RADON, COSINE AND SINE TRANSFORMS ON GRASSMANNIAN MANIFOLDS

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ABSTRACT. Let $G_{n,r}(\mathbb{K})$ be the Grassmannian manifold of k-dimensional $\mathbb{K}$-subspaces in $\mathbb{K}^n$ where $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ is the field of real, complex or quaternionic numbers. We consider the cosine and sine transforms, $\mathcal{C}_{r',r}$ and $\mathcal{S}_{r',r}$, from L^2-functions on $G_{n,r}(\mathbb{K})$ to functions on $G_{n,r'}(\mathbb{K})$ for $r,r'\le n-1$. We prove two Bernstein-Sato type formulas on general root system of type BC for the sine and cosine type functions on the compact torus $\mathbb{R}^r/2\pi Q^\vee$ generalizing our recent results for the hyperbolic sine and cosine functions on the non-compact space $\mathbb{R}^r$. We find the spectral symbol of the cosine and sine transforms and we fine the characterization of their images. Our results generalize those of Alesker-Bernstein and Grinberg. We prove also that the Knapp-Stein intertwining operator for certain induced representations is given by the sine transform and we give the unitary structure of the Stein's complementary series in the compact picture.

1. INTRODUCTION

The present paper is a continuation of our earlier papers [25], [21] on Radon and related transforms and its relation to the Bernstein-Sato type formulas. We study the Radon, cosine and sine transforms on real complex and quaternionic Grassmannian manifolds. Let $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ be the field of real, complex or quaternionic numbers, and $G_{n,r}(\mathbb{K})$ be the Grassmannian manifold of k-dimensional subspaces over $\mathbb{K}$ in $\mathbb{K}^n$. For $1\le r\le r'\le n-1$, $\nu\ge 0$, the Radon, cosine and sine transforms $\mathcal{R}: C^\infty(G_{n,r}(\mathbb{K}))\to C^\infty(G_{n,r'}(\mathbb{K}))$ are defined, for $\eta\in G_{n,r'}(\mathbb{K})$, by

$$(\mathcal{R}_{r',r}f)(\eta)=\int_{\xi\subset\eta} f(\xi)d_\eta\xi, \tag{1.1}$$

$$(\mathcal{C}_{r',r}f)(\eta)=\int_{G_{n,r}} |\mathrm{Cos}(\xi,\eta)|^{2\nu} f(\xi)d_\eta\xi, \tag{1.2}$$

$$(\mathcal{S}_{r',r}f)(\eta)=\int_{G_{n,r}} f(\xi)|\mathrm{Sin}(\xi,\eta)|^{2\nu} d_\eta\xi, \tag{1.3}$$

where $d_\eta\xi$ is certain probability measure on the set $\{\xi\in G_{n,r}(\mathbb{K}):\xi\subset\eta\}$ invariant with respect to the group of unitary transformations of η. The definition of the sine and cosine functions is given in Definition 3.1. In the present paper we shall find the spectral symbols of the cosine and sine transform and find a characterization of their images. This generalizes recent results of [1] where case when $\mathbb{K}=\mathbb{R}$ and $\nu=1$ is

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considered. As application we find the unitary structure of the Stein's complementary series of the group $GL(n, \mathbb{K})$.

In [18] Stein proved that there is a family of unitary irreducible principal series representations of the group $GL(2r, \mathbb{C})$ (with $n = 2r$ is our notation) for non-unitary σ, by computing the Fourier transform of the kernel $|\det(x)|^\alpha$ of the Knapp-Stein intertwining operator; it gave then the unitary structure in the so-called non-compact picture of the induced representations. Vogan [19] proved that the Stein's results can be extended to $GL(2r, \mathbb{K})$ and proved the corresponding existence of the Stein's complementary series. In the present paper we prove that the sine transform for $r = r'$ is the Knapp-Stein intertwining operator for the group $GL(2r, \mathbb{K})$. We find the unitary structure of the Stein's complementary series representations; see Section 6.1. The unitarity and composition series for this family of principal series representation of $GL(n, \mathbb{K})$ are of considerable interests, see [10], [17] and [22], and they have also applications in the theory of valuations of convex sets [1].

The cosine transform on Grassmannians is the compact analogue of the Berezin transform [24] on non-compact symmetric spaces. (See also [16] where sine and cosine transform on real hyperbolic spaces are studied.) The Berezin transform appears naturally in the branching rule of holomorphic representations and quantization, see [24], [25] and references therein. As another application we find the branching rule of certain scalar holomorphic representations on the complex Grassmannian under isometric group of the real or quaternionic Grassmannian as a submanifold of the former; see Section 6.2.

We proceed to give a brief account of our method of obtaining the spectral symbol. On a non-compact real symmetric domain the Berezin transform is an integral kernel being a product of hyperbolic cosine functions [25] in terms of the geodesic coordinates, and the Radon transform being one with kernel a product of hyperbolic sine functions (generally over another symmetric space of smaller rank) [21]. The spectral symbol of the Berezin transform is then the Harish Chandra spherical transform of the hyperbolic cosine function. We find in [25] the spherical transform of the hyperbolic cosine functions on a root system of type BC with general multiplicity. This is proved by deriving certain Bernstein-Sato type formulas for cosine functions. Similar formulas for the hyperbolic sine functions is proved in [21], and is a crucial step in finding the inversion formula for the Radon transform. Using the techniques developed there we derive some corresponding Bernstein-Sato type formula for the sine and cosine functions on the compact torus of a general root system of type BC using the Cherednik operators, and we find the Cherednik-Opdam transform for the sine and cosine transforms. (This method can also be applied Radon transform, and giving a different proof of the result of Grinberg [5], but we will not present the details here.)

Combined with some earlier results of Grinberg on Radon transform we shall find the spectral symbols and images of the cosine and sine transform on Grassmannian manifolds. This generalizes the result of Alesker-Bernstein [1] when the case of real Grassmannian and the sine transform for $\nu = 1$ is considered. We prove further that the sine-transform are intertwining operator for certain principal series representation and we give an independent proof of the existence of the Stein's complementary series.

Our main results in this paper are Theorems 4.1, 4.3, 5.4 and 6.2. Theorem 4.1 gives a Bernstein-Sato type formula for the cosine and sine functions, Theorem 4.3 computes their spherical transforms and Theorem 5.4 gives a characterization of the images of the cosine and sine transforms. In Theorem 6.2 we give an independent proof for the existence the Stein's complementary series and provides a formula for their unitary structure.

After this paper was finished I was informed by Professor Semyon Alesker that Theorem 5.4 (1) was partly proved by him in an unpublished preprint [2] (under the assumption that ν is not a negative half-integer). The method there is quite different, it uses the localization theorem of Beilinson-Bernstein combined with some results by Braden-Grinberg on perverse sheaves and Howe-Lee on the K-type structure of degenerate principal series representations. It is my pleasure to thank him for his comments on an earlier version of this paper, and for sending a copy of his preprint. I thank also Professor B. Rubin for informing me that a special case of Proposition 5.1 (on the real Grassmannian when all m_j are equal) was also obtained in [15] as a consequence of the Hecke equality and Laplace transform on cones using the Gindikin's Gamma function.

2. Grassmannian manifolds $G_{n,r}$ and the irreducible decomposition of $L^2(G_{n,r})$

2.1. Grassmannian manifolds $\mathcal{X} = G_{n,r}$ as symmetric spaces. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the field of real, complex and quaternionic numbers with the standard involution (conjugation) $x \to \bar{x}$ and let $a = \dim_{\mathbb{R}} \mathbb{K} = 1, 2, 4$. Let $M_{n,m} := M_{n,m}(\mathbb{K})$ be the space of all $n \times m$-matrices, also viewed as $\mathbb{K}$-linear transformations from $\mathbb{K}^m$ to $\mathbb{K}^n$. Denote $x^* = \bar{x}^T$, the conjugate transpose, for $x \in M_{n,m}$. Let

$$G := U(n, \mathbb{K}) = \{g \in M_{n,n}; g^*g = I_n\} = O(n),\ U(n),\ Sp(n)$$

be the orthogonal, unitary, and symplectic groups accordingly. Denote, for $x \in M_{n,n}$, $\det_{\mathbb{R}}(x)$ the determinant of x as a real linear transformation on $\mathbb{K}^n = \mathbb{R}^{an}$. (For $\mathbb{K} = \mathbb{C}$ $\det_{\mathbb{R}}(x) = |\det_{\mathbb{C}}(x)|^2$ and for $\mathbb{K} = \mathbb{H}$, $\det_{\mathbb{R}}(x) = \det_{\mathbb{H}}(x)^4$ where $\det_{\mathbb{H}}(x)$ is the so-called Dieudonné determinant.)

