QUASITRACES AND AW^*-ALGEBRAS

Abstract. These are lecture notes of a talk given in the kleines Seminar in Münster. The talk is based on Parts I and II of [BH82], and Sections 3 and 4 of [Haa14].

Contents

1. Introduction

The notion of finiteness is crucial in the study of operator algebras. The weakest form of finiteness is the requirement that $x^*x = 1$ implies $xx^* = 1$, and the strongest is the existence of a separating family of tracial states. For von Neumann algebras, these are equivalent, and this fact plays an important role in the classification of factors by Murray and von Neumann. Cuntz's groundbreaking work on dimension functions [Cun81] was a big step forward in the understanding of the notion of finiteness for C^*-algebras. He defined a partially ordered group $K_0^*(A)$ for a C^*-algebra A, and showed that that its states are identified with what he called *dimension* (or *rank*) *functions* on A.

Blackadar and Handelman clarified the relation between rank functions and (quasi)traces. Their extensive study of quasitraces was the starting point in Haagerup's work [Haa14], where he showed that every quasitrace on a unital exact C^*-algebra is a trace.

There are two goals in this lecture: first, to explain the correspondence between rank functions and quasitraces, and second, to prove the following result:

Theorem. (Blackadar-Handelman; Haagerup). Let A be a unital C^*-algebra and let τ be a quasitrace on A. Then there exist a finite AW^*-algebra M_τ, a faithful normal quasitrace $\overline{\tau}$ on M_τ, and a unital embedding $\psi\colon A \to M_\tau$ such that $\tau = \overline{\tau} \circ \psi$. Moreover, M_τ is the smallest such AW^*-algebra. Finally, if τ is an extreme quasitrace, then M_τ is a factor.

2. Rank functions and quasitraces

We begin with the definition of rank functions. Let A be a unital C^*-algebra. For positive elements $a, b \in A$, we write $a \precsim b$ if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_n b x_n^* \to a$.

Definition 2.1. A function $D\colon A \to [0, 1]$ is said to be a *rank function* if

(1) $D(a + b) = D(a) + D(b)$ whenever $a \perp b$;
(2) $D(a) = D(a^*a) = D(aa^*) = D(a^*)$ for all $a \in A$;
(3) $0 \leq a \leq b$ implies $D(a) \leq D(b)$;
(4) $a \precsim b$ implies $D(a) \leq D(b)$

Moreover, we say that D is

(a) *Normalized*, if $\sup\limits_{a\in A} D(a) = 1$.
(b) *Subadditive*, if $D(a + b) \leq D(a) + D(b)$ for all $a, b \in A$.
(c) *Weakly subadditive*, if $D(a + b) \leq D(a) + D(b)$ whenever $ab = ba$.
(d) *Lower semicontinuous*, if $D^{-1}((\lambda, 1])$ is open in A for $\lambda \in [0, 1)$.

Date: April 26, 2016.

(Dimension functions are defined analogously, using $M_\infty(A)$ as the domain and $[0,\infty)$ as the codomain. We will not need these in this lecture.)

The *kernel* of D is $\ker(D) = \{a \in A \colon D(a) = 0\}$, and D is said to be *faithful* if $\ker(D) = \{0\}$.

Lower semicontinuity guarantees that $\ker(D)$, which is always a 2-sided $*$-ideal, is closed. By Theorem I.1.17 in [BH82], a lower semicontinuous rank function D on A induces a lower semicontinuous faithful rank function on $A/\ker(D)$.

The following is Corollary 4.7 in [Cun81].

Theorem 2.2. (Cuntz). Let A be a simple unital C^*-algebra. Then A is stably finite if and only if there exists a lower semicontinuous, normalized, faithful, subadditive rank function on A.

In analogy to what happens in von Neumann algebras, one would like to conclude that A has a trace. While this is not known to be the case in general, one can conclude that A has a quasitrace (roughly speaking, this is a trace that is only assumed to be linear on commutative subalgebras). Here is the precise definition:

Definition 2.3. Let A be a unital C^*-algebra. A *1-quasitrace* on A is a function $\tau \colon A \to \mathbb{C}$ satisfying

(1) $\tau(x^*x) = \tau(x^*x) \geq 0$ for all $x \in A$;
(2) $\tau(a + ib) = \tau(a) + i\tau(b)$ for $a, b \in A_{\mathrm{sa}}$;
(3) τ is linear on every abelian subalgebra of A.

