Higher rho-invariants for the signature operator
A survey and perspectives

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Basics in noncommutative index theory:
A noncommutative Chern character

Given

- $\mathcal{A}$ a $C^*$-algebra, e. g. $\mathcal{A} = C(B)$, $B$ closed manifold
- $\mathcal{A}_\infty \subset \mathcal{A}$ a “smooth” subalgebra (closed under holomorphic functional calculus, dense, etc.) $\mathcal{A}_\infty = C^\infty(B)$
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one gets

- A $\mathbb{Z}$-graded Fréchet algebra $\hat{\Omega}_*\mathcal{A}_\infty$ of noncommutative differential forms with differential $d : \hat{\Omega}_k\mathcal{A}_\infty \to \hat{\Omega}_{k+1}\mathcal{A}_\infty$
- A Chern character $\text{ch} : K_*(\mathcal{A}) \to H^{dR}_*(\mathcal{A}_\infty)$
- $H^{dR}_*(\mathcal{A}_\infty)$ pairs with continuous reduced cyclic cocycles on $\mathcal{A}_\infty$
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Motivating example used in higher index theory: $\Gamma$ finitely generated group with length function, $\mathcal{A} = C^* \Gamma$, $\mathcal{A}_\infty$ the Connes-Moscovici algebra
Dirac operators over $C^*$-algebras

Given

- $(M, g)$ closed oriented Riemannian manifold
- $E \to M$ hermitian bundle with Clifford action and compatible connection ($\mathbb{Z}/2$-graded, if $\text{dim } M$ even)
- $P \in C^\infty(M, M_n(A_\infty))$ projection

we get an $\mathcal{A}$-vector bundle $\mathcal{F} := P(\mathcal{A}^n \times M) \to M$

and a (odd) Dirac operator $D_\mathcal{F} : C^\infty(M, E \otimes \mathcal{F}) \to C^\infty(M, E \otimes \mathcal{F})$. 

Important example for higher index theory: the Mishchenko $C^*$-Γ-vector bundle: $\mathcal{F} = \tilde{M} \times \Gamma C^* \Gamma$ with $\Gamma = \pi_1(M)$. 

$E = S$ the spinor bundle (gives twisted spin Dirac operator) 

$E = \Lambda^*(T^*M)$ (gives twisted de Rham or signature operator).

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Index theory

The Dirac operator $D_{\mathcal{F}}$ is Fredholm on the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes \mathcal{F})$ with $\text{ind}(D_{\mathcal{F}}) \in K_*(\mathcal{A})$.
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**Proposition (Atiyah-Singer index theorem)**

$$\text{ch}(\text{ind}(D_{\mathcal{F}})) = \int_M \hat{A}(M) \text{ch}(E/S) \text{ch}(\mathcal{F}) \in H^{dR}_*(\mathcal{A}_\infty).$$
Index theory

The Dirac operator $D_F$ is Fredholm on the Hilbert $\mathcal{A}$-module $L^2(M, E \otimes F)$ with $\text{ind}(D_F) \in K_*(\mathcal{A})$.

**Proposition (Atiyah-Singer index theorem)**

$$\text{ch}(\text{ind}(D_F)) = \int_M \hat{A}(M) \text{ch}(E/S) \text{ch}(F) \in H^d_R(\mathcal{A}_\infty).$$

**Application in higher index theory:** If $D_F$ is the signature operator twisted by $\mathcal{F} = \tilde{M} \times_{\Gamma} C^*\Gamma$, then $\text{ind}(D_F)$ is homotopy invariant.

The proposition implies: By pairing $\text{ch}(\text{ind}(D_F))$ with cyclic cocycles one gets higher signatures.

This can be used to prove the Novikov conjecture for Gromov hyperbolic groups (Connes–Moscovici 1990).
Secondary invariants

Let \( A \) be a smoothing symmetric operator on \( L^2(M, E \otimes F) \) such that \( D_F + A \) is invertible. (\( A \) should be odd if \( \text{dim } M \) is even.)