Consider, for $r \le n$, the Stiefel manifold $S_{n,r}$ of all orthonormal r-frames in $\mathbb{K}^n$, and

$$\mathcal{X} = G_{n,r}$$

the Grassmannian manifold of all r-dimensional subspaces over $\mathbb{K}$. $S_{n,r}$ can be realized as the set of all matrices $x \in M_{n,r}$ such that $x^*x = I$, namely K-linear transformations $x \in \mathbb{K}^r \to \mathbb{K}^n$ that are isometries.Each Each $x \in S_{n,r}$ defines uniquely a r-dimensional subspace over $\mathbb{K}$,

$$\xi = \{x\} := x\mathbb{K}^k \subset \mathbb{K}^n \in G_{n,r}.$$

Thus $\mathcal{X}$ is identified as the space of orbits in $S_{n,r}$ under the action of the unitary group $U(r, \mathbb{K})$ on $\mathbb{K}^r$,

$$\mathcal{X} = G_{n,r} = S_{n,r}/U(r, \mathbb{K}).$$

.

We will fix ξ_0 and x_0 as reference points of $G_{n,r}$ and $S_{n,r}$,

$$\xi_0 = \mathbb{K}^r \oplus 0 = \{x_0\} \in G_{n,r}, \qquad x_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}.$$

The group G acts on $S_{n,r}$ and $\mathcal{X}$ by the defining action, they are then realized as as a homogeneous and symmetric space,

$$S_{n,r} = G/U(n-r,\mathbb{K}), \quad \mathcal{X} = G/K = U(n,\mathbb{K})/U(r,\mathbb{K}) \times U(n-r,\mathbb{K}) \tag{2.1}$$

see e.g. [8]. Here $K = U(r,\mathbb{K}) \times U(n-r,\mathbb{K})$ is the isotropic subgroup of ξ_0 consisting block diagonal matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad A \in U(r,\mathbb{K}), \quad D \in U(n-r,\mathbb{K}).$$

Due to the G-isometry between $G_{n,r}$ and $G_{n,n-r}$ we can and will assume in the this paper, if nothing else is said, that

$$2r \le n. \tag{2.2}$$

2.2. **Irreducible decomposition of $L^2(\mathcal{X})$.** To describe the irreducible decomposition of $L^2(\mathcal{X})$ under G we let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. The linear subspace $\mathfrak{p}$ consists of $n \times n$ matrices of the block form

$$p_X = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}, \quad X \in M_{r,n-r}$$

identified with $M_{r,n-r}$ via the mapping $X \mapsto p_X$. We let $\mathfrak{a}$ be the linear subspace of $\mathfrak{p} = M_{r,n-r}$ consisting matrices of the form

$$X = \begin{bmatrix} \mathrm{diag}\{t_1, \cdots, t_r\} & 0_{r,n-2r} \end{bmatrix} = t_1 E_1 + \cdots + t_r E_r, \quad t_1, \cdots, t_r \in \mathbb{R}$$

with the let E_j being the matrix having 1 on the (j,j) position and 0 on the rest positions, $j = 1, \cdots, r$. Let $A = \exp(\mathfrak{a})$. It consists of $n \times n$-matrices $\exp(X)$ of the form, under the decomposition of $\mathbb{K}^n = \mathbb{K}^r \oplus \mathbb{K}^r \oplus \mathbb{K}^{n-2r}$,

$$\exp(X) = \begin{bmatrix} \mathrm{diag}(\cos t, \cdots, \cos t) & \mathrm{diag}(\sin t, \cdots, \sin t) & 0 \\ \mathrm{diag}(-\sin t, \cdots, -\sin t) & \mathrm{diag}(\cos t, \cdots, \cos t) & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{2.3}$$

Let $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$, the non-compact dual of $\mathfrak{g}$. The root system of $R(\mathfrak{g}^*, i\mathfrak{a})$ is of the form

$$R(\mathfrak{g}^*, i\mathfrak{a}) = \{\pm\varepsilon_j \pm \varepsilon_k\} \cup \{\pm\varepsilon_j\} \cup \{\pm 2\varepsilon_j\}$$

with respective multiplicities a, $a-1$ and $a(n-2r)$. Here $\{\varepsilon_j\}$ is the dual basis of $\{iE_j\}$. It is of type C (if $n = 2r$, $a > 1$), type BC (if $n > 2r$, $a > 1$), type B (if $n > 2r$, $a = 1$), or type D (if $n = 2r$, $a = 1$).

Proposition 2.1. Under the action of $U(n)$ the space $L^2(\mathcal{X})$ decomposes as

$$L^2(\mathcal{X}) = \sum_{\underline{\mathbf{m}}}^{\oplus} V^{\underline{\mathbf{m}}} \tag{2.4}$$

with multiplicity free, where each $V^{\underline{\mathbf{m}}}$ has highest weight $\underline{\mathbf{m}} = m_1\varepsilon_1 + \cdots m_r\varepsilon_r$ with $\frac{m_j}{2}$ being integers and

$$m_1 \ge m_2 \cdots \ge m_r \ge 0$$

for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ and

$$m_1 \ge m_2 \cdots \ge |m_r|$$

for $\mathbb{K} = \mathbb{R}$. Each $V^{\underline{\mathbf{m}}}$ consists a unique K-invariant function $\phi_{\underline{\mathbf{m}}}$, called spherical polynomial, normalized so that $\phi_{\underline{\mathbf{m}}}(\xi_0) = 1$.

We will identify the spherical polynomial $\phi_{\underline{\mathbf{m}}}$ as defined on $\mathfrak{a} = \mathbb{R}^r$ via the exponential mapping $\mathfrak{a} \to G_{n,r}$, $X \to \exp(X) \cdot \xi_0$, namely

$$\phi_{\underline{\mathbf{m}}}(t_1, \cdots, t_r) := \phi_{\underline{\mathbf{m}}}(\exp(t_1E_1 + \cdots + t_rE_r) \cdot \xi_0).$$

Finally we recall the polar decomposition of $G_{n,r}$. For any function f on $G_{n,r}$ we have the following formula (which in turn fixes a normalization of the invariant measure)

$$\int_{G_{n,r}} f(\xi)d\xi = \int_K \int_A f(ka) \cdot \xi_0)dkd\mu(a) \tag{2.5}$$

where

$$d\mu(a) = d\mu_R(t) = \prod_{\alpha\in R_+} |2\sin\alpha(it))|^{m_\alpha} dt_1 \cdots dt_r, \quad a = \exp(t) = \exp(t_1E_1 + \cdots + t_rE_r)$$

with m_α the root multiplicity of α and dk being the normalized measure on K; see [9, Chapter I, Theorem 5.10].

3. Radon, cosine and sine transforms

3.1. **Cosine and sine of angles between subspaces.** We fix the standard Euclidean norm on $\mathbb{K}^n = \mathbb{R}^{an}$. For any convex subset S in a d-dimensional real plane we let $\mathrm{vol}_d(S)$ be the corresponding Euclidean volume. The following definition is given in [1] for the real Grassmannians.

Definition 3.1. If $r \le r'$ the cosine of the angle between two subspaces $\xi \in G_{n,r}$ and $\eta \in G_{n,r'}$ is defined by

$$|\mathrm{Cos}(\xi, \eta)| = \left(\frac{\mathrm{vol}_{ra} P_\eta(A)}{\mathrm{vol}_{ra}(A)}\right)^{\frac{1}{a}}, \tag{3.1}$$

where P_η is the orthogonal projection from $\mathbb{K}^n$ to $\eta \subset \mathbb{K}^n$ and $A \subset \xi$ is any convex set of non-zero volume. If $r \le n - r'$ the sine of the angle between two planes $\xi \in G_{n,r}$ and $\eta \in G_{n,r'}$ is defined by

$$|\mathrm{Sin}(\xi, \eta)| = |\mathrm{Cos}(\xi, \eta^\perp)|.$$

For general r and r' we define $|\mathrm{Cos}(\xi, \eta)|$ and $|\mathrm{Sin}(\xi, \eta)|$ by the symmetry condition

$$|\mathrm{Cos}(\xi, \eta)| = |\mathrm{Cos}(\eta, \xi)|, \qquad |\mathrm{Sin}(\xi, \eta)| = |\mathrm{Sin}(\eta, \xi)|.$$

In [4] Grinberg and Rubin introduce also a cosine of angle, $\mathrm{COS}^2(y, x)$, between two elements $x \in S_{n,r}$ and $y \in S_{n,r'}$, for $r \leq r'$, defined to be the $r \times r$-semi-positive definite matrix

$$\mathrm{COS}^2(y, x) = x^* y y^* x. \tag{3.2}$$

The following lemma computes the cosine (3.1) in terms of (3.2)

Lemma 3.2. (1) Let $1 \leq r \leq r'$ and the notations be as above. The two cosine functions (3.1) and (3.2) are related by,

$$|\mathrm{Cos}(\eta, \xi)|^a = (\det_{\mathbb{R}} \mathrm{COS}^2(y, x))^{\frac{1}{2}}$$

where $\eta = y\mathbb{K}^{r'} \in G_{n,r'}$, $\xi = y\mathbb{K}^r \in G_{n,r}$.