We say that τ is *normalized* if $\tau(1) = 1$. For $n \geq 2$, we say that τ is an *n-quasitrace* if it extends to a 1-quasitrace on $M_n(A)$.

We denote by $\mathrm{QT}(A)$ the set of all 2-quasitraces.

Since 1-quasitraces are continuous (not obvious), it follows that a linear quasitrace is in fact a trace. Also, a 2-quasitrace is an n-quasitrace for all $n \in \mathbb{N}$.

Question 2.4. (Kaplansky). Is every 2-quasitrace a trace?

Our next immediate goal is to show that there is a canonical correspondence between rank functions and quasitraces. We start with the commutative case:

Proposition 2.5. Let X be a locally compact Hausdorff space. Then there is a canonical correspondence between subadditive lower semicontinuous rank functions on $C_0(X)$ and countably additive measures on X with σ-compact support, defined on the σ-algebra generated by σ-compact open sets.

Proof. Given μ, set $D_\mu(f) = \mu(\{x \in X \colon f(x) \neq 0\}$. Given D, define μ_D as follows. For $U \subseteq X$ open and σ-compact, find $h \in C_0(X)$ with $U = \{x \in X \colon h(x) \neq 0\}$, and set $\mu_D(U) = D(f)$. Then D is well defined by property (4) in Definition 2.1. ☐

Theorem 2.6. (Theorem II.2.2 in [BH82]). There is a natural affine bijection between weakly subadditive lower semicontinuous rank functions on A and quasitraces on A.

Moreover, the subadditive rank functions correspond to the 2-quasitraces.

Proof. We give a sketch. Suppose D is as in the statement, and define τ_D as follows. If $B \subseteq A$ is commutative, then D induces a positive functional τ_D^B on B by Proposition 2.5. This defines τ_D on normal elements. For general $x \in A$, write $x = a + ib$ with $a, b \in A_{\mathrm{sa}}$, and set $\tau_D(x) = \tau_D(a) + i\tau_D(b)$.

It remains to show that $\tau(x^*x) = \tau(xx^*)$. For this, and by the correspondence given in Proposition 2.5, it suffices to show that for all nonnegative $f \in C(\mathrm{sp}(x^*x) \cup \{0\})$ vanishing at 0, we have $D(f(x^*x)) = D(f(xx^*)$. Represent A on a Hilbert space $\mathcal{H}$, and consider the polar decomposition $x = u|x|$ of x in $\mathcal{B}(\mathcal{H})$. Then $y = u(f(x^*x))^{1/2}$ belongs to A and $y^*y = f(x^*x)$ and $yy^* = f(xx^*)$. Since $D(y^*y) = D(yy^*)$, we are done.

Conversely, given τ, set $D_\tau(a) = \sup\limits_{\varepsilon > 0} \tau(f_\varepsilon(|a|))$ for $a \in A$. Then D_τ is a weakly subadditive lower semicontinuous rank function. We omit the details. ☐

3. Minimal AW^*-completions

In this section, we will explain how every 2-quasitrace on a unital C^*-algebra comes from a AW^*-algebra. We write $\mathrm{QT}(A)$ for the set of all 2-quasitraces on A.

Definition 3.1. Let $\tau \in \mathrm{QT}(A)$. Set $\|x\|_{\tau,2} = \tau(x^*x)^{1/2}$ for $x \in A$.

Lemma 3.2. Let $\tau \in \mathrm{QT}(A)$, and let $x, y \in A$. Then

(1) $\tau(a+b)^{1/2} \leq \tau(a)^{1/2} + \tau(b)^{1/2}$.
(2) $\|x+y\|_{\tau,2}^{2/3} \leq \|x\|_{\tau,2}^{2/3} + \|y\|_{\tau,2}^{2/3}$.
(3) $\|xy\|_{\tau,2} \leq \|x\|\|y\|_{\tau,2}$ and $\|xy\|_{\tau,2} \leq \|x\|_{\tau,2}\|y\|$.

