1 TOPOLOGICAL PARTIAL DYNAMICAL SYSTEMS

Definition 1.1. Let G be a group with unit 1 and X be a topological space. A *topological partial action* of G on X is a pair

$$\theta = \left(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$$

consisting of a family $\{D_g\}_{g\in G}$ of open subsets of X and a family $\{\theta_g\}_{g\in G}$ of homeomorphisms with

$$\theta_g : D_{g^{-1}} \to D_g,$$

such that
(i) $D_1 = X$ and $\theta_1 = id : D_1 \to D_1$,
(ii) $\theta_g \circ \theta_h \subseteq \theta_{gh}$, for all $g,h \in G$.

(Note that $\theta_g \circ \theta_h$ is defined as the map whose domain is the set for which $\theta_g(\theta_h(x))$ makes sense. Technically that domain is the set $\theta_h^{-1}(D_h \cap D_{g^{-1}})$. For elements x in this set we then define $\theta_g \circ \theta_h(x) = \theta_g(\theta_h(x))$. The "$\subseteq$" expresses that θ_{gh} has to be an extension of $\theta_g \circ \theta_h$. When one identifies the functions with their graphs, containment can be viewed equal to being an extension.)

If $D_g = X$ for every $g \in G$, we say θ is a *topological global action*.
We call $\left(X, G, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$ a *topological partial dynamical system*. Note that we will always assume groups to be discrete in this course.

θ can be viewed as a map $\theta : G \to Homeo(X), g \mapsto \theta_g : D_{g-1} \to D_g$

Examples 1.2. Now we will take a look at a few examples of topological partial actions.
(i)
Let X be a topological space. We define the set of all partial homeomorphisms of X as

$$\text{pHomeo}(X) := \{f : C \to D \mid C,D \subseteq X \text{ open subsets, } f \text{ is homeomorphism }\}.$$

Then we have:

Proposition. Let G be a group and X a topological space. A map $\theta : G \to \text{pHomeo}(X)$ is a topological partial action of G on X if an only if the following conditions are fullfilled:
(a) θ_1 is the identity map on X
(b) $\theta_{g^{-1}} = (\theta_g)^{-1}$,
(c) $\theta_g\theta_h\theta_{h^{-1}} = \theta_{gh}\theta_{h^{-1}}$
(d) $\theta_{g^{-1}}\theta_g\theta_h = \theta_{g^{-1}}\theta_{gh}$

Sketch of proof. "$\Rightarrow$" only (c): Observe that the domain of $\theta_{gh}\theta_{h^{-1}}$ can be written as $\theta_h(D_{h^{-1}} \cap D_{(gh)^{-1}})$ which equals $D_h \cap D_{g^{-1}}$ by 2.5 in Exel's book. But this is also the domain of $\theta_g\theta_h\theta_{h^{-1}}$. For any x in $D_h \cap D_{g^{-1}}$ we then have:

$$\theta_{h^{-1}}(x) \in \theta_{h^{-1}}(D_h \cap D_{g^{-1}}) = D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$$

Proposition 2.5.ii in the book says that for y in $D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$ the equation $\theta_g(\theta_h(y)) = \theta_{gh}(y)$ holds. Substituting y for $\theta_{h^{-1}}(x)$ yields (c).
"$\Leftarrow$" only "Extension property" (1.1.(ii)): Let g, h be in G and x be in the domain of $\theta_g\theta_h$. Then $\theta_h(x)$ lies in the domain of $\theta_g\theta_h\theta_{h^{-1}}$ and by (c) also in the domain of $\theta_{gh}\theta_{h^{-1}}$. This means x is in the domain of θ_{gh}. Moreover

$$\theta_g(\theta_h(x)) = \theta_g\Big(\theta_h\big(\theta_{h^{-1}}(\theta_h(x))\big)\Big) = \theta_{gh}(\theta_{h^{-1}}(\theta_h(x))) = \theta_{gh}(x).$$

