The continuous Rokhlin property and permanence of the UCT

Eusebio Gardella

University of Oregon and Fields Institute

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Definition

An action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has the *Rokhlin property* if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in A such that

1. $\lim\limits_{n\to\infty} \|\alpha_\zeta(u_n) - \zeta u_n\| = 0$ uniformly in $\zeta \in \mathbb{T}$.
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Theorem

If $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has the Rokhlin property, then α is the dual action of some automorphism $\check{\alpha}$ of A^α. This automorphism $\check{\alpha}$ is unique up to cocycle equivalence and is approximately inner.

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Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ have the Rokhlin property. If A is a Kirchberg algebra and both A and A^α satisfy the UCT, then the Ext-class of the previous theorem is a complete invariant for α.
Without the UCT, a complete invariant is

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Answering these questions is the main motivation of this work.

The continuous Rokhlin property

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Definition

An *asymptotic morphism* from A to B, is a family $\psi = (\psi_t)_t$ of maps $A \to B$, satisfying:

1. $t \mapsto \psi_t(a)$ is continuous for all a in A
2. For every λ in $\mathbb{C}$ and every a and b in A, we have

$$\lim_{t\to\infty} \|\psi_t(\lambda a + b) - \lambda\psi_t(a) - \psi_t(b)\| = 0,$$

$$\lim_{t\to\infty} \|\psi_t(ab) - \psi_t(a)\psi_t(b)\| = 0, \quad \text{and} \quad \lim_{t\to\infty} \|\psi_t(a^*) - \psi_t(a)^*\| = 0.$$

Asymptotic retraction $A \to A^\alpha$

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Asymptotic retraction $A \to A^\alpha$

If $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has the continuous Rokhlin property, then there exists an asymptotic morphism $\psi = (\psi_t)_t\colon A \to A^\alpha$ such that

$$\lim_{t\to\infty} \|(\psi_t \circ \iota)(a) - a\| = 0$$

for all a in A^α.

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Corollary

If B is separable and nuclear, there is an isomorphism

$$E(B, A) \cong E(B, A^\alpha) \oplus \ker(\psi_*)$$

induced by $\psi_*\colon E(B, A) \to E(B, A^\alpha)$.

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Triviality of the Ext-class

If A is nuclear, then there are isomorphisms

$$K_0(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha) \cong K_1(A).$$

In particular, the Ext-class associated to an action with the *continuous* Rokhlin property is trivial.

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If this is the case, by setting $\varepsilon_{A,B} = \mu_{A,B}^{-1}$, we obtain

$$0 \longrightarrow \mathrm{Ext}(K_*(A), K_{*+1}(B)) \xrightarrow{\varepsilon_{A,B}} KK(A,B) \xrightarrow{\tau_{A,B}} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0.$$

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3. $A \rtimes_\alpha \mathbb{T}$ satisfies the UCT.

Proof: (2) $\iff$ (3) and (2) $\Rightarrow$ (1) are easy.

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For (1) $\Rightarrow$ (2): use $\psi = (\psi_t)_t\colon A \to A^\alpha$ to get a diagram

$$\begin{tikzcd}
KK(A,B) \arrow[r, "\tau_{A,B}"] \arrow[d, bend left, "\iota^*"] & \mathrm{Hom}(K_*(A), K_*(B)) \arrow[d, bend left, "\iota^*"] \\
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which commutes by naturality. An easy diagram chase gives that $\tau_{A^\alpha,B}$ is surjective. The argument for $\mu_{A^\alpha,B}$ is analogous.

Recall the classification of Rokhlin actions on Kirchberg algebras:

Theorem

Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ have the Rokhlin property. If A is a Kirchberg algebra and both A and A^α satisfy the UCT, then its Ext-class is a complete invariant for α. Without the UCT, a complete invariant is

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The range of the invariant for *continuous* Rokhlin actions can be described:

(a) With UCT: every trivial extension arises (and only these).

(b) Without UCT: every Kirchberg algebra is the fixed point algebra of some continuous Rokhlin action, and $KK(\check{\alpha}) = 1$.

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Thank you.