FELL BUNDLES

- Definition
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- Saturation

3.1 Definition: A Fell bundle over a group G is a collection

$$\mathcal{B} = \{B_g\}_{g\in G}$$

of Banach spaces, each of which is called a fiber. In addition, the total space $\mathcal{B} = \bigsqcup_{g\in G} B_g$ is equipped with a multiplication and an involution

$$\cdot\ \mathcal{B}\times\mathcal{B} \longrightarrow \mathcal{B} \qquad\qquad * : \mathcal{B}\longrightarrow\mathcal{B}$$

satisfying the following properties for all $g,h\in G$, $b,c\in\mathcal{B}$:

(a) $B_gB_h \subset B_{gh}$

(b) multiplication is bi-linear from $B_g\times B_h$ to B_{gh}

(c) multiplication on $\mathcal{B}$ is associative

(d) $\|bc\| \leq \|b\|\|c\|$

(e) $(B_g)^* \subseteq B_{g^{-1}}$

(f) involution is conjugate-linear from B_g to $B_{g^{-1}}$

(g) $(bc)^* = c^*b^*$

(h) $b^{**} = b$

(i) $\|b^*\| = \|b\|$

(j) $\|b^*b\| = \|b\|^2$

(k) $b^*b \geqslant 0$ in B_1.

- Note that (a)-(j) imply that B_1 is a C^*-algebra with the restricted operations.
- The positivity in (k) is taken with respect to the standard order relation in B_1.

3.2 Example: GROUP BUNDLE

$$\mathcal{B} = \mathbb{C} \times G$$

where G is any group. The fibers

$$B_g = \mathbb{C} \times \{g\} \quad , \quad g \in G$$

have the usual linear and norm structure, and we define the product and involution as

$$(\lambda, g)(\mu, h) := (\lambda\mu, gh) \qquad , \quad g, h \in G, \ \lambda, \mu \in \mathbb{C}.$$
$$(\lambda, g)^* := (\bar{\lambda}, g^{-1})$$

3.3 Example: SEMI-DIRECT PRODUCT BUNDLE

Fix a C^*-algebraic partial action $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ of a group G on a C^*-algebra A. Define the total space as

$$\mathcal{B} = \{(b, g) \in A \times G : b \in D_g\}$$

We write $b\delta_g$ to refer to (b, g), whenever $b \in D_g$.

The fibers of our bundle are

$$B_g = \{ b\delta_g : b \in D_g\}$$

with the linear and norm structure borrowed from D_g.

The multiplication and involution are defined as

$$(a\delta_g)(b\delta_h) := \theta_g(\theta_{g^{-1}}(a)b)\delta_{gh} \quad , \forall \ a \in D_g, \quad b \in D_h$$
$$(a\delta_g)^* := \theta_{g^{-1}}(a^*)\delta_{g^{-1}} \quad , \ \forall \ g \in G, \quad \forall \ a \in D_g$$

We have that $\mathcal{B}$ is a Fell bundle, called semi-direct product bundle relative to θ.

3.4 Example: FELL BUNDLES GIVEN IN TERMS OF PARTIAL REPRESENTATIONS

Let $u: G \longrightarrow A$ be a $*$-partial representation of a given group G in a unital C^*-algebra A. For each $g \in G$, consider

$$B_g^u := [\{u_{h_1} \cdots u_{h_n} : n \in \mathbb{N}, n \geq 1, h_i \in G, h_1 \cdots h_n = g\}]$$

We have that

$$\mathcal{B}^u = \{B_g^u\}_g$$

is a Fell bundle with the operations borrowed from A.

- $B_g^u B_h^u \subseteq B_{gh}^u$: $\left.\begin{array}{l} u_{g_1} \cdots u_{g_n} \in B_g^u, \ g_1 \cdots g_n = g \in G \\ u_{h_1} \cdots u_{h_m} \in B_h^u, \ h_1 \cdots h_m = h \in G \end{array}\right\} \Rightarrow (u_{g_1} \cdots u_{g_n})(u_{h_1} \cdots u_{h_m}) \in B_{gh}^u$ as $g_1 \cdots g_n h_1 \cdots h_m = gh$

  By taking limits of sums of elements like above, the result follows.

Construction of a C^*-algebra from a given Fell bundle

Fix $\mathcal{B} = \{B_g\}_{g\in G}$ an abitrary Fell bundle.

