## 2 C*-ALGEBRAIC PARTIAL DYNAMICAL SYSTEMS

**Definition 2.1.** Let $A$ be a C*-algebra. Define the set of all partial automorphisms as

$$\mathrm{pAut}(A) = \{\phi : C \to D : \quad C, D \lhd A \text{ closed two-sided ideals}, \quad \phi \text{ *-isomorphism}\}$$

For a partial homoeomorphism $h : U \to V, h \in \mathrm{pHomeo}(X)$ where $X$ is locally compact Hausdorff space we can construct a partial automorphism

$$\phi_h : C_0(V) \to C_0(U)$$

between ideals of $C_0(X)$. Thus we obtain a semigroup isomorphism

$$\mathrm{pHomeo}(X) \to \mathrm{pAuto}(C_0(X)), \quad h \mapsto \phi_{h^{-1}}. \tag{1}$$

**Definition 2.2.** A C*-algebraic partial action of the group $G$ on the C*-algebra $A$ is a pair

$$\theta = \left(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$$

consisting of a family $\{D_g\}_{g\in G}$ of closed two-sided ideals of $A$ and a family $\{\theta_g\}_{g\in G}$ of *-isomorphisms with

$$\theta_g : D_{g^{-1}} \to D_g,$$

such that
(i) $D_1 = X$ and $\theta_1 = id : D_1 \to D_1$,
(ii) $\theta_g \circ \theta_h \subseteq \theta_{gh}$, for all $g, h \in G$.

$\left(A, G, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$ is then called a *C*-algebraic partial dynamical system*.

**Proposition 2.3.** Let $G$ be a group, $A$ be a C*-algebra. A map $\theta : G \to \mathrm{pAut}(X)$ is a C*-algebraic partial action of $G$ on $X$ if an only if the following conditions are fullfilled:
(a) $\theta_1$ is the identity map on $X$
(b) $\theta_{g^{-1}} = (\theta_g)^{-1}$,
(c) $\theta_g \theta_h \theta_{h^{-1}} = \theta_{gh} \theta_{h^{-1}}$
(d) $\theta_{g^{-1}} \theta_g \theta_h = \theta_{g^{-1}} \theta_{gh}$

*Proof.* See 1.2.(i) in the first lecture or 4.5 in Exel's book. ☐

**Corollay 2.4.** If $G$ is a group and $X$ a locally compact Hausdorff space, then (1) induces a natural equivalence between topological partial actions of $G$ on $X$ and C*-algebraic actions of $G$ on $C_0(X)$.

Now we will construct a **crossed product** of the C*-algebra $A$ for a fixed group $G$ and a fixed C*-algebraic partial action $\theta$. For this let us first construct this product as follows:

$$A \rtimes_{alg} G := \{\sum_{g\in G} a_g \delta_g \mid \quad a_g \in D_g, \ \ a_g = 0 \text{ for all but finitely many } g \in G,\}.$$

The $\delta_g$ have to be seen as a sort of placeholder. Technically they indicate that $a_g\delta_g$ can be viewed as a function $a_g\delta_g : G \to A, \ \ h \mapsto \begin{cases} a_g, & \text{if } h = g; \\ 0, & \text{if } h \neq g \end{cases}$.

$A \rtimes_{alg} G$ can be seen as the set of all finitely supported functions $f\colon G \to A$ such that $f(g)$ is in $D_g$.

Addition and scalar multiplication are defined in the obvious way. Multiplication is determined by

$$(a\delta_g)(b\delta_h) = \theta_g(\theta_{g^{-1}}(a)b)\delta_{gh} \quad \forall g,h \in G, a \in D_g, b \in D_h.$$

The involution on $A \rtimes_{alg} G$ is given by

$$(a\delta_g)^* = \theta_{g^{-1}}(a^*)\delta_{g^{-1}}, \quad \forall g \in G, \forall a \in D_g.$$

The resulting algebra $A \rtimes_{alg} G$ is a *-algebra and associative. The problem is that it is not necessarily a C*-algebra. We need to define a norm and complete the algebra with it to achieve this.
(Remark: The associativity of $A \rtimes_{alg} G$ is hard to prove here and is not necessarily true if $A$ is not a C*-algebra.)

**Definition 2.5.** A C*-seminorm on a complex *-algebra $B$ is a seminorm $p : B \to \mathbb{R}_+$ such that for all $a, b \in B$ one has that
(i) $p(ab) \leq p(a)p(b)$
(ii) $p(a^*) = p(a)$
(iii) $p(a^*a) = p(a)^2$

If $B$ is a C*-algebra and $p$ is a C*-seminorm on $B$ we have $p(b) \leq \|b\|$ for all $b \in B$.

