Amenability for Fell bundles

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Amenability & the approximation property

Reconstructing a graded C^*-algebra from its Fell bundle

Recall

Let B be a topologically graded C^*-algebra and $\mathcal{B} = \{B_g\}_{g \in G}$ the associated Fell bundle. There are surjective $*$-homomorphisms

$$\begin{array}{ccccc} C^*(\mathcal{B}) & & \xrightarrow{\quad\Lambda\quad} & & C^*_r(\mathcal{B}) \\ & \searrow^{\varphi} & & \nearrow^{\psi} & \\ & & B & & \end{array} .$$

Question

When do we have $B = C^*(\mathcal{B}) = C^*_r(\mathcal{B})$?

In other words: When can we reconstruct B from its Fell bundle?

Definition

A Fell bundle $\mathcal{B}$ is called *amenable* if we have $C^*(\mathcal{B}) = C^*_r(\mathcal{B})$.

Fell bundles over amenable groups

Example

Let G be amenable. Then every Fell bundle $\mathcal{B}$ over G is amenable.

Proof.

We need to find an inverse to $\Lambda : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$. By Fell's absorption principle, there is a $*$-homomorphism

$$C^*_r(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*_r(G), \quad B_g \ni b \mapsto b \otimes \lambda^G(g).$$

Since G is amenable, the trivial representation $g \mapsto 1$ integrates to a representation $1_G : C^*_r(G) \to \mathbb{C}$. Now

$$C^*_r(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*_r(G) \xrightarrow{\mathrm{id} \otimes 1_G} C^*(\mathcal{B})$$

is an inverse to Λ. □

Constructing wrong-way maps

Here's a general recipe to construct "wrong way maps" $C^*_r(\mathcal{B}) \to C^*(\mathcal{B})$ from a finitely supported function $a : G \to B_1$:

Let $C^*(\mathcal{B}) \subseteq \mathcal{L}(H)$. Fell's absorption principle gives a map

$$\psi : C^*_r(\mathcal{B}) \to \mathcal{L}(H \otimes \ell^2 G), \quad B_g \ni b \mapsto b \otimes \lambda^G(g).$$

Define an operator

$$A : H \to H \otimes \ell^2 G, \quad A\xi := \sum_{g \in G} a(g)\xi \otimes \delta_g.$$

We get a completely positive map

$$V_a : C^*_r(\mathcal{B}) \xrightarrow{\psi} \mathcal{L}(H \otimes \ell^2 G) \xrightarrow{\mathrm{Ad}(A)} \mathcal{L}(H).$$

The approximation property

Lemma (Simple calculation)

Let $a : G \to B_1$ *be finitely supported and* $V_a : C_r^*(\mathcal{B}) \to \mathcal{L}(H)$ *as above. Then we have*

- $\|V_a\| \leq \|A^*A\| = \|\sum_{g\in G} a(g)^*a(g)\|$
- $V_a(b) = \sum_{h\in G} a(gh)^*ba(h) \quad \in C^*(\mathcal{B}), \quad \forall b \in B_g$

Definition (Exel)

A Fell bundle $\mathcal{B}$ has the *approximation property (AP)*, if there is a net $a_i \in C_c(G, B_1)$ (called *Cesaro net*) satisfying

- $\sup_i \|\sum_{g\in G} a_i(g)^*a_i(g)\| < \infty$
- $\sum_{h\in G} a_i(gh)^*ba_i(h) \xrightarrow[i\to\infty]{} b \quad \forall g \in G, b \in B_g.$

(AP) implies amenability

Theorem

Suppose $\mathcal{B}$ has the approximation property. Then $\mathcal{B}$ is amenable.

Proof.

Let $(a_i)_i$ be a Cesaro net. Then the maps $V_{a_i} : C_r^*(\mathcal{B}) \to C^*(\mathcal{B})$ are uniformly bounded with

$$V_{a_i}(b) \xrightarrow[i\to\infty]{} b \quad \forall b \in C_c(\mathcal{B}).$$

So the maps

$$C^*(\mathcal{B}) \xrightarrow{\Lambda} C_r^*(\mathcal{B}) \xrightarrow{V_{a_i}} C^*(\mathcal{B})$$

converge to the identity pointwise. Thus Λ must be injective. □

Does amenability imply the (AP) ?

Example (Buss-Echterhoff-Willett)

There is an action of $G = PSL(2, \mathbb{C})$ on $\mathcal{K}(L^2G)$ with $\mathcal{K}(L^2G) \rtimes G = \mathcal{K}(L^2G) \rtimes_r G$ which does not have the approximation property.

Question

Is there a similar example with G discrete?

Theorem (Buss-Ferraro-Sehnem)

*Let $\alpha : G \curvearrowright C_0(X)$ be a partial action of an **exact** discrete group such that $C_0(X) \rtimes G = C_0(X) \rtimes_r G$. Then the Fell bundle associated to α has the approximation property.*

Different terminologies

Caveat

These papers use a different terminology.

- Buss, Echterhoff, Willett. *Amenability and weak containment for actions of locally compact groups on C^*-algebras* (2021).
- Buss, Ferraro, Sehnem. *Nuclearity for partial crossed products by exact discrete groups* (2020).
- Abadie, Buss, Ferraro. *Amenability and approximation properties for partial actions and Fell bundles* (2021).