3.5 Definition: A section of $\mathcal{B}$ is a function $y : G \to \mathcal{B}$ such that $y_g \in B_g$, $\forall g \in G$.

$$C_c(\mathcal{B}) := \{ y : G \to \mathcal{B} \text{ section} \mid \operatorname{supp} y \text{ is finite} \}$$

Given $y, z \in C_c(\mathcal{B})$, define the convolution product by

$$(y * z)_g := \sum_{h\in G} y_h z_{h^{-1}g}, \quad \forall g \in G$$

and the adjoint operation by

$$(y^*)_g := (y_{g^{-1}})^*, \quad \forall g \in G.$$

- $C_c(\mathcal{B})$ is an associative $*$-algebra.
- If $\mathcal{B}$ is the semi-direct product bundle from Example 3.3, then
$$C_c(\mathcal{B}) \cong A \rtimes_{alg} G$$

Indeed, given $y \in C_c(\mathcal{B})$, we have that $y_g \in B_g = \{ b\delta_g : b \in D_g \}$
$\Rightarrow y_g = a_g \delta_g$ for some $a_g \in D_g$. Consider

$$\varphi : C_c(\mathcal{B}) \longrightarrow A \rtimes_{alg} G$$
$$y \longmapsto \sum_{g\in G} y_g = \sum_{g\in G} a_g \delta_g$$

Let us see that φ is multiplicative (rest is clear):

Consider $y, z \in C_c(\mathcal{B})$ and write $y_g = a_g \delta_g$, $z_g = b_g \delta_g$ $\forall g \in G$

$$\varphi(y * z) = \sum_{g\in G} (y * z)_g = \sum_{g\in G} \Big(\sum_{h\in G} y_h z_{h^{-1}g} \Big) = \sum_{g,h\in G} (a_h \delta_h)(b_{h^{-1}g} \delta_{h^{-1}g}) =$$

$$= \sum_{g,h\in G} \theta_h(\theta_{h}^{-1}(a_h) b_{h^{-1}g}) \delta_g = \sum_{s,g\in G} \theta_h(\theta_h^{-1}(a_h) b_s) \delta_s =$$

$$= \sum_{s,g\in G} (a_h \delta_h)(b_s \delta_s) = \Big(\sum_{h\in G} a_h \delta_h \Big) \Big(\sum_{s\in G} b_s \delta_s \Big) = \varphi(y)\, \varphi(z).$$

3.6 Definition: A *$*$-representation* of a Fell bundle $\mathcal{B}=\{B_g\}_{g\in G}$ in a $*$-algebra C is a collection $\pi=\{\pi_g\}_{g\in G}$ of linear maps

$$\pi_g : B_g \longrightarrow C$$

such that

(i) $\pi_g(b)\pi_h(c)=\pi_{gh}(bc)$

(ii) $\pi_g(b)^* = \pi_{g^{-1}}(b^*)$,

for all $g,h\in G$, and all $b\in B_g$, and $c\in B_h$.

Remark: Given a Fell bundle $\mathcal{B}=\{B_g\}_{g\in G}$, for each g in G, consider (the natural inclusion)

$$j_g : B_g \longrightarrow C_c(\mathcal{B})$$

given by

$$j_g(b)|_h = \begin{cases} b, & \text{if } g=h \\ 0, & \text{otherwise} \end{cases}$$

The collection of maps $j=\{j_g\}_g$ is a $*$-representation of $\mathcal{B}$ in $C_c(\mathcal{B})$.

Recall that we define

$$A\rtimes G = \overline{A\rtimes_{alg} G}^{\|\cdot\|_{max}}$$

where

$$\|\cdot\|_{max} = \sup\{C^*\text{-seminorm}(\cdot) \text{ on } A\rtimes_{alg} G\}$$

We will construct a C^*-algebra from $C_c(\mathcal{B})$ in the same way.

3.7 Proposition: Let p be a C^*-seminorm on $C_c(\mathcal{B})$. Then, for every $y \in C_c(\mathcal{B})$, we have

$$p(y) \leq \sum_{g \in G} \|y_g\|.$$

Proof: For $b \in B_g$, we have

$$p(j_g(b))^2 \underset{\substack{\uparrow \\ p \text{ is } C^*\text{-seminorm}}}{=} p(j_g(b)^* j_g(b)) \underset{\text{def of } *\text{-repres.}}{=} p(j_{g^{-1}}(b^*) j_g(b)) = p(j_1(b^* b))$$

$[B$ C^*-algebra, φ seminorm on $B \Rightarrow \varphi(b) \leq \|b\|, \forall b \in B]$.