Proof. We only sketch (2). For $\lambda > 0$, we have

$$(x+y)^*(x+y) \leq (x+y)^*(x+y) + (\lambda^{1/2}x - \lambda^{-1/2}y)^*(\lambda^{1/2}x - \lambda^{-1/2}y) = (1+\lambda)x^*x + (1+1/\lambda)y^*y.$$

Using (1), it follows that $\|x+y\|_{\tau,2} \leq (1+\lambda)^{1/2}\|x\|_{\tau,2} + (1+1/\lambda)^{1/2}\|y\|_{\tau,2}$. Minimize on λ: we get $\lambda = \left(\frac{\|y\|_{\tau,2}}{\|x\|_{\tau,2}}\right)^{3/2}$ and the minimum value gives

$$\|x+y\|_{\tau,2} \leq \left(\|x\|_{\tau,2}^{2/3} + \|y\|_{\tau,2}^{2/3}\right)^{3/2}.$$

□

Definition 3.3. For a unital C^*-algebra A and a faithful quasitrace τ on A, define a metric d_τ on A by $d_\tau(x,y) = \|x-y\|_{2,\tau}^{2/3}$.

The sum, the involution, and $\tau|_{A_+}$ are continuous in the d_τ metric. Moreover, the product is (jointly) continuous on norm-bounded subsets.

Proposition 3.4. Let τ be a faithful quasitrace on a unital C^*-algebra A. Then the unit ball A_1 of A is closed in the d_τ metric.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A_1 converging to $x \in A$ in d_τ. Set $a_n = x_n^*x_n$ and $a = x^*x$. By the comments above (d_τ-continuity of sum, bounded product and τ on A_+), we deduce that

$$d_\tau\text{-}\lim_{n\to\infty} \tau(f(a_n)) = f(a)$$

for every polynomial f. Let μ_n be the measure on $\mathrm{sp}(a_n)$ given by $\tau|_{C^*(a_n,1)}$, and let μ be the measure on $\mathrm{sp}(a)$ given by $\tau|_{C^*(a,1)}$. We regard them as measures on $J = [0, \max\{1, \|a\|\}]$. Then

$$\text{weak}^*\text{-}\lim_{n\to\infty} \mu_n = \mu$$

as elements in $C(J)^*$. Since μ_n is supported on $[0,1]$ for all n, we deduce that the same holds for μ. Since τ is faithful, we must have $\mathrm{supp}(\mu) = \mathrm{sp}(a)$. Hence $\|x\|^2 = \|a\| \leq 1$, and the proof is finished. □

We turn to AW^*-algebras. Recall that a compact Hausdorff space X is said to be *stonean* if the closure of every open subset of X is open. A Stonean space X is called *hyperstonean* if $C(X)$ is isomorphic, as a Banach space, to the dual of some Banach space E.

Definition 3.5. Let A be a unital C^*-algebra. We say that A is an *AW^*-algebra* if every maximal abelian subalgebra of A has stonean spectrum.

Since the spectrum of an abelian von Neumann algebra is hyperstonean, and a maximal abelian subalgebra of a von Neumann algebra is a von Neumann algebra, it follows that a von Neumann algebra is an AW^*-algebra.

The following will be very relevant. Recall that a C^*-algebra B with a faithful trace is a von Neumann algebra if and only if its norm unit ball is complete in the 2-norm associated to that trace. (And moreover the trace is automatically normal.)

Theorem 3.6. Let τ be a faithful quasitrace on a unital C^*-algebra A. Then A is an AW^*-algebra and τ is normal if and only if the unit ball of A is complete in d_τ.

Proof. We only prove part of the "if" implication. Suppose that A_1 is complete in d_τ, and let B be a maximal abelian subalgebra of A. First, let's observe that B is d_τ-closed in A: if $a \in A$ is a d_τ-limit of elements in b, it follows from continuity of the product on bounded sets that a commutes with every element in B. Since B is maximal abelian, we conclude that $a \in B$, so B is d_τ-closed in A.

By Proposition 3.4, B_1 is d_τ-closed in B, so B_1 is d_τ-closed in A. In particular, B_1 is d_τ-closed in A_1. Since A_1 is d_τ-complete, the same holds for B_1. Now, τ restricts to a (faithful) trace on B, and d_τ-completeness of a bounded set is the same as completeness with respect to the associated 2-norm on B. It follows that B is a von Neumann algebra, and in particular $\mathrm{sp}(B)$ is (hyper)stonean. Hence A is an AW^*-algebra. □

Theorem 3.7. Let M be an AW^*-algebra with a faithful normal quasitrace τ, and let A be a C^*-subalgebra of M. Then the d_τ closure B of A in M is the smallest AW^*-subalgebra of M containing A.