Then one can define

\[
\eta(D_F, A) \in \hat{\Omega}_*A_\infty/[[\hat{\Omega}_*A_\infty, \hat{\Omega}_*A_\infty] + d\hat{\Omega}_*A_\infty
\]

generalizing the classical \( \eta \)-invariant (with \( A = \mathbb{C} \))

\[
\eta(D_F, A) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \text{Tr}(D_F e^{-t(D_F+A)^2}) \, dt.
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\eta(D\mathcal{F}, A) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \text{Tr}(D\mathcal{F} e^{-t(D\mathcal{F} + A)^2}) dt.
$$

Higher $\eta$-invariants were introduced by Lott (1992). The general definition is implicit in work of Lott (1999).
Atiyah–Patodi–Singer index theorem

Let $M$ be an oriented Riemannian manifold with cylindric end $Z = \mathbb{R}^+ \times \partial M$. On $Z$ all structures are assumed of product type. If $\dim M$ is even, then on $Z$

$$D_\mathcal{F}^+ = c(dx)(\partial_x - D_{\mathcal{F}}^\partial) .$$

Let $A$ be a symmetric, smoothing operator on $L^2(\partial M, E^+ \otimes \mathcal{F})$ such that $D_{\mathcal{F}}^\partial + A$ is invertible.

Let $\chi: M \to \mathbb{R}$ be smooth, $\text{supp } \chi \subset Z$; $\text{supp}(\chi - 1)$ compact.

**Proposition (W., 2009)**

$$\text{ch ind}(D_\mathcal{F}^+ - c(dx)\chi(x)A) = \int_M \hat{A}(M) \text{ch}(E/S) \text{ch}(\mathcal{F}) - \eta(D_{\mathcal{F}}^\partial, A) .$$

The proposition generalizes the higher APS index theorem proven by Leichtnam–Piazza (1997–2000).
$M$ oriented Riemannian manifold with $\dim M = 2m - 1$.

Assume that the $m$-th Novikov Shubin invariant of $M$ is $\infty^+$. Let $D_{\mathcal{F}}$ be the signature operator twisted by the Mishchenko bundle $\mathcal{F}$. Let $I$ be the involution which is 1 on forms of degree $< m$ and $-1$ on the complement. Then for $t$ small $D_{\mathcal{F}} + tl$ is invertible.

We define

$$\rho(M) := [\eta(D_{\mathcal{F}}, tl)] \in \hat{\Omega}_* \mathcal{A}_\infty / [\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty] + d \hat{\Omega}_* \mathcal{A}_\infty + \hat{\Omega}_*^{(e)} \mathcal{A}_\infty.$$

Product formula: If both sides are defined, then

$$\rho(M \times N) = \rho(M) \text{ch} (\text{sign}(N)).$$
Generalization to an almost flat setting I 
(Azzali–W., work in progress)

Given \( c \in H^2(\Gamma) \) one may construct a Mishchenko bundle \( \mathcal{F}^{\sigma^s} \) associated to a twisted group \( C^*\)-algebra \( C^*(\Gamma, \sigma^s) \) with \( [\sigma^s] = e^{isc} \in H^2(\Gamma, U(1)) \). The bundle is almost flat. The \( C^*\)-algebras assemble to an upper-semicontinuous field.

Crucial property: If a differential operator associated to that field is invertible at \( s = 0 \), then it is invertible for \( s \) near 0.

In this case we can guarantee that the signature operator twisted by \( \mathcal{F}^{\sigma^s} \) can be perturbed to an invertible operator and define the higher \( \rho \)-invariants. They have the usual properties: Metric independence, product formula.

Some properties follow from the flat case using the Hanke–Schick method of comparison.
Generalization to an almost flat setting II

\((F_n, \nabla^n)\) sequence of Hermitian bundles with connection such that 
\((\nabla^n)^2 \xrightarrow{n \to \infty} 0\).

