Group representations on L^p-spaces

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Joint work with Hannes Thiel from the University of Münster

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Example

Let $\lambda_p \colon G \to \mathcal{B}(L^p(G))$ be given by

$$(\lambda_p)_g(\xi)(h) = \xi(g^{-1}h) \;\; \forall g, h \in G, \; \forall \xi \in L^p(X, \mu).$$

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Example

The integrated form of λ_p is

$$\lambda_p(f)(\xi) = f * \xi$$

for $f \in L^1(G)$ and $\xi \in L^p(X,\mu)$.

Group algebras

Definition (Group L^p-operator algebras)

Let $\lambda_p \colon L^1(G) \to \mathcal{B}(L^p(G))$ be the integrated form of the left regular representation: $\lambda_p(f)\xi = f * \xi$ for $f \in L^1(G), \xi \in L^p(G)$.

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The *full group algebra* $F^p(G)$ is the completion of $L^1(G)$ in

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Duality

For $p > 1$, there are canonical isometric isomorphisms

$$F^p(G) \cong F^{p'}(G) \quad \text{and} \quad F^p_\lambda(G) \cong F^{p'}_\lambda(G).$$

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For example, we don't know whether the canonical map $F^p(G) \to F^p_\lambda(G)$ is surjective!

Proposition (Implicit in work of Herz)

When $p = 1$, we have

$$L^1(G) = F^1_\lambda(G) = F^1(G).$$

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(Not true for $p = 1$ by the proposition above.)

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For the rest: $L^1(G) \to F^p(G) \to C^*(G)$.
(Probably true for arbitrary G when $p \neq 2$.)

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In other words, homomorphism $F^p_\lambda(G) \to F^p_\lambda(H)$ come from homomorphisms $G \to H$.

Consequences

Recall: homomorphism $F^p_\lambda(G) \to F^p_\lambda(H)$ come from homomorphisms $G \to H$, for $p \neq 2$.

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Proof: bicontractive idempotents correspond to unital homomorphisms from $F^p(\mathbb{Z}_2)$.
Whether this is true for $p = 2$ is the content of the Kadison-Kaplansky conjecture.

Thank you.