Globalization in the C^*-context

Zahra Hassanpour

Uniqueness of C^*-globalization

Definition

C*-Partial Action: Let A be a C*-algebra. A partial action $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ on A, such that for every $t \in G$, A_t is a closed two sided ideal and $\alpha_t : A_{t^{-1}} \to A_t$ is a $*$-isomorphism.
In the case, $A_t = A$, for every $t \in G$, α is called a C*-global action.

Definitin

C*-globalization: Let α be a C*-partial action of G on C*-algebra A. A 4-tuple $(B, \beta, I, \mathfrak{i})$, where B is C*-algebra, β is a C*-global action of G on B, I is a C*-ideal of B and $\mathfrak{i} : \alpha \to \beta|_I$ is an isomorphism of C^*-partial actions.

Remark

If α has a C*-globalization, then A is $*$-isomorphic to a C*-subalgebra of B. A C*-globalization of C*-partial action α, is minimal if and only if

$$B = \overline{\Sigma_{t\in G}\alpha_t(A)}$$

Proposition

Let $\alpha = (\{A_t\}_{t \in G}, \{\alpha_t\}_{t \in G})$ be a partial action of group G on the C*-algebra A. Suppose that for $k = 1, 2$ minimal C*-globalization β^k acting on a C*-algebra B^k is given. Then there is an equivariant $*$-isomorphism

$$\phi : B^1 \to B^2$$

such that is the identity on the respective copies of A within B^1 and B^2.

proof: Step1

claim: For every a and b in A,

$$\beta_t^1(a)b = \beta_t^2(a)b, \qquad t \in G.$$

Given $t \in G$, let $\{\nu_i\}_{i\in I}$ be an approximate unit for $A_{t^{-1}}$. Note that $\{\alpha_t(\nu_i)\}_{i\in I}$ is an approximate identity for A_t. Also, since

$$\beta_t^k(a)b \in \beta_t^k(A) \cap A = A_t \cap A = A_t, \qquad t \in G.$$

Now, since for every $t \in G$, A_t is a closed two sided ideal of A, we have

$$\beta_t^k(a)b = \lim_{i\to\infty} \alpha_t(\nu_i)\beta_t^k(a)b = \lim_{i\to\infty} \beta_t^k(\nu_i a)b = \alpha_t(\nu_i a)b, \qquad k = 1, 2.$$

Since, the right hand side does not depend on k, the desired result holds.

Proof: Step2

Suppose that we are given C*-algebra B, such that $B = \overline{\Sigma_{i \in I} J_i}$, where $\{J_i\}_{i \in I}$ is a family of closed two sided ideals of B. Then,

$$\|b\| = \sup_{i \in I} \sup_{x \in J_{i_1}} \|bx\|$$

Proof: Step 3

Construction of the desired equivarinat $*$-isomorphism $\phi : B^1 \to B^2$. Let $a_1, a_2, ..., a_n \in A, t_1, t_2, ..., t_n \in G$. Considering step 2, one can see that the correspondence

$$\Sigma_{i=1}^{n} \beta_{t_i}^{1}(a_i) \mapsto \Sigma_{i=1}^{n} \beta_{t_i}^{2}(a_i)$$

is well-defined and preserves the norms. Also, by minimality of action β_t^k, it extends to an isometric onto mapping $\phi : B^1 \to B^2$. Moreover, the restriction of ϕ to the respective copy of A in B^1 and B^2 is the identity map.

C^*-globalization of C^*-partial Actions Acting on a Commutative C^*-algebra

Proposition

Let β be a C^*-globalization of C*-partial action α. If α acts on a commutative C^*algebra, then so does β.

Proof

Assume that C^*-partial action α acts on commutative C^*-algebra A. Also, β is a minimal C^*-globalization α, acting on B.

Step1: $A \subseteq Z(B)$.

Let $a \in A$ and $b \in B$. Using Cohen-Hewitt, we may write $a = a_1 a_2$, where $a_1, a_2 \in A$. Then, since A can be considered a closed two sided ideal of B,

$$ab = (a_1 a_2)b = a_1(a_2 b) = a_1(ba_2) = (ba_2)a_1 = b(a_1 a_2) = ba.$$

Step2: For every $s, t \in G$, For every $a, b \in A$,

$$\beta_t(a)\beta_s(b) = \beta_t(a\beta_{t^{-1}s}(b)) = \beta_t(\beta_{t^{-1}s}(b)a) = \beta_s(b)\beta_t(a).$$

Step3: From the previous step and minimality of the C^*-globalization β of α, for every $b_1, b_2 \in B$, $b_1 b_2 = b_2 b_1$.