Can we construct an invertible perturbation for the signature operator such that it leads to a well-defined \(\rho\)-invariant?
Generalization to an almost flat setting III

For geometric applications, one wants to assign a $\rho$-invariant to a map $f: M \to B\Gamma$, thus better start with quasirepresentations of $\Gamma$ instead of flat bundles:

Fix a finite set $F \subset \Gamma$ and $\varepsilon > 0$.

A map $\phi: \Gamma \to A$ is an $(F, \varepsilon)$-unitary representation if $\phi(e) = 1$ and it holds that

- $\|\phi(g)\| \leq 1$, $\forall g \in \Gamma$,
- $\|\phi(g^{-1}) - \phi(g)^*\| \leq \varepsilon$, $\forall g \in F$,
- $\|\phi(gh) - \phi(g)\phi(h)\| \leq \varepsilon$, $\forall g, h \in F$.
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For manifolds with psc metrics we define $\rho$-invariants for the spin Dirac operators twisted by almost flat bundles associated to a sequence of $(F_n, \frac{1}{n})$-unitary representations (see work by Dardalat).
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For the signature operator additional conditions may be necessary, e. g. completely positive asymptotic representations of $C^*\Gamma$. 
Higher $\rho$-invariants for homotopy equivalences

Let $M, N$ be odd-dimensional oriented closed Riemannian manifolds, $f : M \to N$ a smooth orientation preserving homotopy equivalence.

$A = C^*\Gamma, \Gamma = \pi_1(N)$.

$\mathcal{F}_N = \tilde{N} \times \Gamma C^*\Gamma$ Mishchenko bundle, $\mathcal{F}_M = f^*\mathcal{F}_N$.

$D_\mathcal{F}$ signature operator on $N \cup M^{op}$ twisted by $\mathcal{F}_N \cup \mathcal{F}_M$. 

There is a smoothing operator $A(f)$ such that $D_\mathcal{F} + A(f)$ is invertible (Hilsum–Skandalis 1992, Piazza–Schick 2007).

Let $\hat{\Omega} \langle e \rangle^\ast(A_\infty) = C\langle g_0 d g_1 ... d g_m | g_0 g_1 ... g_m = e \rangle \subset \hat{\Omega}^\ast A_\infty$.

Definition $\rho(f) := \left[ \eta(D_\mathcal{F}, A(f)) \right] \in \hat{\Omega}^\ast A_\infty / [\hat{\Omega}^\ast A_\infty, \hat{\Omega}^\ast A_\infty] + d \hat{\Omega}^\ast A_\infty + \hat{\Omega} \langle e \rangle^\ast A_\infty$. 

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**Definition**

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Analytic gluing along homotopic boundaries

Assume that $M, N$ are even-dimensional oriented Riemannian manifolds with boundary and a homotopy equivalence $f: \partial M \simeq \partial N$. Then we can define an analytic signature $\text{sign}^{an}(M \cup_{\partial} N^{op})$ by using $A(f)$ for the definition of the boundary conditions for the signature operator.
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The $L$-groups $L_n(\mathbb{Z} \Gamma)$ can be represented by such pairs: More specifically by a normal map $F : W \to [0,1] \times M$ such that the restriction to $\partial_0 W$ is a diffeomorphism and the restriction to $\partial_1 W$ a homotopy equivalence.

Thus we get an induced map $\text{sign}^{an} : L_n(\mathbb{Z} \Gamma) \to K_0(C^* \Gamma)$.
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Thus we get an induced map $\text{sign}^{an} : L_n(\mathbb{Z}\Gamma) \rightarrow K_0(C^*\Gamma)$.

Open question: Does the map for $n = 4k$ agree with the standard map using quadratic forms and periodicity?
Properties

1. By pairing with suitable traces one recovers $\rho_{APS}(N) - \rho_{APS}(M)$ resp. $\rho_{L^2}(N) - \rho_{L^2}(M)$ (Piazza–Schick 2007).
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3. (Product formula) If $N = N_1 \times X$, $M = M_1 \times X$, $f = f_1 \times \text{id}_X$, then $\rho(f) = \rho(f_1) \text{ch} (\text{ind}(D_{\mathcal{F}_X}))$. 