(2) Let $1 \leq r = r' \leq n - r$. Write $\xi \in G_{n,r}$ as $\xi = k \exp(a_1 E_1 + \cdots + a_r E_r) \cdot \xi_0$, $k \in K$. Then

$$|\mathrm{Cos}(\xi, \xi_0)|^\delta = \prod_{j=1}^{r} |\cos t_j|^\delta. \tag{3.3}$$

$$|\mathrm{Sin}(\xi, \xi_0)|^\delta = \prod_{j=1}^{r} |\cos t_j|^\delta. \tag{3.4}$$

Proof. We recall first that if $T : \mathbb{R}^p \to \mathbb{R}^q$, $p \leq q$, is a linear transformation and $B \subset \mathbb{R}^p$ is any convex set of positive volume, then $\mathrm{vol}_p(T(B)) = \det(T^t T)^{\frac{1}{2}} \mathrm{vol}_p(B)$, where T^t is the transpose of T with respect to the Euclidean metrics in $\mathbb{R}^p$ and $\mathbb{R}^q$. Indeed, let $T = U(T^t T)^{\frac{1}{2}}$ be the polar decomposition of T with U being a partial isometry. If $\mathrm{rank}(T) < p$, then both $\mathrm{vol}_p(T(B))$ and $\det(T^t T)^{\frac{1}{2}}$ are zero and the formula is trivially true. If $\mathrm{rank}(T) = p$ then $U^t U = I$ and U is an isometry. We have then

$$\mathrm{vol}_p(T(B)) = \mathrm{vol}_p(U(T^t T)^{\frac{1}{2}}(B)) = \mathrm{vol}_p((T^t T)^{\frac{1}{2}}(B)) = \det((T^t T)^{\frac{1}{2}})\ \mathrm{vol}_p(B),$$

proving the identity.

Now if $\xi = x = x\mathbb{K}^r$ and $\xi = y = y\mathbb{K}^{r'}$, with isometries $x \in S_{n,r}$ and $y \in S_{n,r'}$, we have $P_\eta = yy^*$. Let $A \subset \xi = x\mathbb{K}^r$ be any convex set of nonzero volume. We write $A = x(B)$ with $B \subset \mathbb{K}^r$ a convex set, and $\mathrm{vol}_{ra}(B) = \mathrm{vol}_{ra}(A)$ since x is an isometry. Its image under P_η is $P_\eta(A) = yy^* x(B)$. We apply the previous formula with $T = P_\eta x = yy^* x$, noticing that $T^* T = x^* P_\eta^2 x = x^* P_\eta x = x^* yy^* x$,

$$\mathrm{vol}_{ra} P_\eta(A) = (\det_{\mathbb{R}}(x^* yy^* x))^{\frac{1}{2}} \mathrm{vol}_{ra}(A) = \det_{\mathbb{R}}(\mathrm{COS}^2(x, y))^{\frac{1}{2}} \mathrm{vol}_{ra}(A).$$

This proves the first part, and the second part is then a special case by using the formula (2.3). □

3.2. **Factorization and diagonalization of the cosine and sine transforms.** We define the Radon transform $\mathcal{R}_{r',r} : C^\infty(G_{n,r}) \to C^\infty(G_{n,r'})$ by

$$(\mathcal{R}_{r',r} f)(\eta) = \int_{\xi \in G_{n,r}; \xi \subset \eta} f(\xi) d_\eta \xi, \tag{3.5}$$

if $r < r'$ and

$$(\mathcal{R}_{r',r}f)(\eta) = R^*_{r',r}f(\eta) = \int_{\xi\in G_{n,r};\xi\supset\eta} f(\xi)d_\eta\xi. \tag{3.6}$$

if $r > r'$. Here $d_\eta\xi$ is the unique Riemannian measure on the subset induced from the Riemannian measure on $G_{n,r}$; see [4] for an expression of $\mathcal{R}_{r',r}$ and the measure in terms of the realization (2.1). In particular the Radon transform $\mathcal{R}_{r',r}$ commutes with the actions of $U(n)$ on $C^\infty(G_{n,r})$ and $C^\infty(G_{n,r'})$.

Definition 3.3. Let $1 \le r, r' < n$ and $\nu \ge 0$. We define the sine and cosine transform from $C^\infty(G_{n,r})$ to $C^\infty(G_{n,r'})$, by

$$\mathcal{C}^{(2\nu)}_{r',r}f(\eta) = \int_{G_{n,r}} |\mathrm{Cos}(\eta,\xi)|^{2\nu} f(\xi)d\xi, \quad \mathcal{S}^{(\nu)}_{r',r}f(\eta) = \int_{G_{n,r}} |\mathrm{Sin}(\eta,\xi)|^{2\nu} f(\xi)d\xi.$$

The sine and cosine transforms are related to the Radon transform via the following Lemma. This Lemma in the case when $2\nu = 1$ is well-known and is proved in Lemma 1.7 in [1] of the case when $\mathbb{K} = \mathbb{R}$ and $\nu = 1$. The same method can be applied to the present case for general $\nu \ge 0$, by using a variant of the Cauchy-Kubota formula

$$\mathrm{vol}_{ar}(B)^\delta = c_\delta \int_{\xi\in G_{r',r}} \mathrm{vol}_{ar}(\mathrm{Pr}_\xi\, B)^\delta d\xi,$$

for any convex polygon in $B \subset \xi_0$, which follows easily by the invariance of both sides under translation and linear action by $GL(\xi_0)$. We skip the elementary proof.

Lemma 3.4. Let $1 \le r \le r' < n$, and $\nu \ge 0$. The cosine and sine transforms can be factorized as

$$\mathcal{C}^{(\nu)}_{r,r'} = c_1 \mathcal{C}^{(\nu)}_{r,r}\mathcal{R}_{r,r'}, \qquad \mathcal{S}^{(\nu)}_{r,r'} = c_2 \mathcal{S}^{(\nu)}_{r,r}\mathcal{R}_{r,r'},$$

where $c_1 = c_1(\nu)$ and $c_2 = c_2(\nu)$ are some positive constants.

The constant c_1 and c_2 can be computed by using the integral formula in [20] but we will not need it here.

By taking conjugate of the above formulas we get factorizations for all $1 \le r, r' < n$.

The transforms $\mathcal{C}^{(\nu)}_{r,r'}$ and $\mathcal{S}^{(\nu)}_{r,r'}$ clearly intertwine the action of $U(n)$. Consider the corresponding decomposition of $L^2(G_{n,r'})$ according to Proposition 2.1,

$$L^2(G_{n,r'}) = \sum_{\underline{\mathbf{m}}}^{\oplus} W^{\underline{\mathbf{m}}}. \tag{3.7}$$

Thus $\mathcal{C}^{(\nu)}_{r,r'}$ and $\mathcal{S}^{(\nu)}_{r,r'}$ are diagonal operators up to a normalization. The eigenvalue of $\mathcal{R}_{r,r'}$ has been found earlier by Grinberg [5]; see Theorem 5.2 below. Thus, in view of Lemma 3.4 above, to find the eigenvalues of $\mathcal{C}^{(2\nu)}_{r,r'}$ and $\mathcal{S}^{(2\nu)}_{r,r'}$ we need only consider the case when $r' = r$.

Denote $C^{(\nu)}(\underline{\mathbf{m}})$ the eigenvalue of $\mathcal{C}^{(\nu)}_{r,r}$ on $V^{\underline{\mathbf{m}}}$. In particular,

$$\mathcal{C}^{(\nu)}_{r,r}\phi_{\underline{\mathbf{m}}}(\xi_0) = C^{(\nu)}(\underline{\mathbf{m}})\phi_{\underline{\mathbf{m}}}(\xi_0) = C^{(\nu)}(\underline{\mathbf{m}}). \tag{3.8}$$

Namely

$$C^{(\nu)}(\underline{\mathbf{m}}) = \int_{G_{n,r}} |\mathrm{Cos}(\xi, \xi_0)|^{2\nu} \phi_{\underline{\mathbf{m}}}(\xi) d\xi \tag{3.9}$$

This is an integral of a K-invariant function. We use the polar coordinates (2.5), which further can be written as an integral on the quotient of $\mathbb{R}^r/2\pi Q^\vee$ by the (spherical) coroots lattice $Q^\vee$, namely the lattice generated by $\alpha^\vee = \frac{\hat{\alpha}}{(\alpha,\alpha)}$ for $\alpha \in R$, where $\hat{\alpha}$ is the dual of α, $\lambda(\hat{\alpha}) = (\lambda, \alpha)$; see [9, Chapter I, Theorem 5.10].