Proof. One can show, using an argument similar to the proof of Kaplansky's density theorem, that A_1 is dense in B_1. In particular, B_1 is d_τ-complete and B is an AW^*-algebra by Theorem 3.6. If C is another AW^*-algebra with $A \subseteq C$, then C_1 is d_τ-complete again by Theorem 3.6, and thus C is d_τ-closed. Hence $\overline{A}^{d_\tau} = B \subseteq C$. □

Theorem 3.8. Let A be a unital C^*-algebra and let τ be a faithful quasitrace on A. Then there exist an AW^*-algebra M, a normal faithful quasitrace σ on M, and an embedding $\varphi\colon A \to M$ such that $\tau = \sigma \circ \varphi$.

Proof. Let ω be a free ultrafilter on $\mathbb{N}$, and define a (rarely faithful) quasitrace τ^ω on $\ell^\infty(A)$ by

$$\tau^\omega((x_n)_{n\in\mathbb{N}}) = \lim_\omega \tau(x_n)$$

for $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$. Set

$$J_{\tau,\omega} = \{(x_n)_{n\in\mathbb{N}}\colon \lim_\omega \tau_n(x_n^* x_n) = 0\},$$

which is a closed two-sided ideal in $\ell^\infty(A)$. Set $A_\omega = \ell^\infty(A)/J_{\tau,\omega}$. Then τ^ω induces a faithful quasitrace $\tau_\omega\colon A_{\tau,\omega} \to \mathbb{C}$. If $\varphi\colon A \to \ell^\infty(A) \to A_{\tau,\omega}$ denotes the composition of the canonical maps, then φ is injective because τ is faithful, and it is clear that $\tau = \tau_\omega \circ \varphi$.

We claim that $A_{\tau,\omega}$ is an AW^*-algebra. It is not hard to see that the unit ball of $A_{\tau,\omega}$ is d_{τ_ω}-complete, using a reindexation argument, and the fact that it is the ultrapower of the metric spaces A_1. The result then follows from Theorem 3.6. □

The AW^*-algebra from the previous theorem is not unique, and it is generally too large.

By Theorem 3.7, the d_σ-closure M_τ of $\varphi(A)$ in M is the smallest AW^*-subalgebra of M containing A. With

$$\ell^\infty_\tau(A) = \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(A)\colon (x_n)_{n\in\mathbb{N}} \text{ is a } d_\tau\text{-Cauchy sequence}\}$$

and

$$I_\tau = \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(A)\colon d_\tau\text{-}\lim_{n\to\infty} x_n = 0\},$$

it is easy to see that $M_\tau \cong \ell^\infty_\tau(A)/I_\tau$. Moreover, there is a faithful quasitrace $\overline{\tau}$ on M_τ induced by τ, given by

$$\overline{\tau}((x_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} \tau(x_n).$$

With $\psi\colon A \to \ell^\infty_\tau(A) \to M_\tau$ denoting the canonical inclusion, it follows that $\tau = \overline{\tau} \circ \psi$. This is the AW^*-algebra we were looking for.

Definition 3.9. We call the pair $(M_\tau, \overline{\tau})$ the *(minimal) AW^*-completion* of (A, τ).

For a projection p in a unital C^*-algebra, we write $p^\perp$ for $1 - p$.

Proposition 3.10. Let τ be a faithful normalized quasitrace on a unital C^*-algebra A. If τ is an extreme point in $\mathrm{QT}(A)$, then M_τ is an AW^*-factor (it has trivial center).

Proof. Suppose there is a central projection $p \in Z(M_\tau)\setminus\{0,1\}$. Set $\tau_1(x) = \overline{\tau}(p\psi(x))$ and $\tau_2(x) = \overline{\tau}(p^\perp\psi(x))$ for all $x \in A$. Then $\tau_1 \neq 0 \neq \tau_2$. Since p and $p^\perp$ are central, it follows that $\tau = \tau_1 + \tau_2$. Upon normalizing τ_1 and τ_2, we conclude that τ is not extreme. This contradiction implies the result. □

References

[BH82] B. Blackadar, D. Handelman, *Dimension functions and traces on C^*-algebras*, J. Funct. Anal. **45** (1982), 297–340.

[Cun81] J. Cuntz, *Dimension functions on simple C^*-algebras*, Math. Ann. **233** (1981), 145153.

[Haa14] U. Haagerup, *Quasitraces on exact C^*-algebras are traces*, C. R. Math. Rep. Acad. Sci. Canada **36** (2014), 67–92.