Exel's book		Buss & others	Meaning
Amenability	$\Leftrightarrow$	Weak containment	$C^*(\mathcal{B}) = C^*_r(\mathcal{B})$
Approximation property	$\Leftrightarrow$	Amenability	$\exists$ Cesaro net $a_i : G \to B_1$

Application to Fourier coefficients

Reconstructing an element from its Fourier coefficients

Remark

We saw in Holger's talk that an element $b \in C_r^*(\mathcal{B})$ is completely determined by its Fourier coefficients $E_g(b) \in B_g, g \in G$. A stronger question is whether we have a convergent series

$$b = \sum_{g \in G} E_g(b).$$

Here convergence could be interpreted as convergence of the net

$$\left(\sum_{g \in F} E_g(b) \right)_{\substack{F \subseteq G \\ \text{finite}}}.$$

Classical Fourier analysis

Caveat

Consider a continuous function $f \in C(\mathbb{T}) \cong C_r^*(\mathbb{Z})$. The Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)z^n$ does not need to converge uniformly to f. However, the *Cesaro sums* converge:

Theorem (Fejér)

Let $f \in C(\mathbb{T})$. Then we have

$$\sigma_N(f) := \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=-n}^{n}\hat{f}(k)z^k\right) \xrightarrow[N\to\infty]{\|\cdot\|_\infty} f.$$

Remark

Consider the Cesaro net $a_N := \frac{1}{\sqrt{N}}\mathbf{1}_{[0,N-1]} \in C_c(\mathbb{Z})$. Then we have $\sigma_N(f) = V_{a_N}(f)$.

Generalized Cesaro sums

Proposition

Let $\mathcal{B}$ be a Fell bundle admitting a Cesaro net $a_i : G \to B_1$. Then the maps

$$S_i : C^*(\mathcal{B}) \to C^*(\mathcal{B}), \quad S_i(b) = \sum_{g,h \in G} a_i(gh)^* E_g(b) a_i(h)$$

have the following properties:

1. $S_i(b) \to b$ *for all* $b \in C^*(\mathcal{B})$
2. $S_i \circ E_g = 0$ *for all but finitely many* $g \in G$ *(depending on* i*)*
3. $S_i(b) = \sum_{g \in G} S_i(E_g(b))$ *for all* $b \in C^*(\mathcal{B})$

Remark

$S_i = V_{a_i} \circ \Lambda$

Proof.

1. We already saw that $S_i \to \mathrm{id}$ in the proof that amenability implies the (AP).
2. Let $g \in G$ with

$$S_i \circ E_g(b) = \sum_{h \in G} a_i(gh) E_g(b) a_i(h) \neq 0.$$

   Then $g \in \operatorname{supp} a_i (\operatorname{supp} a_i)^{-1}$.
3. We have $S_i(b) = \sum_{g \in G} S_i(E_g(b))$ for all $b \in C_c(\mathcal{B})$. By 2. both sides make sense for $b \in C^*(\mathcal{B})$. So they are equal by continuity.

□

Nuclearity

Nuclearity

Theorem (Abadie-Vicens)

Let $\mathcal{B}$ be a Fell bundle with the approximation property such that B_1 is nuclear. Then $C_r^*(\mathcal{B})$ is nuclear.

Proof (Sketch)

Let A be any C^*-algebra. One proves $C_r^*(\mathcal{B}) \otimes_{\max} A = C_r^*(\mathcal{B}) \otimes A$ in the following steps:

- Define maximal and minimal tensor product Fell bundles $\mathcal{B} \otimes_{\max} A$ and $\mathcal{B} \otimes A$.
- Construct natural isomorphisms $C^*(\mathcal{B} \otimes_{\max} A) \cong C^*(\mathcal{B}) \otimes_{\max} A$ and $C_r^*(\mathcal{B} \otimes A) \cong C_r^*(\mathcal{B}) \otimes A$.

Nuclearity

- Prove that $\mathcal{B} \otimes_{\max} A$ and $\mathcal{B} \otimes A$ also have the approximation property and that they agree when B_1 is nuclear.
- Identify the quotient map $C_r^*(\mathcal{B}) \otimes_{\max} A \to C_r^*(\mathcal{B}) \otimes A$ with the isomorphism

$$C_r^*(\mathcal{B}) \otimes_{\max} A \overset{AP}{=} C^*(\mathcal{B}) \otimes_{\max} A \cong C^*(\mathcal{B} \otimes_{\max} A)$$
$$\overset{nuc.}{=} C^*(\mathcal{B} \otimes A) \overset{AP}{=} C_r^*(\mathcal{B} \otimes A) \cong C_r^*(\mathcal{B}) \otimes A$$

$\square$

Nuclearity

Theorem (Buss-Echterhoff-Willett)

Let $\mathcal{B}$ be a Fell bundle over a ***discrete*** *group G. The following are equivalent:*

1. *$C_r^*(\mathcal{B})$ is nuclear.*
2. *B_1 is nuclear and $\mathcal{B}$ has the approximation property.*

Remark

The proof uses quite a lot of theory!

Thank you for your attention!