Lemma 3.5. Let $\delta \geq 0$. The eigenvalue $C^{(\nu)}(\underline{\mathbf{m}})$ and $S^{(\nu)}(\underline{\mathbf{m}})$ of the cosine respectively sine transforms are given by

$$C^{(\nu)}(\underline{\mathbf{m}}) = \int_{\mathbb{R}^r/2\pi Q^\vee} \prod_{j=1}^r |\cos t_j|^{2\nu} \phi_{\underline{\mathbf{m}}}(t_1, \cdots, t_r) d\mu_R(t) \tag{3.10}$$

$$S^{(\delta)}(\underline{\mathbf{m}}) = \int_{\mathbb{R}^r/2\pi Q^\vee} \prod_{j=1}^r |\sin t_j|^{2\nu} \phi_{\underline{\mathbf{m}}}(t_1, \cdots, t_r) d\mu_R(t) \tag{3.11}$$

4. Spherical transform on compact torus associated with general root system of Type BC

4.1. **Bernstein-Sato type formulas for cosine and sine functions.** We consider the root system R on an Euclidean space $i\mathfrak{a} = i\mathbb{R}^r$ $(i = \sqrt{-1})$ of type BC

$$R = \{\pm\varepsilon_j \pm \varepsilon_k, j \neq k\} \cup \{\pm\varepsilon_k\} \cup \{\pm 2\varepsilon_k^*\} \tag{4.1}$$

with general non-negative multiplicities a, $2b$ and ι for the respective sets of roots. Here $\{\varepsilon_j\}$ is the dual basis in $(i\mathfrak{a})^*$ of a fixed orthonormal basis $\{iE_j\}$ of $i\mathfrak{a}$. It is understood that the roots with multiplicity zero will not appear (e.g if $a = 1$ the third set is empty), and thus the type D or type B will be considered as a special case. Denote by W the Weyl group. We will compute the spherical transform of certain Weyl group invariant sine and cosine functions, by using the harmonic analysis of the Cherednik operators developed by Opdam [14]. We follow the presentation there loc. cit., however with our roots being twice of the roots there and our multiplicities half of the ones there.

Let $Q^\vee$ be the set of be the (spherical) coroot lattice. We define

$$|\mathrm{Cos}(t)| := |\prod_{j=1}^r \cos t_j|, \qquad |\mathrm{Sin}(t)| := |\prod_{j=1}^r \sin t_j|,$$

with

$$t = t_1 E_1 + \cdots + t_r E_r = (t_1, \cdots, t_r) \in \mathbb{R}^r/2\pi Q^\vee$$

We let $D_j = D_{iE_j}$, $j = 1 \cdots, r$, be the trigonometric Cherednik operators acting on the functions on $\mathbb{R}^r/2\pi Q^\vee$,

$$D_j = \partial_j - ia\sum_{k<j}\frac{1}{1-e^{-2i(t_k-t_j)}}(1-s_{kj}) + ia\sum_{j<k}\frac{1}{1-e^{-2i(t_j-t_k)}}(1-s_{jk})+$$
$$+ia\sum_{k\neq j}\frac{1}{1-e^{-2i(t_j+t_k)}}(1-\sigma_{jk}) + 2i\iota\frac{1}{1-e^{-4it_j}}(1-\sigma_j)$$
$$+2ib\frac{1}{1-e^{-2it_j}}(1-\sigma_j) - i\rho_j,$$

where s_{kj}, σ_{kj} and σ_j are the reflections corresponding to the roots $\varepsilon_j - \varepsilon_k$, $\varepsilon_j + \varepsilon_k$,and ε_j. Let $\phi_{\underline{\mathbf{m}}}$ be the Heckman-Opdam Jacobi polynomials on the root system ([7], [6], and [14]), and

$$\hat{f}(\underline{\mathbf{m}}) := \int_{\alpha/2\pi Q^\vee} f(t)\phi_{\underline{\mathbf{m}}}(t)d\mu_R(t)$$

be the spherical (or Jacobi) transform. Our objective is to find the spherical transform of the functions $|\mathrm{Cos}(t)|^{2\nu}$, $|\mathrm{Sin}(t)|^{2\nu}$. We establish first certain Bernstein-Sato type formulas.

Theorem 4.1. Let $\delta \geq 0$ and $\mathcal{M}_\delta$ be the following operator

$$\mathcal{M}_\delta : = \prod_{j=1}^{r}(D_j^2 + (\delta+\rho_1)^2).$$

Then the following Bernstein-Sato type formulas hold

$$\mathcal{M}_\delta|\mathrm{Cos}(t)|^\delta = \prod_{j=1}^{r}(\delta + a(j-1))(\delta+\iota-1+a(r-j))|\mathrm{Cos}(t)|^{\delta-2}$$

and

$$\mathcal{M}_\delta|\mathrm{Sin}(t)|^\delta = \prod_{j=1}^{r}(\delta + a(j-1))(\delta+\iota+2b-1+a(r-j))|\mathrm{Sin}(t)|^{\delta-2}.$$

Proof. In [25, Theorem 2.1] and [21, Theorem 3.1] the following formulas are proved for the hyperbolic sine and cosine functions, defined on $i\mathfrak{a}$,

$$\prod_{j=1}^{r}\big((iD_j)^2 - (\delta+\rho(\xi_1))^2\big)\big(\prod_{j=1}^{r}\cosh x_j\big)^\delta$$
$$= \prod_{j=1}^{r}\big(\delta + a(j-1)\big)\big(1-\delta-\iota-a(r-j)\big)\big(\prod_{j=1}^{r}\cosh x_j\big)^{\delta-2}, \quad x=(x_1,\cdots,x_r)\in\mathbb{R}^r,$$

$$\prod_{j=1}^{r}\big((iD_j)^2 - (\delta+\rho_1)^2\big)\big(\prod_{j=1}^{r}\sinh x_j\big)^\delta$$
$$= \prod_{j=1}^{r}\big(\delta + a(j-1)\big)\big(\delta-1+\iota+2b+a(r-j)\big)\big(\prod_{j=1}^{r}\sinh x_j\big)^{\delta-2}, \quad x=(x_1,\cdots,x_r)\in\mathbb{R}^r_+,$$

where iD_j is acting on the variable x. The function $(\sinh x)^\delta$ and $(\cosh x)^\delta$ are analytic functions on the right hand plane $\Re x > 0$, so are the products $\prod_{j=1}^r (\sinh x_j)^\delta$ and $\prod_{j=1}^r (\cosh x_j)^\delta$ on the product of the right half plane $\{t; \Re x_j > 0\}$. All the identities has a limit at the points $x = it$, $t \in \mathbb{R}^r$, $t_j \neq 0, \frac{\pi}{2}$. Our result follows then by taking the limit, observing also that $\sinh^2(it_j) = (-1)\sin t$. □

Remark 4.2. The above theorem can also be proved directly by a straightforward but long computations. First, we can choose a dense open subset of the orbit $W\backslash\mathbb{R}^r/2\pi Q^\vee$ of the Weyl group so that the functions $\sin t_j > 0, \cos t_j > 0$. Furthermore the operator $\mathcal{M}_\delta$ can be factorized as follows,

$$\mathcal{M}_\delta = \prod_{j=1}^r (D_j^2 + (\delta+\rho_1)^2) = \prod_{j=1}^r (-iD_j + (\delta+\rho_1))(iD_j + (\delta+\rho_1)) \tag{4.2}$$

and we may compute successively the action of the factors. We have

$$\prod_{l=1}^j (iD_l + (\delta+\rho_1))\mathrm{Cos}(t)^\delta = \prod_{l=1}^j (\delta + a(l-1))\mathrm{Cos}(t)^\delta \prod_{l=1}^j \frac{e^{it_j}}{\cos t_l},$$

$$\prod_{l=j}^r (-iD_l + (\delta+\rho_1))\mathrm{Cos}(t)^{\delta-1}\prod_{l=1}^r e^{it_l} = \prod_{l=j}^r (\delta - 1 + \iota + a(r-l))\mathrm{Cos}(t)^{\delta-1}\prod_{l=1}^r e^{it_j}\prod_{l=j}^r \frac{e^{-it_l}}{\cos t_l}$$

$$\prod_{l=1}^j (iD_l + (\delta+\rho_1))\mathrm{Sin}(t)^\delta = \prod_{l=1}^j (\delta + a(l-1))|\mathrm{Sin}(t)|^\delta \prod_{l=1}^j \frac{e^{it_j}}{\sin t_l},$$

$$\prod_{l=j}^r (-iD_l + (\delta+\rho_1))\mathrm{Sin}(t)^{\delta-1}\prod_{l=1}^r e^{it_l} = \prod_{l=j}^r (\delta - 1 + \iota + 2b + a(r-l))\mathrm{Sin}(t)^{\delta-1}\prod_{l=1}^r e^{it_j}\prod_{l=j}^r \frac{e^{-it_l}}{\cos t_l},$$

by similar computations as in [25] and [21], which together with the factorization (4.2) implies our theorem.