This shows that $\theta_g\theta_h \subseteq \theta_{gh}$
The remaining parts of the proof are left as an exercise. While not explicitly shown here, it is worth to mention that property (b) is essential for the correct definition of domain and range of the θ_g. □

(ii) **Restriction of a topological global action on a topological space X**
Let X be a topological space, G be a group, $H \subseteq G$ a subgroup and let $\eta : G \to \mathrm{Homeo}(X)$ be a topological global action. Define

$$D_g := \begin{cases} X, & \text{if } g \in H; \\ \emptyset, & \text{if } g \notin H \end{cases} \quad \textit{and} \quad \theta_g := \begin{cases} \eta_g, & \text{if } g \in H; \\ \emptyset, & \text{if } g \notin H \end{cases}$$

to obtain a topological partial action $\theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ of G on X.

(iii) **Restriction of a topological global action on Y to a topological partial action on $X \subseteq Y$**
Let Y be a topological space, $\eta : G \to \mathrm{Homeo}(Y)$ a global action. For any open subset $X \subseteq Y$ define $D_g = X \cap \eta_g(X) \subseteq X$.
$\Rightarrow \eta_g(D_{g^{-1}}) = D_g$ for all $g \in G$
$\Rightarrow$ One can define $\eta_g|_{D_{g^{-1}}} =: \theta_g : D_{g^{-1}} \to D_g$.
Thus we get a topological partial action on X from a topological global action on Y via restriction.
Note that $\eta_g(X)$ is open in Y as η_g is a homeomorphism. So D_g is indeed an open subset of X. The restrictions θ_g are of course homeomorphisms so we indeed have a topological partial action.

Definition 1.3. Let η be a topological global action of G on a space Y, and let θ be the topological partial action obtained by restricting η to an open subset $X \subseteq Y$. If $Orb(X) = \bigcup_{g\in G} \eta_g(X)$, the *orbit* of X, coincides with Y, we say that η is a topological globalization of θ.

Definition 1.4. Let G be a group and suppose for $i = 1, 2$ we have partial actions $\theta^i = \big(\{D^i_g\}_{g\in G}, \{\theta^i_g\}_{g\in G}\big)$ on a set X^i. A map

$$\phi : X^1 \to X^2$$

is called G-equivariant if for all $g \in G$ we have:
(i) $\phi(D^1_g) \subseteq D^2_g$
(ii) $\phi(\theta^1_g(x)) = \theta^2_g(\phi(x))$ for all $x \in D^1_{g^{-1}}$

If moreover ϕ is bijective and ϕ^{-1} is also G-equivariant, we say that ϕ is an equivalence of partial actions and that θ^1 is equivalent to θ^2. Two topological partial actions are topologically equivalent if, in addition, ϕ is a homeomorphism.
Note: On global actions if ϕ is G-equivariant and bijective, then ϕ^{-1} is automatically G-equivariant. This is not necessarily true for partial actions.
For example take the group $\mathbb{Z}_2$ and let $X^1 = X^2$ be a non-empty space. Then define $D^1_{-1} = \emptyset$ and $D^2_{-1} = X$ with θ^i being the identity function.Then $\phi = id : X^1 \to X^2$ is G-equivariant but $\phi^{-1} : X^2 \to X^1$ is not because $\phi^{-1}(X^2) = id(X^2) \nsubseteq \emptyset$. Thus (i) is not fullfilled for ϕ^{-1}.

Theorem 1.5. Every topological partial action admits a topological globalization, unique up to topological equivalence.

Proof. Existence:
Let θ be a topological partial action of a group G on X. Define $\tilde{X} = {G \times X}/{\sim}$ where $\sim$ is the following equivalence relation:

$$(g,x) \sim (h,y) \Leftrightarrow x \in D_{g^{-1}h}, \text{ and } \theta_{h^{-1}g}(x) = y.$$

We can identify X with its image in $\tilde{X}$ under the injective map $\iota : X \to \tilde{X}, x \mapsto [1,x]$.
Furthermore we have a well-defined global action τ of G on $\tilde{X}$ given by

$$\tau_g\big([h,x]\big) = [gh,x] \quad \forall g,h \in G, \quad \forall x \in X.$$

[h,x] is the equivalence class of (h,x) $\in G \times X$.