We have that $p \circ j_1$ is a seminorm on B_1, thus

$$p(j_g(b))^2 = p(j_1(b^* b)) \leq \|b^* b\| = \|b\|^2$$

So, we have that $p(j_g(b)) \leq \|b\|, \forall b \in B_g$. $(*)$

Note that for $y \in C_c(\mathcal{B})$ we can write

$$y = \sum_{g \in G} j_g(y_g) \qquad \left(j_g(y_g)|_h = \begin{cases} y_g, & \text{if } g = h \\ 0, & \text{otherwise} \end{cases} \right)$$

Thus,

$$p(y) = p\Big(\sum_{g \in G} j_g(y_g)\Big) \underset{p \text{ seminorm}}{\leq} \sum_{g \in G} p(j_g(y_g)) \underset{(*)}{\leq} \sum_{g \in G} \|y_g\|.$$

$\blacksquare$

Given $y \in C_c(\mathcal{B})$, we define

$$\|y\|_{max} := \sup \{p(y) : p \text{ is } C^*\text{-seminorm on } C_c(\mathcal{B})\}$$

From the previous prop. we have that $\|y\|_{max} < \infty$, $\forall y \in C_c(\mathcal{B})$.

Moreover, $\|\cdot\|_{max}$ defines a C^*-seminorm on $C_c(\mathcal{B})$

3.8 Definition: Consider the ideal $\mathcal{N} := \{ y \in C_c(\mathcal{B}) : \|y\|_{max} = 0 \} \triangleleft C_c(\mathcal{B})$

On $C_c(\mathcal{B})/\mathcal{N}$, $\|\cdot\|_{max}$ induces a C^*-norm $\|\cdot\|^{\sim}_{max}$.

The cross sectional C^*-algebra of $\mathcal{B}$ is defined as

$$C^*(\mathcal{B}) := \overline{\frac{C_c(\mathcal{B})}{\mathcal{N}}}^{\|\cdot\|^{\sim}_{max}}$$

We denote by $k : C_c(\mathcal{B}) \longrightarrow C^*(\mathcal{B})$ the canonical mapping arising from the completion process

$$\begin{tikzcd} C_c(\mathcal{B}) \arrow[r, "\pi"] \arrow[dr, "k"'] & \frac{C_c(\mathcal{B})}{\mathcal{N}} \arrow[d, "i"] \\ & C^*(\mathcal{B}) \end{tikzcd}$$

inclusion

$$k := i \circ \pi$$

Since k is a $*$-homomorphism and $j = \{j_g\}_{g \in G}$ is a $*$-representation of $\mathcal{B}$ in $C_c(\mathcal{B})$, if we set $\forall\, g \in G$

$$\hat{j}_g := k \circ j_g : \mathcal{B}_g \longrightarrow C^*(\mathcal{B})$$

we have that $\hat{j} = \{\hat{j}_g\}_g$ is a $*$-representation of $\mathcal{B}$ in $C^*(\mathcal{B})$.

$\hat{j}$ is called the universal representation.

3.9 Proposition: Consider $\theta = (\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ a C^*-algebraic partial action of a group G on a C^*-algebra A, and let $\mathcal{B}$ be the corresponding semi-direct product bundle (Example 3.3). Then $A \rtimes G$ is naturally isomorphic to $C^*(\mathcal{B})$.

3.10 **Proposition:** Let $\pi = \{\pi_g\}_{g\in G}$ be a representation of the Fell bundle $\mathcal{B}$ in a C^*-algebra C. Then $\exists!$ $*$-homomorphism $\varphi : C^*(\mathcal{B}) \longrightarrow C$ such that $\varphi \circ \hat{j}_g = \pi_g \quad \forall\, g \in G$.

$$\begin{tikzcd} B_g \arrow[r, "\pi_g"] \arrow[dr, "\hat{j}_g"'] & C \\ & C^*(\mathcal{B}) \arrow[u, "\varphi"'] \end{tikzcd}$$

φ is called the integrated form of π.

Proof: Define $\varphi_0 : C_c(\mathcal{B}) \longrightarrow C$ as

$$\varphi_0(y) = \sum_{g\in G} \pi_g(y_g) \quad , \forall\, y \in C_c(\mathcal{B})$$

We have that φ_0 is a $*$-homomorphism.

Then $p(y) := \|\varphi_0(y)\|$, $\forall\, y \in C_c(\mathcal{B})$ defines a C^*-seminorm on $C_c(\mathcal{B})$.