4.2. **Spherical transform for cosine and sine functions.** We let N_ν and N'_ν be the following normalization constants,

$$N_\nu = \int_{\mathbb{R}^r/2\pi Q^\vee} |\mathrm{Cos}|^{2\nu}(t)d\mu(t) \tag{4.3}$$

and

$$N'_\nu = \int_{\mathbb{R}^r/2\pi Q^\vee} |\mathrm{Sin}|^{2\nu}(t)d\mu(t). \tag{4.4}$$

Their exact values can be evaluated by using the Macdonald formula for generalized Beta-integrals (see [13, Ex. 7, Sect. 10, Chapt. VII]),

$$\begin{aligned} N_\nu = 2^{ar(r-1)+2rb+2r\iota} r! \prod_{1\le i<j\le r} \frac{\Gamma(\frac{a}{2}(j-i+1))}{\Gamma(\frac{a}{2}(j-i))} \times \\ \frac{\Gamma_a(1+b+\frac{\iota-1}{2}+\frac{a}{2}(r-1))\Gamma_a(\nu+1+\frac{\iota-1}{2}+\frac{a}{2}(r-1))}{\Gamma_a(\nu+1+b+\iota+a(r-1))} \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
N'_\nu = 2^{ar(r-1)+2rb+2r\iota} r! \prod_{1\le i<j\le r} \frac{\Gamma(\frac{a}{2}(j-i+1)}{\Gamma(\frac{a}{2}(j-i)} \times \\
\frac{\Gamma_a(1+\frac{\iota-1}{2}+\frac{a}{2}(r-1))\Gamma_a(\nu+1+\frac{\iota-1}{2}+\frac{a}{2}(r-1))}{\Gamma(\nu+1+b+\iota+a(r-1)-\frac{a}{2}(j-1))}.
\end{aligned} \tag{4.6}
$$

Here $\Gamma_a(\alpha)$ is the Gindikin's Gamma function

$$\Gamma_a(\alpha) = \prod_{j=1}^{r} \Gamma(\alpha - \frac{a}{2}(j-1)).$$

We recall also the Pochammer symbol $(\nu)_k = (\nu)(\nu+1)\cdots(\nu+k-1)$.

Theorem 4.3. The spherical transform of the functions $|\mathrm{Cos}(t)|^{2\nu}$ and $|\mathrm{Sin}(t)|^{2\nu}$ are given by

$$c_{\nu,r}(\underline{\mathbf{m}}) := \widehat{|\mathrm{Cos}|^{2\nu}}(\underline{\mathbf{m}}) = N_\nu \prod_{j=1}^{r} \frac{(\nu+1+\frac{a}{2}(j-1)-\frac{m_j}{2})_{\frac{m_j}{2}}}{(\nu+1+\iota+b+a(r-1)-\frac{a}{2}(j-1))_{\frac{m_j}{2}}}$$

and

$$s_{\nu,r}(\underline{\mathbf{m}}) := \widehat{|\mathrm{Sin}|^{2\nu}}(\underline{\mathbf{m}}) = N'_\nu \prod_{j=1}^{r} \frac{(\nu+1+\frac{a}{2}(j-1)-\frac{m_j}{2})_{\frac{m_j}{2}}}{(\nu+1+\iota+b+a(r-1)-\frac{a}{2}(j-1))_{\frac{m_j}{2}}} \phi_{\underline{\mathbf{m}}}(\frac{\pi}{2},\cdots,\frac{\pi}{2}).$$

For systems of Type C or Type D

$$s_{\nu,r}(\underline{\mathbf{m}}) := \widehat{|\mathrm{Sin}|^{2\nu}}(\underline{\mathbf{m}}) = N'_\nu \prod_{j=1}^{r} \frac{(-\nu-\frac{a}{2}(j-1))_{\frac{m_j}{2}}}{(\nu+1+\iota+b+a(r-1)-\frac{a}{2}(j-1))_{\frac{m_j}{2}}}.$$

We need first the following elementary result, it is proved in [23] for the cosine function in the case when the root system corresponds to (the compact dual of) a bounded symmetric domain, and the same proof can be used, noticing also that the sine function $\sin t$ takes maximum at $t = \frac{pi}{2}$ instead of 0 for the cosine function.

Lemma 4.4. Suppose ϕ be a bounded and continuous function on $\mathbb{R}^r/2\pi Q^\vee$. Then

$$\lim_{\nu\to\infty} \frac{1}{N_\nu} \int_{\mathbb{R}^r/2\pi Q^\vee} |\mathrm{Cos}|^{2\nu}(s)\phi(s)d\mu(s) = \phi(0)$$

and

$$\lim_{\nu\to\infty} \frac{1}{N'_\nu} \int_{\mathbb{R}^r/2\pi Q^\vee} |\mathrm{Sin}|^{2\nu}(s)d\mu(s) = \phi(\frac{\pi}{2},\cdots,\frac{\pi}{2}).$$

The following lemma asserts that $\phi_{\underline{\mathbf{m}}}(\frac{\pi}{2},\cdots,\frac{\pi}{2})$ is always nonzero.

Lemma 4.5. (1) Suppose R is a root system of type C, or D with general non-negative root multiplicity. Then

$$\phi_{\underline{\mathbf{m}}}(\frac{\pi}{2},\cdots,\frac{\pi}{2}) = \prod_{j=1}^{r}(-1)^{m_j}$$

(2) Suppose R is the root system of the Grassmannian manifold $G_{n,r}$. Then

$$\phi_{\underline{\mathbf{m}}}(\frac{\pi}{2}, \cdots, \frac{\pi}{2}) \neq 0$$

Use integral formula and the isomorphism between $G_{n,r}$ and $G_{n,n-r}$.

Proof. (1) From the formula for D_j we see that, if R is a root system of type C, or D, then $\{D_j\}$ are invariant under the map $(t_1, \cdots, t_r) \mapsto (t_1 + \frac{\pi}{2}, \cdots, t_r + \frac{\pi}{2})$. Moreover it is easy to prove that the polynomial $f(t_1, \cdot, t_r) := \phi_{\underline{\mathbf{m}}}(t_1 + \frac{\pi}{2}, \cdots, t_r + \frac{\pi}{2})$ is also invariant under the Weyl group. We prove this for type C, the type D is the same. We need only to prove that for the simple reflections σ_r and $s_{j,j+1}$. First notice that by definition $\phi_{\underline{\mathbf{m}}}$ is invariant under the mappings $(t_1, \cdots, t_r) \mapsto (t_1, \cdots, t_r + \pi)$.

We have

$$\begin{aligned}
(\sigma_r f)(t_1, \cdot, t_r) &= f(t_1, \cdot, -t_r) = \phi_{\underline{\mathbf{m}}}(t_1 + \frac{\pi}{2}, \cdots, -t_r + \frac{\pi}{2}) \\
&= \phi_{\underline{\mathbf{m}}}(t_1 + \frac{\pi}{2}, \cdots, -(t_r + \frac{\pi}{2}) + \pi) \\
&= \phi_{\underline{\mathbf{m}}}(t_1 + \frac{\pi}{2}, \cdots, -(t_r + \frac{\pi}{2})) \\
&= \phi_{\underline{\mathbf{m}}}(t_1 + \frac{\pi}{2}, \cdots, (t_r + \frac{\pi}{2})) \\
&= f(t_1, \cdot, t_r)
\end{aligned}$$

and that $s_{jj+1}f = f$ is trivially true. Thus f is an eigenfunction of the Weyl group invariant polynomials of $\{D_j\}$ and having the same eigenvalue as that of $\phi_{\underline{\mathbf{m}}}$. Namely $f = c\phi_{\underline{\mathbf{m}}}$. Comparing the leading coefficients proving our result.