Consider the product topology on $G \times X$ and equip $\tilde{X}$ with its quotient topology.
Claim: τ is a topological globalization of θ
For this we have to show that the τ_g are homeomorphisms, ι is a homeomorphism onto its image and $\iota(X)$ is open in $\tilde{X}$.
τ_g is homeomorphism: This is clear because the map $(h,x) \mapsto (gh,x)$ ist a homeomorphism respecting the equivalence relation "$\sim$". This means if $(h,x) \sim (j,y)$ then $(gh,x) \sim (gj,y)$ (easily seen by inserting into the definition of the equivalence relation).
ι is homeomorphism: Let $\pi : G \times X \to {G \times X}/{\sim}$ be the quotient map. Then $\iota(x) = \pi(1,x)$ for all $x \in X$, so ι is continuous.
$\Rightarrow$ For ι to be a homeomorphism, it suffices to show that it is an open map
Let $U \subseteq X$ be an open subset. We want to show that $\iota(U)$ is open in $\tilde{X}$.
$\iota(U)$ open in $\tilde{X}$ $\quad \Leftrightarrow \quad$ $\pi^{-1}\big(\iota(U)\big)$ open in $G \times X$ $\quad$ (by definition of the quotient map).
Moreover

$$(g,x) \in \pi^{-1}\big(\iota(U)\big) \Leftrightarrow (g,x) \sim (1,y) \text{ for some } y \in U$$

This is equivalent to $x \in D_{g^{-1}}, \theta_g(x) \in U$ or $x \in \theta_g^{-1}(D_g \cap U)$.
Finally we obtain $\pi^{-1}\big(\iota(U)\big) = \bigcup_{g \in G} \{g\} \times \theta_g^{-1}(D_g \cap U)$, which is open in $G \times X$
$\Rightarrow \iota(U)$ is open in $\tilde{X}$.

$\Rightarrow \iota$ is a homeomorphism onto its image and $\iota(X)$ is open subset of $\tilde{X}$.
In conclusion τ is a topological globalization of θ and the existence is shown.

Uniqueness:
Let η be another topological globalization of θ, acting on the space $Y \supseteq X$. We want to construct a topological equivalence $\tilde{\phi} : \tilde{X} \to Y$.
Define a map $\phi : G \times X \to Y, \quad (g,x) \mapsto \eta_g(x)$. We then have

$$\phi(g,x) = \phi(h,y) \Leftrightarrow \eta_g(x) = \eta_h(y) \Leftrightarrow x = \eta_{g^{-1}h}(y).$$
$$\Leftrightarrow x \in \eta_{g^{-1}h}(X) \cap X = D_{g^{-1}h} \text{ and } y = \eta_{h^{-1}g}(x) = \theta_{h^{-1}g}(x)$$

This is the same as having the equivalence $(g,x) \sim (h,y)$, so there exists an injective map $\tilde{\phi} : \tilde{X} \to Y$.
As Y is globalization, we have $Y = Orb(X) \Rightarrow \tilde{\phi}$ ist surjective. By definition $\tilde{\phi}$ is G-equivariant and coincides with the identity on the copies of X in $\tilde{X}$ and Y.
It only remains to show that $\tilde{\phi}$ is a homeomorphism.
Let $y \in Y$. As Y coincides with the orbit of X, there exists $g \in G$, such that $y \in \eta_g(X)$
$\Rightarrow$ for any $y' \in \eta_g(X)$ we have $x' = \eta_g^{-1}(y') \in X$
$\Rightarrow \tilde{\phi}^{-1}(y') = \tilde{\phi}^{-1}(\eta_g(x')) = \tau_g(\tilde{\phi}^{-1}(x')) = \tau_g(x') = \tau_g(\eta_g^{-1}(y'))$
As the map $\eta_g(X) \to \tau_g(X), \quad y' \mapsto \tau_g(\eta_g^{-1}(y'))$ is continuous and coincides with $\tilde{\phi}^{-1}$ on $\eta_g(X)$ which is an open neighbourhood of y, we see that $\tilde{\phi}^{-1}$ is continuous at y.
Similar: $\tilde{\phi}$ is continuous
$\Rightarrow \tilde{\phi}$ ist a homeomorphism.
$\Rightarrow \eta$ and τ are topologically equivalent. □