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4. The following diagram is well-defined and commutes

\[
\begin{array}{ccc}
L_{n+1}(\mathbb{Z}\Gamma) & \xrightarrow{\text{sign}^a_n} & S(N) \\
\downarrow \rho & & \downarrow \rho \\
K_{n+1}(C^*\Gamma) & \xrightarrow{\text{ch}} & \hat{\Omega}^* A_\infty / [\hat{\Omega}^* A_\infty, \hat{\Omega}^* A_\infty] + d \hat{\Omega}^* A_\infty + \hat{\Omega}^* \langle e \rangle A_\infty.
\end{array}
\]
Applications

\[ \text{dim } N = 4k - 1, \ k \geq 2, \ \Gamma = \pi_1(N) \text{ with torsion} \]

**Proposition (Chang–Weinberger 2003)**

There are homotopy equivalences \( f_i : M_i \to N, \ i \in \mathbb{N} \) such that
\[ \rho_{L^2}(M_i) \neq \rho_{L^2}(M_j), \ i \neq j. \]
Thus \( [(M_i, f_i)] \) are distinct in \( S(N) \).

**Corollary**

Let \( X \) be a closed manifold with a non-zero higher signature. Assume that
\( \pi_1(X), \Gamma \) are Gromov hyperbolic.
Then \( [(M_i \times X, f_i \times \text{id}_X)] \) are distinct in \( S(N \times X) \) and distinguished by
\[ \rho(f_i \times \text{id}_X). \]
Proposition

Assume that $N$ is a closed oriented connected odd-dimensional manifold whose fundamental group is a product $\Gamma = \Gamma_1 \times \Gamma_2$. We assume that the following conditions hold:

1. $\Gamma_1$ contains a nontrivial torsion element,
2. there are $k, m \in \mathbb{N}$ with $k \geq 2$ and $\dim N + 1 - m = 4k$ such that $H^m(B\Gamma_2, \mathbb{Q}) \neq 0$,
3. $\Gamma_2 = \mathbb{Z}^m$ or $\Gamma_1, \Gamma_2$ have property $(RD)$ and $\Gamma_2$ has in addition property $(PC)$.

Then $S(N)$ is infinite.

Proof adapts methods by Leichtnam–Piazza for psc manifolds.
**Heuristic application**

**Heuristic Prop.**

Assume that $\Gamma$ has torsion. Then

$$\mathbb{Z}\{\langle g \rangle \mid g \neq e, g \text{ has finite order and } \langle g \rangle \text{ of polynomial growth}\}$$

acts freely on $S(N)$ if $\dim N = 4k - 1, \ k \geq 2$.

**Heuristic proof.**

The trace $\tau_{\langle g \rangle}$ is continuous on the Connes–Moscovici algebra $A_\infty \subset C^*\Gamma$.

Let $n$ be the order of $g$. The projections $P_{\langle g \rangle} = \frac{1}{n} \sum_{k=1}^{n} g^k$ define elements in $L_0(\mathbb{Q}\Gamma) \otimes \mathbb{Q} \cong L_0(\mathbb{Z}\Gamma) \otimes \mathbb{Q} \cong L_{4k}(\mathbb{Z}\Gamma) \otimes \mathbb{Q}$.

We get lifts $p_{\langle g \rangle} \in L_{4k}(\mathbb{Z}\Gamma)$, and $\tau_{\langle g \rangle}(p_{\langle h \rangle}) \neq 0$ if and only if $\langle g \rangle = \langle h \rangle$.

Thus $\tau_{\langle g \rangle}$ can be used to build a dual basis to the images of the $p_{\langle g \rangle}$ in $K_0(C^*\Gamma) \otimes \mathbb{C}$. 
Heuristic proof cont.