(2) Consider the case of Grassmannian manifolds. We need only to treat the case when the corresponding root system R is of type B or BC, namely $G_{n,r}(\mathbb{K})$ for $n-r > r$. Write

$$\xi_1 := \exp(\frac{\pi}{2}E_1 + \cdot + \frac{\pi}{2}E_r) \cdot \xi_0 = 0 \oplus \mathbb{K}^r \oplus 0 \in G_{n,r}$$

We have, by the integral formula for spherical polynomials, [9, Chapter IV, Proposition 2.2], that

$$\phi_{\underline{\mathbf{m}}}(\xi_1)^2 = \int_K \phi_{\underline{\mathbf{m}}}(\exp(\frac{\pi}{2}E_1 + \cdot + \frac{\pi}{2}E_r)k\exp(\frac{\pi}{2}E_1 + \cdot + \frac{\pi}{2}E_r) \cdot \xi_0)dk. \tag{4.7}$$

We will find the set of integration in the above integral, namely

$$\{\exp(\frac{\pi}{2}E_1 + \cdot + \frac{\pi}{2}E_r)k\exp(\frac{\pi}{2}E_1 + \cdot + \frac{\pi}{2}E_r) \cdot \xi_0; k \in K\}. \tag{4.8}$$

Let $k = \operatorname{diag}(A, D) \in U(r) \otimes U(n-r)$, and write D as a 2×2-block matrix with under the decomposition of $\mathbb{K}^{n-r} = \mathbb{K}^r \oplus \mathbb{K}^{n-2r}$,

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

Using the formula (2.3) for the exponential we find that the element $\exp(\frac{\pi}{2}E_1 + \cdot + \frac{\pi}{2}E_r)k\exp(\frac{\pi}{2}E_1 + \cdot + \frac{\pi}{2}E_r) \cdot \xi_0$ is

$$\{D_{11}v \oplus 0 \oplus D_{21}v;\ v \in \mathbb{K}^r\} \subset \eta_1 := \mathbb{K}^r \oplus 0 \oplus \mathbb{K}^{n-2r} \subset \mathbb{K}^r \oplus \mathbb{K}^r \oplus \mathbb{K}^{n-2r}.$$

Namely the set (4.8) is

$$\{\xi \in G_{n,r}; \xi \subset \eta_1\}$$

So that the right hand side of (4.7) is precisely the Radon transform of $\phi_{\underline{\mathbf{m}}}$, namely

$$\phi_{\underline{\mathbf{m}}}(\xi_1)^2 = \mathcal{R}_{n-r,r}\phi_{\underline{\mathbf{m}}}(\eta_1).$$

However by the K-invariance we have further

$$\phi_{\underline{\mathbf{m}}}(\xi_1)^2 = \mathcal{R}_{n-r,r}\phi_{\underline{\mathbf{m}}}(\eta_1) = \mathcal{R}^t_{n-r,r}\mathcal{R}_{n-r,r}\phi_{\underline{\mathbf{m}}}(\xi_0)$$

It follows from the main result in [5] (see Theorem 5.2 below) that $\mathcal{R}^t_{n-r,r}\mathcal{R}_{n-r,r}\phi_{\underline{\mathbf{m}}}(\xi_0)$ is a non-zero constant of $\phi_{\underline{\mathbf{m}}}(\xi_0) = 1$ since $r \le n-r$. Thus $\phi_{\underline{\mathbf{m}}}(\xi_1) \neq 0$. □

Remark 4.6. One may follow the argument in the proof and find the constant $\phi_{\underline{\mathbf{m}}}(\xi_1)^2$ by using the result of Grinberg. It would be interesting to evaluate the constant for a general root system of positive multiplicity. For the real projective space $G_{n,1} = P^{n-1}(\mathbb{R})$ the second claim (2) is proved in [9, Chapter I, Lemma 4.9].

We prove now Theorem 4.3.

Proof. We perform the spherical transform on the Bernstein-Sato formula for the cosine function in Theorem 4.1 with $\delta = 2\nu+2$. Using the self-adjoint property of the operator $\mathcal{M}_{2\nu}$ and that (see [14]),

$$\mathcal{M}_{2\nu+2}\phi_{\underline{\mathbf{m}}} = \prod_{j=1}^{r}((2\nu+2+\rho_1)^2 - (m_j+\rho_j)^2)$$

we find that (suppressing the subindex r in $c_{\nu,r}(\underline{\mathbf{m}})$)

$$\frac{c_\nu(\underline{\mathbf{m}})}{N_\nu} = \frac{N_{\nu+1}}{N_\nu}\frac{\prod_{j=1}^{r}((2\nu+2+\rho_1)^2-(m_j+\rho_j)^2)}{\prod_{j=1}^{r}(2\nu+2+a(j-1))(2\nu+1+\iota+a(r-j))}\frac{1}{N_{\nu+1}}c_{\nu+1}(\underline{\mathbf{m}}).$$

After a simplification we get

$$\frac{c_\nu(\underline{\mathbf{m}})}{N_\nu} = \prod_{j=1}^{r}(1-\frac{\frac{m_j}{2}}{\nu+1+\frac{a}{2}(j-1)})(1+\frac{\frac{m_j}{2}}{\nu+1+b+\iota+a(r-1)-\frac{a}{2}(j-1)})\frac{c_{\nu+1}(\underline{\mathbf{m}})}{N_{\nu+1}}.$$

Iterating the result produces furthermore

$$\frac{c_\nu(\underline{\mathbf{m}})}{N_\nu} = \frac{c_{\nu+l+1}(\underline{\mathbf{m}})}{N_{\nu+l+1}}\prod_{j=1}^{r}\prod_{k=0}^{l}(1-\frac{\frac{m_j}{2}}{\nu+k+1+\frac{a}{2}(j-1)})$$
$$(1+\frac{\frac{m_j}{2}}{\nu+k+1+b+\iota+a(r-1)-\frac{a}{2}(j-1)}).$$

However $\frac{c_{\nu+l+1}(\underline{\mathbf{m}})}{N_{\nu+l+1}} \to \phi_{\underline{\mathbf{m}}}(0) = 1$ according to Lemma 4.4. Therefore,

$$\frac{c_\nu(\underline{\mathbf{m}})}{N_\nu} = \prod_{j=1}^{r}\prod_{k=0}^{\infty}(1-\frac{\frac{m_j}{2}}{\nu+k+1+\frac{a}{2}(j-1)})$$
$$(1+\frac{\frac{m_j}{2}}{\nu+k+1+b+\iota+a(r-1)-\frac{a}{2}(j-1)}),$$

which can also be written in terms of the Gamma function ([3, p.5])

$$\prod_{j=1}^{r}\frac{\Gamma(\nu+1+\frac{a}{2}(j-1))\Gamma(\nu+1+b+\iota+a(r-1)-\frac{a}{2}(j-1))}{\Gamma(\nu+1+\frac{a}{2}(j-1)-\frac{m_j}{2})\Gamma(\nu+1+b+\iota+a(r-1)-\frac{a}{2}(j-1)+\frac{m_j}{2})}.$$

This proves our formula for the cosine function. The sine function is done similarly. □

5. Spectral symbols and range characterization

5.1. **Diagonalization of the transforms.** Theorem 4.3 applied to Lemma 3.5 gives then

Proposition 5.1. Let $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ be the field of real, complex and quaternionic numbers. Let $1\le r\le n-1$. The eigenvalue of $\mathcal{C}^{(\nu)}_{r,r}$ and $\mathcal{S}^{(\nu)}_{r,r}$ are given by $c_{\nu,r}(\underline{\mathbf{m}})$ and $s_{\nu,r}(\underline{\mathbf{m}})$ in Theorem 4.3 with $\iota=a-1$, $b=a(n-2r)$.

To state our result on the spectral symbol of $\mathcal{C}^{(\nu)}_{r,r'}$ and $\mathcal{S}^{(\nu)}_{r,r'}$ for different r and r' we recall first the following result of Grinberg [5], reformulated slightly differently here.

Theorem 5.2. Let $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ be the field of real, complex and quaternionic numbers. Let $1\le r<r'\le n-1$. Then the operator $\mathcal{R}_{r',r}$ defines a bounded operator from $L^2(G_{n,r})$ into $L^2(G_{n,r'})$. The operator $\mathcal{R}^*_{r',r}\mathcal{R}_{r',r}$ is a diagonal operator under the decomposition,

$$\mathcal{R}^*_{r',r}\mathcal{R}_{r',r}f=c(\underline{\mathbf{m}})f,\quad f\in V^{\underline{\mathbf{m}}},$$

with an explicit formula for the eigenvalue $c(\underline{\mathbf{m}})$. The closure in $L^2(G_{n,r'})$ of the image of the operator $\mathcal{R}_{r',r}$ on $L^2(G_{n,r})$ is

$$\sum_{\underline{\mathbf{m}}\in L_{r,r'}}W^{\underline{\mathbf{m}}}$$

where $L_{r,r'}$ is the subset of those $\underline{\mathbf{m}}$ for which $m_j=0$ if $j\ge\min\{r,r'\}$.