**Proposition 2.6.** Let $p$ be a C*-seminorm on $A \rtimes_{alg} G$. Then for every $a = \sum_{g\in G} a_g\delta_g$ in $A \rtimes_{alg} G$ we have that

$$p(a) \leq \sum_{g\in G} \|a_g\|.$$

*Proof.* $A\delta_1$ is isomorphic to $A$ so one has $p(a\delta_1) \leq \|a\|$ for all $a$ in $A$. It follows

$$p(a_g\delta_g)^2 = p\big((a_g\delta_g)(a_g\delta_g)^*\big) = p(a_g a_g^* \delta_1) \leq \|a_g a_g^*\| = \|a_g\|^2,$$

so the statement follows from the triangle inequality. □

Now we define the seminorm $\|.\|_{max}$ on $A \rtimes_{alg} G$ by

$$\|a\|_{max} = \sup\{p(a) : \ \ p \text{ is a C*-seminorm on } A \rtimes_{alg} G\}.$$

In fact $\|.\|_{max}$ is a norm but we will not focus on that now.

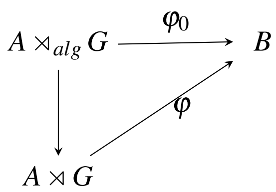
**Definition 2.7.** The C*-algebraic crossed product of a C*-algebra $A$ by a group $G$ under a C*-algebraic partial action $\theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ is defined as

$$A \rtimes G := \overline{A \rtimes_{alg} G}^{\|\cdot\|_{max}}.$$

**Proposition 2.8.** Let $B$ be another C*-algebra and let

$$\varphi_0 : A \rtimes_{alg} G \to B$$

be a *-homomorphism. The there exists a unique *-homomorphism $\varphi : A \rtimes G \to B$ such that the diagram




commutes.

*Proof.* Notice that $p(x) := \|\varphi_0(x)\|$ defines a C*-seminorm on $A \rtimes_{alg} G$ which is therefore bounded by $\|.\|_{max}$. Thus $\varphi_0$ is continuous and hence extends the completion.
A bounded linear function $f : X \to Y$ can be uniquely extended to $\tilde{f} : \tilde{X} \to Y$ □

## 3 COVARIANT REPRESENTATIONS

In this section we will take a look at covariant representations and especially their connection to partial actions.

**Definition 3.1.** Let $G$ be a discrete group an let $B$ be a unital C*-algebra. A partial representation of $G$ in $B$ is a map $\rho : G \to B$ such that:
(i) $\rho(e) = 1_B$.
(ii) $\rho(g)\rho(h)\rho(h^{-1}) = \rho(gh)\rho(h^{-1})$ for all $g$ and $h$ in $G$.
(iii) $\rho(g^{-1})\rho(g)\rho(h) = \rho(g^{-1})\rho(gh)$ for all $g$ and $h$ in $G$.
(iv) $\rho(g^{-1}) = \rho(g)^*$ for all $g$ in $G$.

Note that if the domains of a partial action are unital, then the map $g \mapsto 1_g\delta_g$ is a partial representation of $G$ on the crossed product, where $1_g$ denotes the unit of $D_g$. To prove this just check the above conditions:

.

A covariant representation of a partial dynamical system $\left(A, G, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$ in a C*-algebra $B$ is a pair $(\pi, \rho)$ where
(i) $\pi : A \to B$ is a homomorphism,
(ii) $\rho : G \to B$ is a partial representation
such that

$$\rho(g)\pi(a)\rho(g^{-1}) = \pi(\theta_g(a)) \quad \text{for all } a \in D_{g^{-1}}, \text{ for all } g \in G.$$

**Proposition 3.2.** Given a covariant representation $(\pi, \rho)$ of a C*-algebraic partial dynamical system $\left(A, G, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$ in a unital C*-algebra $B$, then there exists a unique *-homomorphism

$$\pi \times \rho : A \rtimes G \to B$$

such that $(\pi \times \rho)(a\delta_g) = \pi(a)\rho(g)$, for all $g \in G$ and all $a \in D_g$.

To prove this result we will use the following short lemma:

**Lemma 3.3.** Given the notation above, we have

$$\pi(a)\rho(g)\rho(g^{-1}) = \pi(a) = \rho(g)\rho(g^{-1})\pi(a)$$

for all $a \in D_g$ and $g \in G$.

*Proof.* Write $a$ as $\theta_g(b)$ for some $b \in D_{g^{-1}}$. Then

$$\pi(a)\rho(g)\rho(g^{-1}) = \pi(\theta_g(b))\rho(g)\rho(g^{-1}) = \rho(g)\pi(b)\rho(g^{-1})\rho(g)\rho(g^{-1})$$

$$=\rho(g)\pi(b)\rho(g^{-1}) = \pi(\theta_g(a)) = \pi(a).$$

The right side of the equation works similar. □

*Proof of Proposition 3.2* . Multiplicativity of $(\pi \times \rho)$:

$$\begin{aligned}((\pi\times\rho)(a\delta_g))\cdot((\pi\times\rho)(b\delta_h)) &= \pi(a)\rho(g)\pi(b)\rho(h)\\ &\overset{3.3}{=} \rho(g)\rho(g^{-1})\pi(a)\rho(g)\pi(b)\rho(h)\\ &= \rho(g)\pi(\theta_{g^{-1}}(a)b)\rho(h)\\ &= \rho(g)\pi(\theta_{g^{-1}}(a)b)\rho(g^{-1})\rho(g)\rho(h)\\ &= \pi(\theta_g(\theta_{g^{-1}}(a)b))\rho(gh)\\ &= (\pi\times\rho)((a\delta_g)\cdot(b\delta_h)).\end{aligned}$$

□

Now we will see the main result of this chapter.

