The Equivariant Cuntz semigroup

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Joint work with Luis Santiago from the University of Aberdeen

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One shows that $\mathrm{Cu}(A)$ is a partially ordered abelian semigroup, and that $A \mapsto \mathrm{Cu}(A)$ is a functor from the category of C^*-algebras to a certain category **Cu** of such semigroups.

The representation semiring

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$\mathrm{Cu}(G)$ is a unital **Cu**-semiring under direct sum and tensor product.

The Equivariant Cuntz semigroup

Recall: $\mathrm{Cu}(G)$ is the set of equivalence classes of separable unitary representations of G.

Definition

Let G be a compact group, let A be a C^*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$. The *equivariant Cuntz semigroup* $\mathrm{Cu}^G(A, \alpha)$ is defined using G-invariant positive elements in $A \otimes \mathcal{K}(\mathcal{H}_\mu)$, where μ ranges over all unitary representations of G, and $A \otimes \mathcal{K}(\mathcal{H}_\mu)$ has the diagonal action of G.

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Cu^G resembles K_*^G.

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1. if $x \in S$, then $1 \cdot s = s$;
2. if $x, y \in S$ and $s, t \in \mathrm{Cu}(G)$ satisfy $x \leq y$ and $r \leq s$, then $r \cdot x \leq s \cdot y$;

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$$\sup_{n\in\mathbb{N}}(r_n \cdot x_n) = \left(\sup_{n\in\mathbb{N}} r_n\right) \cdot \left(\sup_{n\in\mathbb{N}} x_n\right);$$

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Alternative pictures

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$\mathrm{Cu}^G(A,\alpha)$ is naturally isomorphic in $\mathbf{Cu}^G$ to the direct limit

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When G is abelian, the $\mathrm{Cu}(G)$-semimodule structure is easy to describe: an element $\chi \in \widehat{G}$ acts via

$$\chi \cdot s = \mathrm{Cu}(\widehat{\alpha}_\chi)(s)$$

for $s \in \mathrm{Cu}(A \rtimes_\alpha G)$.

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Let G act on a locally compact space X. Then $X \to X/G$ induces

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Then (1) implies (2), and if G is a Lie group and X is compact, then also (2) implies (1).

Cu^G as an invariant for group actions

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Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. An α-*cocycle* is a strongly continuous function $\omega\colon G \to \mathcal{U}(M(A))$ such that $\omega_{gh} = \omega_g \alpha_g(\omega_h)$ for all $g, h \in G$.

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In some special cases, actions with isomorphic Cu^G are cocycle conjugate.

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Classification of finite group actions

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A C^*-algebra is in Robert's class $\mathcal{R}$ if it is a direct limit of 1-dimensional NCCW-complexes with trivial K_1-groups.

This includes all AF- and all AI-algebras.

Theorem (Robert, 2010)

Unital algebras in $\mathcal{R}$ are classified by their Cuntz semigroup.

(Instead of A being unital, can require $A \otimes \mathcal{K}$ having an approximate identity of projections.)

Classification of finite group actions

Definition

Let $A \in \mathcal{R}$, and write it $A = \varinjlim A_n$ as in the definition. If G is finite, an action $\alpha\colon G \to \mathrm{Aut}(A)$ is called *locally representable* if it is the direct limit of inner actions on A_n.

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$$\rho\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(B,\beta) \ \text{ with } [1_A] \mapsto [1_B]$$

there are a β-cocycle ω and $\phi\colon (A,\alpha) \to (B,\beta^\omega)$ lifting ρ.

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Here e_α is the projection $e_\alpha = \frac{1}{|G|}\sum_{g\in G} u_g$ in the crossed product $A \rtimes_\alpha G$, and similarly for e_β.

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(Unitality can be replaced by $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ having an approximate identity of projections.)

Thank you.