# Rokhlin dimension for compact group actions

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Some of this is joint work with Ilan Hirshberg and Luis Santiago.

For positive elements $a, b \in A$, write $a \perp b$ if $ab = ba = 0$.

Let $A$ and $B$ be $C^*$-algebras, and let $\varphi\colon A \to B$ be a cp map. We say that $\varphi$ has *order zero* if $a, b \in A_+$ and $a \perp b$ imply $\varphi(a) \perp \varphi(b)$.

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Let $A$ and $B$ be $C^*$-algebras. A cpc order zero map $\varphi\colon A \to B$ induces the homomorphism $\rho_\varphi\colon C_0((0,1]) \otimes A \to B$ determined by

$$\rho_\varphi(\mathrm{id}_{(0,1]} \otimes a) = \varphi(a) \quad a \in A.$$

Conversely, if $\rho\colon C_0((0,1]) \otimes A \to B$ is a homomorphism, then the induced cpc order zero map $\varphi_\rho\colon A \to B$ is

$$\varphi_\rho(a) = \rho(\mathrm{id}_{(0,1]} \otimes a) \quad a \in A.$$

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We say that $\alpha$ has *Rokhlin dimension $d$*, written $\dim_{\mathrm{Rok}}(\alpha) = d$, if $d$ is the least integer such that there exist equivariant completely positive contractive order zero maps

$$\varphi_0, \ldots, \varphi_d \colon C(G) \to A_{\infty,\alpha} \cap A'$$

such that $\varphi_0(1) + \ldots + \varphi_d(1) = 1$.
If one can choose the maps $\varphi_0, \ldots, \varphi_d$ to have commuting ranges, then we say that $\alpha$ has *Rokhlin dimension with commuting towers $d$*, and denote it $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) = d$.

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It is not hard to check that this definition agrees with that of Hirshberg-Winter-Zacharias for finite groups.

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Let $\alpha\colon G \to \mathrm{Aut}(A)$ be any action.

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2. Let $I$ be an $\alpha$-invariant ideal in $A$, and set $\overline{\alpha}\colon G \to \mathrm{Aut}(A/I)$. Then
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Furthermore,

3. Let $(A_n, \iota_n, \alpha^{(n)})_{n\in\mathbb{N}}$ be an equivariant direct system. Set $A = \varinjlim A_n$ and $\alpha = \varinjlim \alpha^{(n)}$. Then
$$\dim_{\mathrm{Rok}}(\alpha) \leq \liminf_{n\to\infty} \dim_{\mathrm{Rok}}(\alpha^{(n)}).$$

# Rokhlin dimension and subgroups

Let $G$ be a finite dimensional compact group, let $H$ be a closed subgroup of $G$, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then

$$\dim_{\mathrm{Rok}}(\alpha|_H) \leq (\dim(G) - \dim(H) + 1)(\dim_{\mathrm{Rok}}(\alpha) + 1) - 1.$$

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The key point is the existence of local cross-sections $G/H \to G$. One assembles these to get cpc order zero equivariant maps $C(H) \to C(G)$, which are then combined with the ones corresponing to $\alpha$ to get the result.

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Rokhlin dimension can increase when passing to subgroups. Also, there are examples of circle actions such that all of the restrictions to $\mathbb{Z}_n \subseteq \mathbb{T}$ have Rokhlin dimension zero, but the action has infinite Rokhlin dimension.

# Free actions on spaces and Rokhlin dimension

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1. If $\alpha$ has finite Rokhlin dimension, then $G \curvearrowright X$ is free.
2. If $G \curvearrowright X$ is free, then $\dim_{\mathrm{Rok}}(\alpha) < \infty$. Moreover, if $\dim(X) < \infty$, we have
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Let $G$ be a Lie group, and let $G \curvearrowright X$ be a free action with $\dim(X) = \dim(G)$. Then $\alpha\colon G \to \mathrm{Aut}(C(X))$ has the Rokhlin property. We even have $X \cong X/G \times G$ equivariantly.

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A bit of notation: We write $K_*^G(A)$ for the equivariant $K$-theory of $A$, and $I_G$ for the augmentation ideal in $R(G)$, which is the kernel of the dimension homomorphism $R(G) \to \mathbb{Z}$.

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If $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) < \infty$, then $\alpha$ has *discrete $K$-theory*, this is, there is $n \in \mathbb{N}$ such that

$$I_G^n \cdot K_*^G(A) = 0.$$

This uses (still unpublished) work of Hirshberg and Phillips.

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Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action.

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3. We say that $\alpha$ is *totally $K$-free* if for every closed subgroup $H$ of $G$, the restriction $\alpha|_H$ is $K$-free.

# Comparison with other notions of freeness

For a Lie group $G$, we have the following implications:

$$
\begin{tikzcd}
 & X-\text{Rokhlin property} \arrow[d, Leftrightarrow] & \dim_{\mathrm{Rok}}(\alpha) < \infty \arrow[r, Rightarrow] & \text{outerness} \\
\text{Rok prop} \arrow[r, Rightarrow] & \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty \arrow[r, Rightarrow] \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \text{Discrete } K\text{-theory} & \\
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None the above arrows can be reversed in full generality, and presumably there are no other implications between the stated conditions.

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3. If $A$ is a Kirchberg algebra and $G = \mathbb{Z}_2$ (and possibly also if $G$ is any finite group), then $\dim_{\mathrm{Rok}}(\alpha) < \infty$ is equivalent to outerness (Barlak-Enders-Matui-Szabo-Winter).

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4. If $G$ is finite, $A$ has strict comparison and at most countably many extreme traces, then $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$ implies the weak tracial Rokhlin property (G-Hirshberg-Santiago).

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1. (G. for arbitrary $G$; H-W-Z when $G$ is finite.) We have

$$\mathrm{dr}(A \rtimes_\alpha G) \leq (\mathrm{dr}(A) + 1)(\dim_{\mathrm{Rok}}(\alpha) + 1) - 1$$

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2. (G-Hirshberg-Santiago) If $A \otimes \mathcal{D} \cong A$ for $\mathcal{D}$ s.s.a. and $\dim_{\mathrm{Rok}}^{c}(\alpha) < \infty$,

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$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty \Leftrightarrow \alpha \text{ has the Rokhlin property.}$$

Thank you.