Warning: The quotient topology on $\tilde{X}$ may not be Hausdorff, even if X is Hausdorff!
The next proposition delivers a helpful condition for this problem. For the proof we need one further definition.

Definition 1.6. Let $\theta = \left(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$ be a partial action of G on X. The graph of θ is defined as the set

$$\text{Graph}(\theta) = \left\{(y,g,x) \in X \times G \times X : x \in D_{g^{-1}}, \theta_g(x) = y\right\}.$$

A useful characteristic of the graph is the following. For a global action η of G on Y where θ is its restriction to $X \subseteq Y$ we have

$$\text{Graph}(\theta) = \text{Graph}(\eta) \cap (X \times G \times X). \tag{1}$$

Now we will see how to tell if the quotient topology is Hausdorff:

Proposition 1.7. A topological partial action θ of a group G on a Hausdorff space X admits a Hausdorff globalization if and only if its graph is closed in $X \times G \times X$, where G is given the discrete topology.

Proof. " $\Rightarrow$ " : Let η be a globalization of θ on a Hausdorff space Y. We have

$$\text{Graph}(\eta) = \{(y,g,x) \in Y \times G \times Y : \underbrace{x \in D_{g^{-1}}}_{D_{g^{-1}}=Y,\text{ can be omitted}}, \eta_g(x) = y\}$$

$\Rightarrow$ Graph(η) is closed in $Y \times G \times Y$ and with (1) it follows that Graph(θ) is closed in $X \times G \times X$.

" $\Leftarrow$ " : Now assume that Graph(θ) is closed and let η be the topological globalization of θ acting on Y. We need to show that Y is Hausdorff.
Take $y_1, y_2 \in Y$ with $\eta_{g_1}(x_1) = y_1 \neq y_2 = \eta_{g_2}(x_2)$.

$$\begin{aligned}&\Rightarrow \eta_{g_2^{-1}g_1}(x_1) \neq x_2\\ &\Rightarrow (x_2, g_2^{-1}g_1, x_1) \notin \text{Graph}(\theta)\end{aligned}$$

We can therefore conclude that there exist open subsets $U_1, U_2 \subseteq X$ with $x_i \in U_i$ such that

$$\left(U_2 \times \{g_2^{-1}g_1\} \times U_1\right) \cap \text{Graph}(\theta) = \emptyset.$$

As $X \underset{open}{\subseteq} Y$, the U_i are open in Y and it follows that $\eta_{g_i}(U_i) \underset{open}{\subseteq} Y$.
This means that y_i equals $\eta_{g_i}(x_i)$ which lies in $\eta_{g_i}(U_i)$. To show that Y is Hausdorff it now suffices to show that $\eta_{g_1}(U_1)$ and $\eta_{g_2}(U_2)$ are disjoint.
Assumption: There is a y that lies in the intersection $\eta_{g_1}(U_1) \cap \eta_{g_2}(U_2)$.

$$\begin{aligned}&\Rightarrow y = \eta_{g_1}(z_1) = \eta_{g_2}(z_2), \quad z_i \in U_i\\ &\Rightarrow \eta_{g_2^{-1}g_1}(z_1) = z_2\end{aligned}$$

But the z_i are also elements of X so the results holds for the partial action as well: $\theta_{g_2^{-1}g_1}(z_1) = z_2$.
From this it follows that $(z_2, g_2^{-1}g_1, z_1)$ lies in $\left(U_2 \times \{g_2^{-1}g_1\} \times U_1\right) \cap$ Graph(θ). But this is an empty set as seen above, so we have a contradiction ↯
Thus we can conclude that $\eta_{g_1}(U_1)$ and $\eta_{g_2}(U_2)$ are disjoint and Y is Hausdorff. □

From this we get another result for the case that D_g is closed for all $g \in G$:

Proposition 1.8. Let X be a topological Hausdorff space. A topological partial dynamical system $\left(X, G, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$ such that each D_g is closed, always admits a Hausdorff globalization. If, in addition, X is compact, the converse also holds.