Note that the L^2-bounded result was not stated in [5]. However it follows directly from the explicit formula for the eigenvalue $c(\underline{\mathbf{m}})$ there.

Using Lemma 3.4, Proposition 5.1 and Theorem 5.2 we get

Corollary 5.3. Suppose $\nu>0$. Let $1\le r,r'\le n-1$ be such that $2r,2r'\le n$. The eigenvalue of $\mathcal{C}^*_{r,r'}\mathcal{C}_{r,r'}$ and $\mathcal{S}^*_{r,r'}\mathcal{S}_{r,r'}$ on the space $V^{\underline{\mathbf{m}}}$ are given respectively by

$$c(\underline{\mathbf{m}})c_{\nu,r}(\underline{\mathbf{m}})c_{\nu,r'}(\underline{\mathbf{m}}),\qquad s(\underline{\mathbf{m}})s_{\nu,r}(\underline{\mathbf{m}})s_{\nu,r'}(\underline{\mathbf{m}}),$$

where $c_{\nu,r}$ is given in Proposition 5.1 and $c(\underline{\mathbf{m}})$ in Theorem 5.2. In particular $\mathcal{C}_{r,r'}$ and $\mathcal{S}_{r,r'}$ are bounded operators from $L^2(G_{n,r'})$ to $L^2(G_{n,r})$.

5.2. **Characterization of the image of the transforms.** The following theorem follows immediately from Corollary 5.3 and Lemma 4.5.

Theorem 5.4. Let $1\le r,r'<n-1$ and $\nu>0$.

(1) Let $\mathbb{K}=\mathbb{R}$. If $\nu\notin\frac{\mathbb{Z}}{2}$. Then the the closures in $L^2(G_{n,r'})$ of images of $\mathcal{C}_{r',r}$ and $\mathcal{S}_{r',r}$ on $L^2(G_{n,r})$ are given by

$$\sum_{\underline{\mathbf{m}}\in L_{r,r'}} W^{\underline{\mathbf{m}}}. \tag{5.1}$$

where $L_{r,r'}$ is given in Theorem 5.2. In particular the image is dense is $r'\le r$. If $\nu\in\frac{\mathbb{Z}}{2}$ then the closure of their images in $L^2(G_{n,r'})$ is given by

$$\sum_{\underline{\mathbf{m}}\in L_{r,r'}\cap L_\nu} W^{\underline{\mathbf{m}}},$$

where L_ν is the subset of $\underline{\mathbf{m}}$ such that

$$\frac{m_j}{2}<\nu+1+\frac{1}{2}(j-1),\quad \text{if } \nu+1+\frac{1}{2}(j-1) \text{ is an integer},\ j=1,\cdots,r. \tag{5.2}$$

(2) Let $\mathbb{K}=\mathbb{C},\mathbb{H}$. If $\nu\notin Z$. Then the closure of the images are as in (5.1). If $\nu\in\mathbb{Z}$ then the closure of images in $L^2(G_{n,r'})$ is given by

$$\sum_{\underline{\mathbf{m}}\in L_{r,r'}\cap L_\nu} W^{\underline{\mathbf{m}}}$$

L_ν is the subset of $\underline{\mathbf{m}}$ such that

$$\frac{m_j}{2}<\nu+1+\frac{1}{2}(j-1),\quad j=1,\cdots,r. \tag{5.3}$$

6. The sine-transform as Knapp-Stein intertwining operator. Cosine transform and branching of holomorphic representations

In this section we give applications of our results to the existence of the Stein's complementary series and on the branching rule of holomorphic representations.

6.1. **Stein's complementary series.** We fix $n=2r$ in this subsection. Let $GL_n=GL(n,\mathbb{K})$ be the linear group over $\mathbb{K}$ and $G=U(n,\mathbb{K})$ as before. Consider the parabolic subgroup P of GL_n consisting of block matrices of the form

$$p=\begin{bmatrix}A & B\\ 0 & D\end{bmatrix}$$

where $A,B,D\in M_{r,r}(\mathbb{K})$. The Langland decomposition of P is $P=LN=MAN$, with the nilpotent groups N and its opposite $\bar{N}$ consisting of upper triangular matrices

$$n_X=\begin{bmatrix}I & X\\ 0 & I\end{bmatrix},\quad n_Y=\begin{bmatrix}I & 0\\ Y & I\end{bmatrix},\quad X,Y\in M_{r,r}, \tag{6.1}$$

both identified with $M_{r,r}$, whereas $\bar{N}$ being the the corresponding lower triangular matrices; the group L consists of diagonal matrices

$$p=\begin{bmatrix}A & 0\\ 0 & D\end{bmatrix}$$

with

Let e^ρ on $P=LN$ be defined by

$$e^\rho(p)=|\det{}_{\mathbb{R}}(AD^{-1})|^{\frac{r}{2}}$$

Let δ_t be the representation on P defined by

$$\delta_t(p) = |\det_{\mathbb{R}}(AD^{-1})|^t$$

Let $\mathrm{Ind}_P^{GL_n}(t)$ be the induced representation [11, Chapter VII] consisting of measurable functions f on G such

$$f(gp) = (\delta_t(p)e^{\rho}(p))^{-1}f(g)$$

and that $f|_G \in L^2(G)$. When realized on $\bar{N} = M_{r,r} = M_{r,r}(\mathbb{K})$ the Knapp-Stein intertwining operator from $\mathrm{Ind}_P^{GL_n}(t)$ to $Ind_P^{GL_n}(-t)$ is given by [18],

$$\mathcal{I}_t F(\bar{n}_X) = \int_{M_{r,r}} |\det_{\mathbb{R}}(X-Y)|^{(-r+2t)} F(n_Y)dY.$$

The induced representation has also a realization on the space the space $L^2(G/G\cap L)$. Now $\mathcal{X} = G/G\cap L = G/K$ is precisely the Grassmannian manifold $\mathcal{X} = G_{n,r}$ The intertwining map from the K-picture to the N-picture is, by the definition, given by the $KMAN$ decomposition of $\bar{N}$, namely

$$f(\xi) \in L^2(\mathcal{X}) \to F(\bar{n}) = \delta_t(p(\bar{n}))^{-1}e^{-\rho)(p(\bar{n}))}f(\kappa(\bar{n})\xi_0) \tag{6.2}$$

where $\bar{n} = k(\bar{n})p(\bar{n})$ is the Iwasawa $G = KP = KMAN$ decomposition of $\bar{n}$ in G.

Lemma 6.1. In the compact picture the Knapp-Stein (formal) intertwining operator $\mathcal{I}_t$ is given by (up to a non-zero constant)

$$\mathcal{J}_t f(\xi) = \int_{G_{n,r}} |\mathrm{Sin}(\xi,\eta)|^{2\nu}f(\eta)d\eta;$$

where

$$\boxed{\nu = -\frac{a}{2}(r-2t)}. \tag{6.3}$$

It is well-defined on the space of the space of algebraic (i.e. finite) sum of the irreducible representations $V^{\mathbf{m}}$ in $L^2(\mathcal{X})$ for $\nu \ge 0$.