Thus,

$$\|\varphi_0(y)\| = p(y) \leq \sup\{\, p(y) : p \text{ is } C^*\text{-seminorm on } C_c(\mathcal{B}) \,\} = \|y\|_{max}$$

$\Rightarrow \varphi_0$ is continuous $\Rightarrow \exists!$ $*$-homomorphism $\varphi : C^*(\mathcal{B}) \longrightarrow C$ that extends φ_0. ∎

Saturation

Consider $\mathcal{B} = \{B_g\}_g$ a Fell bundle. We have (from the definition)

$$B_g B_h \subseteq B_{gh} \quad , \forall g, h \in G$$

$\Rightarrow$ $\underbrace{[B_g B_h]}_{\text{closed linear span}}$ is a closed subspace of B_{gh}.

If $[B_g B_h] = B_{gh}$, $\forall g, h \in G$, we say that the Fell bundle $\mathcal{B}$ is saturated.

3.11 Lemma: $[B_g B_{g^{-1}} B_g] = B_g \quad , \forall g \in G.$

We have that $[B_g B_{g^{-1}}] \subseteq B_1$ is a closed ideal in B_1

3.12 Proposition: A Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ is saturated if and only if $[B_g B_{g^{-1}}] = B_1$, $\forall g \in G$.

Proof: $(\Rightarrow)$: $\mathcal{B}$ saturated $\Rightarrow$ $[B_g B_h] = B_{gh}$, $\forall g, h \in G$

$\Rightarrow$ $[B_g B_{g^{-1}}] = B_1$, $\forall g \in G$.

$(\Leftarrow)$: Assume that $[B_g B_{g^{-1}}] = B_1$, $\forall g \in G$, then for $g, h \in G$

$$B_{gh} \underset{\substack{\uparrow \\ 3.11}}{=} [\underbrace{B_{gh} B_{(gh)^{-1}}}_{\subseteq B_1} B_{gh}] \subseteq [B_1 B_{gh}] \underset{\substack{\uparrow \\ \text{hypothesis}}}{=} [[B_g B_{g^{-1}}] B_{gh}] = [B_g \underbrace{B_{g^{-1}} B_{gh}}_{\subseteq B_{g^{-1}gh}}] \subseteq [B_g B_h] \subseteq B_{gh}$$

$\Rightarrow$ $[B_g B_h] = B_{gh}$, ie, $\mathcal{B}$ is saturated. $\blacksquare$

3.13 Proposition: Consider $\theta = (\{D_g\}_g, \{\theta_g\}_g)$ a C^*-algebraic partial action of the group G on the C^*-algebra A, and let $\mathcal{B}$ be the associated semi-direct product bundle. Then

$$\mathcal{B} \text{ is saturated} \iff \theta \text{ is a global action.}$$

Proof: Consider $g \in G$, $a \in D_g$, $b \in D_{g^{-1}}$. Then

$$(a\delta_g)(b\delta_{g^{-1}}) = \theta_g(\underbrace{\theta_{g^{-1}}(a)b}_{\in D_{g^{-1}}})\,\delta_1 \underset{\substack{\uparrow \\ \theta_g : D_{g^{-1}} \to D_g \text{ *-isom.}}}{=} \theta_g(\theta_{g^{-1}}(a))\,\theta_g(b)\,\delta_1 \underset{\substack{\uparrow \\ \theta_{g^{-1}} = (\theta_g)^{-1}}}{=} a\,\theta_g(b)\,\delta_1$$

$$\Rightarrow (D_g\delta_g)(D_{g^{-1}}\delta_{g^{-1}}) = D_g \underbrace{\theta_g(D_{g^{-1}})}_{D_g \leftarrow \theta_g \text{ surj.}}\delta_1 = D_g D_g \delta_1 \underset{\substack{\uparrow \\ \text{Cohen-Hewitt factorization Theorem}}}{=} D_g\delta_1 \qquad (*)$$

Therefore, $\forall\, g \in G$

$$[\mathcal{B}_g\mathcal{B}_{g^{-1}}] \underset{\substack{\uparrow \\ \text{def}}}{=} [(D_g\delta_g)(D_{g^{-1}}\delta_{g^{-1}})] \underset{\substack{\uparrow \\ (*)}}{=} [D_g\delta_1] \underset{\substack{\uparrow \\ D_g \text{ closed}}}{=} D_g\delta_1 \qquad (**)$$

($\Rightarrow$) Assume that $\mathcal{B}$ is saturated. Then $\forall\, g \in G$

$$\{b\delta_1 : b \in D_1\} \underset{\substack{\uparrow \\ \text{def}}}{=} \mathcal{B}_1 \underset{\substack{\downarrow \\ 3.12}}{=} [\mathcal{B}_g\mathcal{B}_{g^{-1}}] \underset{\substack{\uparrow \\ (**)}}{=} D_g\delta_1 \underset{\substack{\uparrow \\ \text{def}}}{=} \{b\delta_1 : b \in D_g \cap D_1\} \Rightarrow D_g = D_1 = A$$

$\Rightarrow \theta$ is a global action.