Corollary

Let α be a partial action of a group G on a LCH space X. Denote be α' the C^*-partial action of G on $C_0(X)$ corresponding to α. A necessary and sufficient condition for α' to admit a (top)globalization is that the globalization of α takes place on a Hausdorff space.

Proof

Let (β, Y) be a globalization of α and Y is Hausdorff. Then the corresponding action β' of G on B is a C^*-globalization of α'.
On the other hand, if (β', B) is a C^*-globalization of α', then B is a commutative C^*-algebra, hence isomorphic to $C_0(Y)$, for some Hausdorff space Y. Denoting β the global action of G on Y, corresponding to β'. β is a globalization of α.

Theorem

Every C^*-algebraic partial action is Morita-Rieffel equivalent to one admitting a globalization. More precisely, every C^*-algebraic partial action is Morita-Reiffel equivalent to the dual action Δ on the restricted smash product for the corresponding semi-direct product bundle (which admits a globalization).

sketch of proof

Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a C*-partial action of the group G on C*-algebra A. Consider its semi-direct product bundle $\mathcal{B}$. By the definition of Morita-Rieffel equivalence, the structure of a Hilbert $A - \mathcal{B}_\flat G$-bimodule and a set theorical partial action $\gamma = (\{M_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$ of G on M is required such that

$$(M, G, \{M_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$$

satisfies the properties of an imprimitivity system.
Let M be the subspace of $\mathcal{B}_\flat G$ given by

$$M = \overline{\Sigma_{h\in G} B_h \otimes e_{1,h}}.$$

sketch of proof

1. **Left A-module structure of M:** A is identified with $B_1 \otimes e_{1,1}$, via
$$a \in A \mapsto a\delta_1 \otimes e_{1,1}.$$
2. **Right $\mathcal{B}_\flat G$-module structure of M:** M is a right ideal in $\beta_\flat G$.
$$(B_h \otimes e_{1,h})(B_{k^{-1}}, B_l \otimes e_{k,l}) = \delta_{h,k}(B_h B_{h^{-1}} B_l \otimes e_{1,l}) \subset B_l \otimes e_{1,l} \subset M.$$
3. **A-valued inner product:** Given $\xi, \eta \in M$, $\xi\eta^* \in B_1 \otimes e_{1,1}$.
$$\xi\eta^* = \langle \xi, \eta \rangle_A \delta_1 \otimes e_{1,1}.$$
4. **$\mathcal{B}_\flat G$-valued inner product:** Given $\xi, \eta \in M$,
$$\langle \xi, \eta \rangle_{\mathcal{B}_\flat G} = \xi^* \eta.$$

sketch of proof

The structure of partial action on M Given $t \in G$,

$$M_t = \overline{\Sigma_{t\in G}[B_t B_{t^{-1}} B_s] \otimes e_{1,hs}}.$$

- M_t is a Hilbert $A - \mathcal{B}_\flat G$- bimodule.

$$(B_t B_{t^{-1}} B_s) \otimes e_{1,s}(B_{s^{-1}} B_r) \otimes e_{s,r} \subset M_t.$$

Observe that $B_t = A_t \delta_t$, hence

$$[B_t B_{t^{-1}}] = [A_t \delta_t A_{t^{-1}} \delta_{t^{-1}}] = [A_g \alpha_t (A_t^{-1}) \delta_1] = A_t \delta_1.$$

$$[B_t B_t^{-1} B_s] = [A_t A_s \delta_s] = (A_t \cap A_s)\delta_s, \qquad s, t \in G.$$

Consequently, $M_t = \overline{\Sigma_{s\in G}(A_t \cap A_s)\delta_s \otimes e_{1,s}}$. $\gamma_t : M_{t^{-1}} \to M_t$, given by

$$\gamma_t(a\delta_s) \otimes e_{1,s} = \theta_t(a)\delta_{ts} \otimes e_{1,ts}.$$

Theorem

Let α and β be Morita-Rieffel equivalent

$$\alpha = (A, G, \{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G}), \qquad \beta = (B, G, \{B_t\}_{t\in G}, \{\beta_t\}_{t\in G})$$

Then

- $A \rtimes_{red} G$ and $B \rtimes_{red} G$ are Rieffel-Morita equivalent.
- $A \rtimes G$ and $B \rtimes G$ are Rieffel-Morita equivalent.

