*_r(\mathcal{B}) \otimes_{\min} C^*_r(G)$$

is not injective since it factors through $C^*_r(\mathcal{B})$!

Proof of the Theorem

Let $\pi : C^*(\mathcal{B}) \hookrightarrow \mathcal{L}(H)$ be faithful and write

$$\psi := \pi \otimes \lambda^G : C_r^*(\mathcal{B}) \hookrightarrow \mathcal{L}(H \otimes \ell^2 G).$$

Denote the right regular representation by

$$\rho^G : C_r^*(G) \to \mathcal{L}(\ell^2 G), \quad \rho_g^G(\xi)(h) := \xi(hg).$$

Now check that ψ and $1 \otimes \rho^G$ commute. The composition

$$C^*(\mathcal{B}) \xrightarrow{\mathcal{S}} C_r^*(\mathcal{B}) \otimes_{\max} C_r^*(G) \xrightarrow{\psi \times (1 \otimes \rho^G)} \mathcal{L}(H \otimes \ell^2 G) \xrightarrow{(1 \otimes e_{11}) - (1 \otimes e_{11})} \mathcal{L}(H)$$

is equal to π and thus faithful.

Graded C^*-algebras

Graded C^*-algebras

Definition

A (G-)*graded* C^*-algebra is a C^*-algebra B together with a choice of linearly independent closed linear subspaces $B_g \subseteq B, g \in G$ satisfying

- $B_g^* = B_{g^{-1}}$
- $B_g B_h \subseteq B_{gh}$
- $\overline{\sum_{g \in G} B_g} = B$

Example

If B is a graded C^*-algebra, then $\mathcal{B} := \{B_g\}_{g \in G}$ is a Fell bundle.

Question

When do we have $C^*(\mathcal{B}) = B$ or $C_r^*(\mathcal{B}) = B$?

Reconstructing a graded C^*-algebra from its Fell bundle

Remark

Let B be a graded C^*-algebra with associated Fell-bundle $\mathcal{B} := \{B_g\}_{g\in G}$. Then there is a surjective $*$-homomorphism

$$\varphi : C^*(\mathcal{B}) \to B, \quad B_g \ni b \mapsto b.$$

Question

When do we have a $*$-homomorphism

$$B \to C^*_r(\mathcal{B}), \quad B_g \ni b \mapsto \Lambda(b) \quad ?$$

Definition

A graded C^*-algebra B is called *topologically graded*, if there is a bounded linear map $F : B \to B_1$ such that $F|_{B_1} = \mathrm{id}$ and $F|_{B_g} = 0, \quad g \neq 1.$

Topologically graded C^*-algebras lie between $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$

Theorem

Let B be a graded C^*-algebra. The following conditions are equivalent:

1. *B is topologically graded (i.e. $\exists$ bounded linear $F : B \to B_1$ with $F|_{B_1} = \mathrm{id}$ and $F|_{B_g} = 0, g \neq 1$)*
2. *There exists a surjective $*$-homomorphism*

$$\psi : B \to C^*_r(\mathcal{B}), \quad B_g \ni b \mapsto \Lambda(b)$$

In this case, $F : B \to B_1$ is a conditional expectation.

Proof of the theorem

$'' \Leftarrow ''$: Suppose we have $\psi : B \to C_r^*(\mathcal{B})$ as above. Then $F := E_1 \circ \psi$ does the trick.

$'' \Rightarrow ''$: Suppose we have a map $F : B \to B_1$ as above. Define

$$\langle b, c \rangle_{B_1} := F(b^* c), \quad b, c \in B.$$

Let $X := \overline{B}^{\langle , \rangle_{B_1}}$ be the separated completion of B. We have a representation $L : B \to \mathcal{L}_{B_1}(X)$ by left multiplication. The map

$$U : X \to \ell^2(\mathcal{B}), \quad B_g \ni b \mapsto b$$

defines an isometry of Hilbert-B_1-modules such that

$$UL(b) = \Lambda(b)U, \quad \forall b \in B_g.$$

Define $\psi := \mathrm{Ad}(U) \circ L : B \to C_r^*(\mathcal{B})$.

Conclusion

Theorem

Let B be a topologically graded C^*-algebra with associated Fell bundle $\mathcal{B} = \{B_g\}_{g \in G}$. We have a commutative diagram

$$\begin{array}{ccc} C^*(\mathcal{B}) & \xrightarrow{\quad \Lambda \quad} & C^*_r(\mathcal{B}) \\ & {\scriptstyle \varphi} \searrow \quad \nearrow {\scriptstyle \psi} & \\ & B & \end{array}$$

Faithfulness of the conditional expectation

Corollary

Let B be a topologically graded C^*-algebra with conditional expectation $F : B \to B_1$ and $\psi : B \to C_r^*(\mathcal{B})$ as before. Then

$$\ker(\psi) = \{x \in B : F(x^*x) = 0\}.$$

Moreover, F is faithful if and only if ψ is an isomorphism.

Proof.

This follows from faithfulness of $E_1 : C_r^*(\mathcal{B}) \to B_1$ and $F = E_1 \circ \psi$. □

A non-topologically graded C^*-algebra

Example

Let $X \subsetneq \mathbb{T}$ be an infinite, closed proper subset. Then $C(X)$ is a graded C^*-algebra with grading subspaces

$$C(X)_n := \{az^n, a \in \mathbb{C}\}.$$

However, $C(X)$ is not topologically graded.

Proof.

Suppose $F : C(X) \to \mathbb{C}$ was continuous and bounded with $F\left(\sum_n a_n z^n\right) = a_0$. By using density of polynomials, we get

$$F(f|_X) = \int_{\mathbb{T}} f(z)dz, \quad \forall f \in C(\mathbb{T}).$$

Pick $0 \neq f \in C(\mathbb{T})_{\geq 0}$ with $f|_X = 0$. We get $F(f|_X) \neq 0$, contradiction! □

Thank you for your attention!