($\Leftarrow$) Assume that θ is global ($D_g = A$, $\forall\, g \in G$). Then,

$$\mathcal{B}_1 \underset{\substack{\uparrow \\ \text{def}}}{=} D_1\delta_1 \underset{\substack{\uparrow \\ \theta \text{ global}}}{=} D_g\delta_1 \underset{\substack{\uparrow \\ (**)}}{=} [\mathcal{B}_g\mathcal{B}_{g^{-1}}] \Rightarrow \mathcal{B} \text{ is saturated.} \qquad \blacksquare$$

3.14 Definition: Let G be a group and A a unital C^*-algebra. A unitary group representation is a map $u: G \longrightarrow A$ such that

(i) $u_1 = 1$ (ii) $\underbrace{u_g u_h = u_{gh}}$ (iii) $u_{g^{-1}} = (u_g)^*$

- $u_g u_h u_{h^{-1}} = u_{gh} u_{h^{-1}}$
- $u_{g^{-1}} u_g u_h = u_{g^{-1}} u_{gh}$

} for $*$-partial rep.

3.15 Proposition: Given a $*$-partial representation u of a group G in a nonzero unital C^*-algebra A, consider its associated Fell bundle $\mathcal{B}^u$ (Example 3.4). Then $\mathcal{B}^u$ is saturated if and only if u is a unitary group representation.

Proof: ($\Leftarrow$) Assume that u is a unitary group representation. Then, $\forall g \in G$

$$\mathcal{B}^u_g \underset{\text{def}}{=} [\{\underbrace{u_{h_1} \cdots u_{h_n}}_{u_{h_1 \cdots h_n} = u_g} : n \geq 1,\ h_i \in G,\ h_1 \cdots h_n = g\}] = [\{u_g\}] = \mathbb{C} u_g \quad (*)$$

Thus,

$$[\mathcal{B}^u_g \mathcal{B}^u_{g^{-1}}] \underset{(*)}{=} [(\mathbb{C} u_g)(\mathbb{C} u_{g^{-1}})] = [\mathbb{C} u_g u_{g^{-1}}] = [\mathbb{C} u_1] = \mathbb{C} u_1 \underset{(*)}{=} \mathcal{B}^u_1$$

$\overset{\text{Prop. 3.12}}{\Longrightarrow}$ $\mathcal{B}^u$ is saturated.

($\Rightarrow$) Assume that $\mathcal{B}$ is saturated, and denote $e_g := u_g u_{g^{-1}}$, $\forall g \in G$. Then

$$\mathcal{B}^u_1 = [\mathcal{B}^u_g \mathcal{B}^u_{g^{-1}}] \underset{(16.8.iii)}{=} [\mathcal{B}^u_1 u_g u_{g^{-1}} \mathcal{B}^u_1] = [\mathcal{B}^u_1 e_g \mathcal{B}^u_1] \underset{(**)}{=} [e_g \mathcal{B}^u_1]$$

Thus,

$$[\{u_{h_1} \cdots u_{h_n} : h_1 \cdots h_n = 1\}] = \mathcal{B}^u_1 = [e_g \mathcal{B}^u_1] = [\{e_g u_{h_1} \cdots u_{h_n} : h_1 \cdots h_n = 1\}] \Rightarrow u_g u_{g^{-1}} = e_g = 1$$

Then, $\forall h, g \in G$

$$u_h u_g = \underbrace{u_h u_{h^{-1}}}_{= e_h = 1} u_h u_g \underset{\substack{u \text{ is } *\text{-partial} \\ \text{repres.}}}{=} \underbrace{u_h u_{h^{-1}}}_{= e_h = 1} u_{hg} = u_{hg} \Rightarrow u \text{ is a unitary group representation.}$$

Now let us show $(**)$, ie, $[\mathcal{B}^u_1 e_g \mathcal{B}^u_1] = [e_g \mathcal{B}^u_1]$:

- Claim 1: $u_g e_h = e_{gh} u_g \quad \forall g, h \in G$.