Proof

An imprimitivity system for α and β:

$$\gamma = (M, G, \{M_t\}_{t\in G}, \{\gamma_t\}_{t\in G})$$

The linking algebra of M:

$$L = \begin{bmatrix} A & M \\ M^* & B \end{bmatrix}$$

The partial action of G on L:

$$\lambda = (\{L_t\}_{t\in G}, \{\lambda_t\}_{t\in G})$$

where, for every $t \in G$,

$$L_t = \begin{bmatrix} A_t & M_t \\ M_t^* & B_t \end{bmatrix} \qquad \lambda_t : \begin{bmatrix} a & \xi \\ \eta^* & b \end{bmatrix} \mapsto \begin{bmatrix} \alpha_t(a) & \gamma_t(\xi) \\ \gamma_t(\eta) & \beta_t(b) \end{bmatrix}$$

Proof

Since A is a closed subspace of L that is λ invariant:

$$A \rtimes_{red} G \subseteq L \rtimes red G.$$

Claim: $A \rtimes_{red} G$ is a full hereditary subalgebra of $L \rtimes_{red} G$.
Consider the formal left or right multiplication of

$$e_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

by elements of L to define a multiplier of L.
The inclusion of L in $L \rtimes_{red} G$ is a non-degenerate $*$-homomorphism that can be extended to a $*$-homomorphism

$$\mathcal{M}(L) \to \mathcal{M}(L \rtimes_{red} G)$$

denote the image of $e_{1,1}$ under this map by $e_{1,1}\delta_1$.

proof

Given $t \in G$ and

$$x = \begin{bmatrix} a & m \\ n^* & b \end{bmatrix} \in L_t.$$

Notice that

$$(e_{1,1}\delta_1)(x\delta_t)(e_{1,1}\delta_1) = (\lambda_t\lambda_{t^{-1}}(e_{1,1}x)e_{1,1})\delta_t = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \delta_t$$

Hence,

$$(e_{1,1}\delta_1)(L \rtimes_{red} G)(e_{1,1}\delta_1) = A \rtimes_{red} G \qquad (1)$$

Consequently, $A \rtimes_{red} GL \rtimes_{red} GA \rtimes_{red} GA \rtimes_{red} G \subset A \rtimes_{red} G$.
In other words, $A \rtimes_{red} G$ is a hereditary subalgebra of $A \rtimes_{red} G$.

Proof

Claim: $A \rtimes_{red} G$ is a hereditary subalgebra of $L \rtimes_{red} G$. In other words, considering

$$L^{\rtimes} = L \rtimes_{red} G, \qquad A^{\rtimes} = A \rtimes_{red} G, \qquad p = e_{1,1}\delta_t$$

We first check that:

$$[L^{\rtimes} A^{\rtimes} L^{\rtimes}] = [L^{\rtimes} p L^{\rtimes}].$$

Note that $[L^{\rtimes} p L^{\rtimes}]$. is an ideal of $L^{\rtimes}$. So:

$$[L^{\rtimes} p L^{\rtimes}] = [L^{\rtimes} p L^{\rtimes} L^{\rtimes} p L^{\rtimes}] = [L^{\rtimes} p L^{\rtimes} p L^{\rtimes}] = [L^{\rtimes} A^{\rtimes} L^{\rtimes}].$$

Proof

Given $t \in G$, $x = \begin{bmatrix} a & m \\ n^* & b \end{bmatrix} \in L, x' = \begin{bmatrix} a' & m' \\ n'^* & b' \end{bmatrix} \in L_t$, We have:

$$(x\delta_1)(e_{1,1}\delta_1)(x\delta_t) = xe_{1,1}x'\delta_t = \begin{bmatrix} aa' & am' \\ n^*a' & \langle n, m'\rangle_B \end{bmatrix} \delta_t$$

This implies that $\begin{bmatrix} [AA_t] & [AM_t] \\ [(A_tM)^*] & [\langle M, M_t\rangle_B] \end{bmatrix} \delta_g \subseteq [L^{\rtimes} A^{\rtimes} L^{\rtimes}]$. Observe that:

- A_t is an ideal of A, so $[AA_t] = A_t$.
- M_t is a left A-module, so $[AM_t] = M_t$.
- Given $\xi \in M$, $\xi = \lim_{n\to\infty} \langle \xi, \xi\rangle^{1/n}\xi = \xi$, so $M_t \subset [A_tM]$.
- $B_t = [\langle M_t, M_t\rangle_B] \subset [\langle M, M_t\rangle_B]$

So, $[L^{\rtimes} A^{\rtimes} L^{\rtimes}]$ contains $L\delta_t$, for every $t \in G$, consequently, $L \rtimes_{red} G$.
Hence, $A \rtimes_{red} G$ is Morita-Rieffel equivalent to $L \rtimes_{red} G$.
Similarly, $B \rtimes_{red} G$ is Morita-Rieffel equivalent to $L \rtimes_{red} G$.

Proof

claim: $A \rtimes G \subset L \rtimes G$. Consider the semi direct product bundles $\mathfrak{A}, \mathfrak{L}$ associated to actions α and λ. We show that there is a conditional expextion

$$P = \{P_t\}_{t \in G} : \mathfrak{L} \to \mathfrak{A}.$$

Hence, the calim holds. Since $\mathfrak{L}$ is faithfully represented in $C^*_{red}(\mathfrak{L})$ (via $\Lambda \circ \kappa$), We can work with elements of $C^*_{red}(\mathfrak{L})$ or equivalently $L \rtimes G$. For every $t \in G$, consider

$$P_t : x \in L \rtimes G \mapsto (e_{1,1}\delta_1)x(e_{1,1}\delta_1) \in A \rtimes G.$$

By Equation 1, the map P_t is well-defined. One can easily check that $P = \{P_t\}_{t \in G}$ is a conditional expectation. Hence, the claim holds.

Theorem

Let

$$\alpha = (A, G, \{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$$

be a C*-algebraic partial action admitting a globalization η, acting on a C*-algebra B. Then:

- $A \rtimes_{red} G$ is a full hereditary subalgebra of $B \rtimes_{red} G$ in a natural way, hence, $A \rtimes_{red} G$ and $B \rtimes_{red} G$ are Morita-Rieffel equivalent.
- $A \rtimes G$ is a full hereditary subalgebra of $B \rtimes G$ in a natural way, hence, $A \rtimes G$ and $B \rtimes G$ are Morita-Rieffel equivalent.

Proof

Since A is a closed subspace of B that is β invariant:

$$A \rtimes_{red} G \subseteq B \rtimes redG.$$

$A \rtimes_{red} G$ is a full sublagebra of $B \rtimes G$:
Let

$$A^{\rtimes} = A \rtimes_{red} G, \qquad B^{\rtimes} = A \rtimes_{red} G.$$

Consider

$$J = [B^{\rtimes} A^{\rtimes} B^{\rtimes}].$$

Given $s, t \in G$, we have

$$[B\delta_s A\delta_1 B\delta_{s^{-1}t}] = [B_{\beta_s}(AB)\delta_t] = \beta_s(A)\delta_t.$$

So

$$B\delta_g \subset J.$$

Consequently,

$$J = B \rtimes_{red} G.$$

Characterizing Partial Actions
Admitting C*-globalization

Definition

$*$-**Partial Action:** Let A be a $*$-algebra. A partial action $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ on A, such that for every $t \in G$, A_t is a $*$-ideal and $\alpha_t : A_{t^{-1}} \to A_t$ is a $*$-homeomorphism.
In the case, $A_t = A$, for every $t \in G$, α is called a $*$-global action.

Definition

$*$-**globalization:** Let α be a $*$-partial action. A 4-tuple $(B, \beta, I, \mathfrak{i})$, where B is a $*$-algebra, β is a $*$-global action of G on B, I is a $*$-ideal of B and $\mathfrak{i} : \alpha \to \beta|_I$ is an isomorphism of partial actions.

A $*$-globalization $(B, \beta, I, \mathfrak{i})$ of $*$-partial action α of G on $*$-algebra A is said minimal if

$$[I] = span\{\beta_t(I) : t \in G\} = B.$$

Also, it is said to be non degenerate if B is a non degenerate $*$-algebra $(bB = 0 \to b = 0)$.

Theorem

Let α be a C^*-partial action of G on C^*-algebra A. TFAE:

- α has a $*$-globalization.
- For every $(t, a, b) \in G \times A \times A$, there is a unique $u \in A_t$, such that for every $c \in A_{t^{-1}}$, $\alpha_t(c)u = \alpha_t(ca)b$.