Proof. Let $\{(y_i, g_i, x_i)\}_i$ be a net in Graph(θ) with $\{(y_i, g_i, x_i)\}_i$ converging to (y, g, x). It follows that g_i becomes eventually constant because G has discrete topology.
So we can assume without loss of generelaty that $g_i = g$ for all i.
This means that $x_i \in D_{g^{-1}}$ for all i and thus x lies also in $D_{g^{-1}}$. $\Rightarrow y = \lim_i y_i = lim_i \theta_g(x_i) = \theta_g(x)$
$\Rightarrow (y, g, x) \in$ Graph(θ)
So Graph(θ) is closed and by 1.7 we know that Y is Hausdorff.

Now let X be a compact topological Hausdorff space and let θ admit a Hausdorff globalization η. Then D_g can be written as $D_g = \eta_g(X) \cap X$. As η_g is a homeomorphism, $\eta_g(X)$ is also compact and D_g is compact as an intersection of compact spaces. Thus D_g is closed as a compact subset of a Hausdorff space. □

We will close with more examples of topological partial actions:

Examples 1.9. (i) **Global an Partial Bernoulli Action**
Let G be a group and consider the compact topological space $\{0,1\}^G$. One can identify $\{0,1\}^G$ with $\mathscr{P}(G)$, the power set of G by the characteristic function $\chi_\omega : G \to \{0,1\}$ where $\omega \in \mathscr{P}(G)$ and

$$\chi_\omega(g) = \begin{cases} 1, & g \in \omega; \\ 0, & g \notin \omega \end{cases}.$$

Note that $\mathscr{P}(G)$ is Hausdorff. We write $g\omega = \{gh : h \in \omega\}$ and look at the map

$$\eta_g : \mathscr{P}(G) \to \mathscr{P}(G), \quad \omega \mapsto g\omega,$$

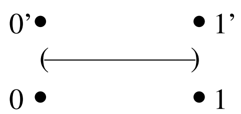
which is a homeomorphism for each $g \in G$ and η is a topological global action of G on $\{0,1\}^G$. We call η the global Bernoulli action of G.

Now consider the the set $\Omega_1 = \left\{\omega \in \{0,1\}^G : 1 \in \omega\right\}$ which is an compact open subset of $\{0,1\}^G$.
We define the partial Bernoulli action of a group G as the topological partial action θ of G on Ω_1 obtained by restricting the global Bernoulli action to Ω_1 as we did in 1.2 (iii).
By defining $D_g = \Omega_1 \cap \eta_g(\Omega_1)$ as demanded in 1.2 (iii), we see that

$$D_g = \left\{\omega \in \Omega_1 : g \in \omega\right\} = \left\{\omega \in \{0,1\}^G : 1, g \in \omega\right\}$$

As Ω_1 is compact open subset, D_g is therefore compact for every $g \in G$, thus $\mathrm{Graph}(\theta)$ is automatically closed by the propositions above.
The partial Bernoulli action becomes an important tool later on.

(ii) **Example where Graph(θ) is not closed**
Let $G = \mathbb{Z}_2$ and $X = [0,1]$. Let $D_{-1} = (0,1)$ and define $\theta_{-1}(x) = 1 - x$. Then $\mathrm{Graph}(\theta)$ is not closed and thus the globalization (constructed like $\tilde{X}$ in the proof of the theorem) is not Hausdorff. In fact the globalization looks like this:



It can be described as $[0,1]$ with two additional points $0'$ and $1'$ which can not be seperated from 0 and 1 respectively.