Proof: $u_g e_h = u_g u_h u_{h^{-1}} \underset{\text{def of } *\text{-rep.}}{=} u_{gh} u_{h^{-1}} = 1 u_{gh} u_{h^{-1}} = \overbrace{u_{(gh)(gh)^{-1}} u_{gh}}^{u_{gh} u_{(gh)^{-1}} u_{gh}} u_{h^{-1}} \underset{\text{def of } *\text{-rep}}{=}$

$$= u_{gh} \underbrace{u_{(gh)^{-1}} u_{gh} u_{h^{-1}}}_{u_{(gh)^{-1}} u_g} = u_{gh} u_{(gh)^{-1}} u_g = e_{gh} u_g. \quad \square$$

- Claim 2: $e_g e_h = e_h e_g \quad \forall\, g,h \in G$

<u>Proof</u>: $e_g e_h = u_g u_{g^{-1}} e_h \underset{\substack{\uparrow \\ \text{Claim 1}}}{=} u_g e_{g^{-1}h} u_{g^{-1}} = e_g u_g u_{g^{-1}} = e_h e_g$ $\square$

- Claim 3: Given $h_1, \dots, h_n \in G$, we have

$$u_{h_1} \cdots u_{h_n} = e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_n} u_{h_1 \cdots h_n}.$$

<u>Proof</u>: We prove by induction in n.

For $n = 1$:

$$u_{h_1} = 1 u_{h_1} = u_1 u_{h_1} = u_{h_1 h_1^{-1}} u_{h_1} = u_{h_1} u_{h_1^{-1}} u_{h_1} = e_{h_1} u_{h_1}$$

Assuming $n \geq 2$ and that the result is valid for $n-1$:

$$\begin{aligned}
u_{h_1} u_{h_2} \cdots u_{h_n} &= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} u_{h_1 \cdots h_{n-1}} u_{h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} \cdot 1 \cdot u_{h_1 \cdots h_{n-1}} u_{h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} u_{(h_1 \cdots h_{n-1})(h_1 \cdots h_{n-1})^{-1}} u_{h_1 \cdots h_{n-1}} u_{h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} u_{h_1 \cdots h_{n-1}} \cdot u_{(h_1 \cdots h_{n-1})^{-1}} u_{h_1 \cdots h_{n-1}} u_{h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} u_{h_1 \cdots h_{n-1}} \cdot u_{(h_1 \cdots h_{n-1})^{-1}} u_{h_1 \cdots h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} e_{h_1 \cdots h_{n-1}} u_{h_1 \cdots h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} u_{h_1 \cdots h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} 1 \cdot u_{h_1 \cdots h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} u_{(h_1 \cdots h_n)(h_1 \cdots h_n)^{-1}} u_{h_1 \cdots h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} u_{h_1 \cdots h_n} u_{(h_1 \cdots h_n)^{-1}} u_{h_1 \cdots h_n} \\
&= e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_{n-1}} e_{h_1 \cdots h_n} u_{h_1 \cdots h_n}
\end{aligned}$$

$\square$

Now, consider $a = u_{h_1} \cdots u_{h_n}$, $b = u_{g_1} \cdots u_{g_m} \in \mathcal{B}_1^u$ $(h_1 \cdots h_n = 1 = g_1 \cdots g_m)$. Then

$$a e_g b = u_{h_1} \cdots u_{h_n} e_g u_{g_1} \cdots u_{g_m} \underset{\substack{\uparrow \\ \text{claim 3}}}{=} e_{h_1} e_{h_1h_2} \cdots e_{h_1h_2 \cdots h_n} e_g u_{g_1} \cdots u_{g_m}$$

$$\overset{\text{Claim 2}}{=} e_g e_{h_1} e_{h_1h_2} \cdots e_{h_1 \cdots h_n} u_{g_1} \cdots u_{g_m} = e_g \underbrace{u_{h_1} \cdots u_{h_n} u_{g_1} \cdots u_{g_m}}_{\in \mathcal{B}_1^u} \in e_g \mathcal{B}_1^u$$

$\Rightarrow [\mathcal{B}_1^u e_g \mathcal{B}_1^u] \subseteq [e_g \mathcal{B}_1^u]$.

On the other hand, note that $u_1 = 1 \in \mathcal{B}_1^u$. Thus, $[e_g \mathcal{B}_1^u] \subseteq [\mathcal{B}_1^u e_g \mathcal{B}_1^u]$.

$\blacksquare$