Theorem

Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a C*-partial action. Then, the following are equivalent:

1. α has a C*-globalization.
2. α has a $*$-globalization.
3. For every $(t, a, b) \in G \times A \times A$, there is a $u \in A_t$, such that for every $c \in A_{t^{-1}}$, $\alpha_t(c)u = \alpha_t(ca)b$.

Proof: Step1

Let $(B, \beta, I, \mathfrak{i})$ be a $*$-globalization of α. Consider A^G, the $*$-algebra of all functions from G to A. Consider $\pi : B \to M(A^G)$ defined by

$$\pi(b)f|_r = \mathfrak{i}^{-1}(\beta_r(b)\mathfrak{i}(f|_r)).$$

Also, consider the canonical action of G on $M(A^G)$,
$\Theta : G \to \mathrm{Auto}(M(A^G))$, defined by

$$\Theta_t(L, R) = (\theta_t \circ L \circ \theta_{t^{-1}}, \theta_t \circ R \circ \theta_{t^{-1}}).$$

Where, θ_t is the automorphism of A^G, defined by: $\theta_t(f)|_r = f|_{rt}$.

Step 2

The set of bounded functions from G to A, A_b^G is a C*-algebra with the $*$-algebra structure inherited from A^G and the sup norm. Define:

$$C := \{T \in M(A^G) : T(A_b^G) \cap T^* A_b^G \subset A_b^G\}.$$

We have:

1. C is Θ invariant.
2. $\pi(B) \subseteq C$.

Step2

Note that π is injective. Assume $b \in ker(\pi)$. Given $a \in A, g \in G$, consider $a\delta_r \in A^G$ taking the value a at r and 0, otherwise. Then

$$0 = \mathfrak{i}(\pi(b)a\delta_r|_r) = \beta_r(b)\mathfrak{i}(a).$$

This implies that $bB = \operatorname{span} b\beta_r(\mathfrak{i}(A)) = 0$. Hence, $b = 0$. Since $(B, \beta, I, \mathfrak{i})$ is a non-degenerate $*$-globalization.
Consider $M(A_b^G)$ as a C*-algebra and let

$$\rho : \pi(B) \to M(A_b^G), \qquad \rho(T)f = Tf.$$

ρ is injective: In order to show it, it suffices to show that $\rho \circ \pi$ is injective. Let $b \in B$, given that $[I] = B$, we have there are $t_1, t_2, ..., t_n \in I$ and $a_1, a_2, ..., a_n \in I$, such that $b = \Sigma_{i=1}^n \beta_{t_i}(b_i)$. Given $r \in G$ and $c \in A$

$$b\beta_r(c) = \beta_r(\beta_{r^{-1}}(b)c) = \beta_r(\rho \circ \pi(b)\delta^c_{r^{-1}}|_{r^{-1}}) = 0 \to \ bB = 0 \ \to b = 0.$$

Step2

Given $t \in G$, set

$$\psi_t : A_b^G \to A_b^G, \qquad \psi_t(f)|_r = f|_{rt}.$$

Also, let

$$\Psi_t : M(A_b^G) \to M(A_b^G) \qquad \Psi(T) = \psi_t \circ T \circ \psi_{t^{-1}}.$$

- Ψ is a C*-global action of G on $M(A_b^G)$.
- $\rho \circ \pi : \beta \to \Psi$ is a morphism of C^*-partial actions.

Step2

- Let $D = \overline{\rho \circ \pi(B)}$.

  D is a Ψ invariant C*-subalgebra of $M(A_b^G)$.
- Let $\gamma = \Psi|_D$.
- Let $J = \rho(\pi(A))$.

  J is a C*-ideal of D because $\rho \circ \pi|_A$ has a closed range and J is an ideal of $\rho \circ \pi(B)$.

We have $\gamma|_J = \Psi|_D|_{rho \circ \pi}(A) = \Psi|_{\rho \circ \pi(A)} = \Psi|_{\rho \circ \pi(B)}|_{\rho \circ \pi(A)}$. Besides,

$$\rho : \Theta|_{\pi(B)} \to \Psi|_{\rho \circ \pi(B)} \text{is an isomorphism.}$$

Then,

$$\rho \circ \pi|_A : \alpha \to \Psi|_J \text{ is an isomorphism of partial actions.}$$

Then, $(D, \gamma, J, \rho \circ \pi|_A)$ is a C*-globalization of α.

Thank you for your attention!