Now the assertion follows from the diagram

\[
\begin{array}{ccc}
L_{4k}(\mathbb{Z}\Gamma) & \longrightarrow & S(N) \\
\downarrow & & \downarrow \rho \\
K_0(C^*\Gamma) \otimes \mathbb{C} & \xrightarrow{\text{ch}} & \hat{\Omega} A_\infty/[\hat{\Omega} A_\infty, \hat{\Omega} A_\infty] + d \hat{\Omega} A_\infty + \hat{\Omega}^{<e>} A_\infty \\
\downarrow \tau\langle g \rangle & & \downarrow \tau\langle g \rangle \\
\mathbb{C} & = & \mathbb{C}'.
\end{array}
\]

Problem: Need that sign\(^an\) agrees with the standard map \(L_{4k}(\mathbb{Z}\Gamma) \rightarrow K_0(C^*\Gamma)\).

Remark: This application is inspired by arguments of Weinberger–Yu and also follows in stronger form from their work.
Comparing sign\textsuperscript{an} to the standard map: A Kaminker–Miller type strategy

Let $M, N$ be manifolds with homotopic boundary.

1. Associate to them simplicial complexes $M^s, N^s$ with a Poincaré duality structure (and further appropriate properties). There is a “homotopy equivalence” $F_M: M^s \to M$ at the level of complexes (using the Whitney map).
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2. We should be able to define $\text{sign}^{an}(M^s \cup \partial N^s, \text{opp})$, $\text{sign}^{an}(M^s \cup_{F_M} M^{opp})$, $\text{sign}^{an}(N^s \cup_{F_N} N^{opp})$, $\text{sign}^{an}(M \cup \partial N^{opp})$ using the Hilsum–Skandalis formalism.
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3. Then the following additivity property should hold:

$$\text{sign}^{an}(M^s \cup F_M M^{opp}) + \text{sign}^{an}(M \cup \partial N^{opp}) = \text{sign}^{an}(M^s \cup \partial N^{s,opp}) + \text{sign}^{an}(\cdots)$$
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$$\text{sign}^\text{\textit{an}}(M^s \cup F_M M^{\text{opp}}) + \text{sign}^\text{\textit{an}}(M \cup \partial N^{\text{opp}}) = \text{sign}^\text{\textit{an}}(M^s \cup \partial N^s, \text{opp}) + \text{sign}^\text{\textit{an}}(N^s \cup F_N N^{\text{opp}}).$$

4. It follows that $\text{sign}^\text{\textit{an}}(M \cup \partial N^{\text{opp}}) = \text{sign}^\text{\textit{an}}(M^s \cup \partial N^s, \text{opp}).$

5. Identify $\text{sign}^\text{\textit{an}}(M^s \cup \partial N^s, \text{opp}) \in K_0(C^*\Gamma)$ with the image of the Ranicki $L$-theoretic signature in $K_0(C^*\Gamma).$
Generalization to the almost flat setting

should be possible for twisted group $C^*$-algebras as well as almost flat bundles by using the Hilsum–Skandalis framework
Generalization to the almost flat setting

- should be possible for twisted group $C^*$-algebras as well as almost flat bundles by using the Hilsum–Skandalis framework
- In the almost flat setting the relation to the ordinary resp. higher $\rho$-invariants cannot be proven by directly adapting the methods from the flat case: Limit process by Piazza–Schick uses flatness of the connection.
Further open questions

What is the connection with

- the Higson–Roe map “from surgery to analysis” (2004)?

\[
\begin{align*}
L_{n+1}(\mathbb{Z}\Gamma) & \longrightarrow S(N) \longrightarrow \mathcal{N}(N) \longrightarrow L_n(\mathbb{Z}\Gamma) \\
& \downarrow \downarrow \downarrow \\
K_{n+1}(C_r^*\Gamma) & \longrightarrow K_{n+1}(D_\Gamma^*N) \longrightarrow K_n(B\Gamma) \longrightarrow K_n(C_r^*\Gamma)
\end{align*}
\]

- the analytic version by Piazza–Schick?

- other interpretations of $\rho$-invariants (Higson–Roe, Deeley–Goffeng, Weinberger–Xie–Yu)?