Proof. Let $S(\xi,\eta)$ denote temporarily the kernel of the intertwining operator in the compact realization on $L^2(\mathcal{X})$, namely let

$$\mathcal{J}_t f(\xi) = \int_{\mathcal{X}} S(\xi,\eta)f(\eta)d\eta.$$

The kernel $S(\xi,\eta)$ is then uniquely determined by $S(\xi_0,\eta)$ (recalling that $\xi_0 = \mathbb{K}^r \oplus 0 \in \mathcal{X}$) by the transitivity of $G = U(n,\mathbb{K})$ on $\mathcal{X}$ and by the intertwining property. By definition we have

$$\mathcal{J} f(\xi_0) = \int_{\mathcal{X}} S(\xi_0,\eta)f(\eta)d\eta. \tag{6.4}$$

On the other hand,

$$\mathcal{J} f(\xi_0) = \mathcal{I}_t F(0) = \int_{M_{r,r}} |\det_{\mathbb{R}}(Y)|^{(-r+2t)}F(n_Y)dY$$

where the function f on $\mathcal{X}$ and F on $\bar{N}$ is given by (6.2). Perform change of variable $Y \in M_{r,r} \mapsto \eta = k(n_Y)\xi_0 \in \mathcal{X}$ according to (6.2), and comparing with (6.4) we find that

$$S(\xi_0, \eta) = |\det_{\mathbb{R}}(Y)|^{(-r+2t)} \left(\delta_t(p(n_Y))e^{\rho}(p(n_Y))\right)^{-1} (\mathrm{Jac}_{Y\mapsto\eta})^{-1}$$

where $\mathrm{Jac}_{Y\mapsto\eta}$ is the Jacobian of $Y \to \eta$, namely

$$d\eta = \mathrm{Jac}_{Y\mapsto\eta} dY$$

We will find the three quantities $\delta_t(p(n_Y))$, $e^{\rho}(p(n_Y))$ and the Jacobian. Consider therefore the $GMAN$ decomposition of $n_Y \in \bar{N} = M_{r,r}(\mathbb{K})$ (under the identification (6.1))

$$n_Y = \begin{bmatrix} I_r & 0 \\ Y & I_r \end{bmatrix} = k(n_Y)p(n_Y), \quad k(n_Y) \in G,\ p(n_Y) \in P = MAN.$$

A easy matrix computation shows that the change of variable $Y \to \eta = k(n_Y)\xi_0 \in \mathcal{X}$ is given by

$$\eta = k(n_Y)\xi_0 = \begin{bmatrix} I_r & 0 \\ Y & I \end{bmatrix} \xi_0 = \{v \oplus Yv; v \in \mathbb{K}^r\} \in \mathcal{X}, \tag{6.5}$$

and that

$$e^{\rho}(p(n_Y)) = \det_{\mathbb{R}}(1 + Y^*Y)^r, \qquad \delta_t(p(n_Y)) = \det_{\mathbb{R}}(1 + Y^*Y)^{\frac{t}{2}}.$$

The measure $d\eta$, and equivalently the Jacobian, is given by

$$d\eta = \det_{\mathbb{R}}(I + Y^*Y)^{-r} dY,$$

by direct computation, e.g. using Proposition 3.3 (2) in [20]. Now the sine $\mathrm{Sin}(\eta, \xi_0)$ of the angel between η and ξ_0 is by (6.5) and Lemma 3.2

$$|\mathrm{Sin}(\xi, \xi_0)|^a = \frac{|\det_{\mathbb{R}}(x)|}{\det_{\mathbb{R}}(1 + x^*x)|^{\frac{1}{2}}}.$$

It follows finally that

$$S(\xi, \xi_0) = |\mathrm{Sin}(\xi, \xi_0)|^{-a(r-2t)},$$

as claimed. This completes the proof. □

We have thus given an independent proof of the existence proof Stein's complementary series and we give the corresponding unitary structure.

Theorem 6.2. Let $\nu = -\frac{a}{2}(r-2t)$. The Knapp-Stein intertwining operator $\mathcal{J}_t$, is well-defined for $\nu \geq 0$ on the algebraic sub of the subspace $V^{\mathbf{m}}$ and intertwines the actions of $\mathfrak{gl}(n, \mathbb{K})$ by the induced representations $\mathrm{Ind}_P^{GL_n}(t)$ and $Ind_P^{GL_n}(-t)$. Its eigenvalues are given by

$$N'_\nu \prod_{j=1}^{r} \frac{\left(\frac{a}{2}(r+1-j-2t)\right)_{\frac{m_j}{2}}}{\left(\frac{a}{2}(r+j-1+2t)\right)_{\frac{m_j}{2}}}. \tag{6.6}$$

The sesqui-linear form

$$(f, g)_\nu := \frac{1}{N'_\nu}(\mathcal{J}_t f, g)$$

for

$$0 < t < \frac{1}{2},$$

is well-defined, and is a $\mathfrak{gl}(n, \mathbb{K})$-invariant positive definite Hermitian inner product. The completion of the pre-Hilbert space is a unitary representation of of $GL(n, \mathbb{K})$

For more details on unitarity and composition series of the whole family of the induced representations see also [10], [17] and [22].

6.2. **Branching of holomorphic representations.** Finally we give an application of our result to the case branching of holomorphic representations on compact Hermitian symmetric spaces. The non-compact case has been studied intensively, see [24] and references therein. The result below can be deduced from the general theory of (discrete) branching of highest weight representations, see e.g. [12]. We only indicate here a concrete approach using the cosine transform. To keep the presentation of paper rather explicit we will only treat the case when the Hermitian symmetric spaces is the complex Grassmannian manifold with real form being the real or quaternionic Grassmannians.

We fix $\mathbb{K} = \mathbb{C}$ and let $\mathcal{X}_1$ be the complex Grassmanian manifold $U(2r_1+b_1)/U(r_1) \times U(r_1 + b_1)$ equipped with the $U(2r_1 + b_1)$-invariant Hermitian metric. Let $\mathcal{X}$ be the real Grassmannian

$$\mathcal{X} = G/K = O(r+b)/O(r) \times O(r+b), \quad r = r_1$$

or the quaternionic Grassmannian

$$\mathcal{X} = G/K = Sp(r+b)/Sp(r) \times Sp(b)), \qquad r = \frac{r_1}{2}, \qquad b = \frac{b_1}{2}$$

when r_1 and b_1 are even integers. Then $\mathcal{X}$ can be realized as a totally geodesic real form of the manifold $\mathcal{X}_1$. We realize the space $V_1 = M_{r_1+b_1,r_1}(\mathbb{C})$ as a dense subset of $\mathcal{X}_1$ by the identification

$$z \in V_1 \mapsto \{y \oplus zy; y \in \mathbb{C}^{r_1}\} \in \mathcal{X}_1;$$

similarly $V = M_{r+b,r}(\mathbb{K})$ can be realized as a dense subset of $\mathcal{X}$ via

$$z \in V \mapsto \{y \oplus zy; y \in \mathbb{K}^r\} \in \mathcal{X}.$$

For any positive integer α there is an corresponding weighted Bergman space on the compact space $\mathcal{X}_1$, denoted by $H_\alpha(\mathcal{X}_1)$, with the reproducing kernel $\det(I + w^*z)^\alpha$, $z, w \in V \subset \mathcal{X}_1$, which forms an irreducible unitary representation of $U(2r_1 + b_1)$. The elements in the space $H_\alpha(\mathcal{X}_1)$ will be identified with holomorphic polynomials on V_1. Consider $H_\alpha(\mathcal{X}_1)$ as a unitary representation of $G \subset U(2r_1 + b_1)$ and we will find the explicit irreducible decomposition.

Firstly as representation space of G we can realize the space $L^2(\mathcal{X})$ as on the subset V. Indeed the $L^2(\mathcal{X})$ is G-equivalent to

$$L^2(V, \det(I + x^*x)^{-\frac{2r_1+b_1}{2}} dm(x))$$

where $dm(x)$ is the Lebesgue measure.

The restriction mapping

$$\mathcal{T}: H_\alpha(\mathcal{X}_1) \to C^\infty(\mathcal{X}), f \mapsto f(x)\det(I + x^*x)^{-\frac{\alpha}{2}}$$

defines then an intertwining map, where G acts on $C^\infty(\mathcal{X})$ by the defining action. The operator $\mathcal{T}\mathcal{T}^*$ on $L^2(\mathcal{X}) = L^2(V, \det(I + x^*x)^{-\frac{2r_1+b_1}{2}} dm(x))$ will be called the Berezin transform and is of the form

$$\mathcal{T}\mathcal{T}^* f(x) = \int_V \det(I+x^*x)^{-\frac{\alpha}{2}} \det(I+x^*y)^\alpha f(y) \det(I+y^*y)^{-\frac{\alpha}{2}} \det(I+y^*y)^{-\frac{2r_1+b_1}{2}} dm(y);$$

see [24]. The kernel can be expressed in terms of the cosine functions and which in turn gives the branching rule of the representation. The first part of the next Proposition follows by similar computation as in the previous subsection, the second part follows from Theorem 5.4 and some abstract arguments.

Proposition 6.3. (1) The Berezin transform $\mathcal{T}\mathcal{T}^*$ on $\mathcal{X} = G/K = G_{n,r}(\mathbb{K})$ is the cosine transform $C_{r,r}^{(\nu)}$ with $\nu = \frac{\alpha}{2}$ for $\mathbb{K} = \mathbb{R}$ and $\nu = \alpha$ for $\mathbb{K} = \mathbb{H}$.

(2) The representation $\mathcal{H}_\alpha(\mathcal{X}_1)$ is decomposed under G with multiplicity free, as

$$\mathcal{H}_\alpha(\mathcal{X}_1) = \sum_{\underline{\mathbf{m}} \in L_\nu} V^{\underline{\mathbf{m}}}$$

where L_ν is given in (5.2) and (5.3).

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