**Theorem 3.4.** Let $\theta = \left(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right)$ be a C*-algebraic partial action of a group $G$ on a C*-algebra $A$ and let

$$\psi : A \rtimes G \to \mathscr{L}(H)$$

be a *-representation, where $H$ is a Hilbert space and $\psi$ is non-degenerate (closed linear span of $\psi(A \rtimes G)H$ equals $H$).
Then there exists a unique covariant representation $(\pi,\rho)$ of $\theta$ in $\mathscr{L}(H)$, such that:
(i) $\pi$ is a non-degenerate representation of $A$
(ii) $\rho(g)\rho(g^{-1})$ is the orthogonal projection onto the closed linear span of $\pi(D_g)H$
(iii) $\psi = \pi \times \rho$.

*Proof.* For simplicity we will assume that $A$ is unital with unital element $1_A$ and that the $D_g$ are also unital with unital element $1_g$. Then $(1_A\delta_1)$ is a unital element in $A \rtimes G$. Also for representations of unital algebras non-degeneracy is equivalent to unitality.

Definition of $\pi$ and condition (i):
The first step in our proof is to define $\pi$ an check that it is unital.
Define $\pi$ as the representation of $A$ on $H$ given by

$$\pi(a) = \psi(a\delta_1).$$

As $\psi$ is unital and $A \rtimes_{alg} G$ is dense in $A \rtimes G$ we know that $\psi$ restricted to $A \rtimes_{alg} G$ is also unital. Thus $\pi$ is unital.

Definition of $\rho$ and condition (ii):
For each $g \in G$ define

$$H_g := \pi(1_g)H$$

and note that $e_g := \pi(1_g)$ is the orthogonal projection onto $H_g$.
Note that $1_g 1_h = 1_h 1_g$ because $D_g$ and $D_h$ are two-sided ideals and as $\psi$ is multiplicative it follows that $e_g e_h = e_h e_g$. Define $\rho$ as follows:

$$\rho_g := \psi(1_g \delta_g).$$

Then we see immediately that $\rho$ is a partial representation that fulfills

$$e_g = \rho_g \rho_g^* = \rho_g \rho_{g^{-1}}$$

which verifies condition (ii) of our theorem.

$(\pi, \rho)$ is covariant representation:
What needs to be show is

$$\rho_g \pi(a) \rho_{g^{-1}} = \pi(\theta_g(a)), \quad \text{for any } g \in G \text{ and } a \in D_{g^{-1}}.$$

We have

$$\rho_g \pi(a) \rho_{g^{-1}} = \psi(1_g \delta_g a 1_{g^{-1}} \delta_{g^{-1}}) = \psi(\theta_g(a) 1_g) = \pi(\theta_g(a))$$

so we indeed have a covariant representation.

Condition (iii):
We want to show

$$(\pi \times \rho)(a\delta_g) = \psi(a\delta_g), \quad \text{for every } g \in G \text{ and every } a \in D_g.$$

Write $a = a1_g$ and we get

$$\begin{aligned} \psi(a\delta_g) &= \psi(1_g \delta_g \theta_{g^{-1}}(a) \delta_1) = \psi(1_g \delta_g) \pi(\theta_{g^{-1}}(a)) \\ &= \pi(a) \rho_g = (\pi \times \rho)(a\delta_g) \end{aligned}$$

With this we have proven condition (iii) and the existence of a covariant representation $(\pi \times \rho)$ was shown.

Uniqueness of the covariant representation:

Assume there is another covariant representation $(\pi', \rho')$ as demanded in the theorem such that

$$\pi' \times \rho' = \psi.$$

For every $a$ in $A$ we then have $\pi'(a) = \psi(a\delta_1) = \pi(a)$, so $\pi'$ and $\pi$ must coincide.
For a $g$ in $G$ and $\xi$ in $H_{g^{-1}}$ we write $\xi = \pi(a)\eta$ for some $a$ in $D_{g^{-1}}$ and $\eta$ in $H$. Then

$$\begin{aligned}
&\rho'_g(\xi) = \rho'_g \pi(a)\eta = \rho'_g \pi'(a)\eta = (\pi'(a^*)\rho'_{g^{-1}})^*\eta \\
=&\big((\pi' \times \rho')(a^*\delta_{g^{-1}})\big)^*\eta = \big((\pi \times \rho)(a^*\delta_{g^{-1}})\big)^*\eta \\
=&\rho_g \pi(a)\eta = \rho_g(\xi)
\end{aligned}$$

which means that $\rho'_g$ coincides with $\rho_g$ on $H_{g^{-1}}$.
By condition (ii) we know that $\rho'_g$ and $\rho_g$ vanish on $H_{g^{-1}}^{\perp}$ so $\rho'_g$ coincides with $\rho_g$ on $H$. thus $\rho' = \rho$ and uniqueness is shown. □