Soliton scattering in the hyperbolic relativistic Calogero-Moser system

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The hyperbolic relativistic Calogero-Moser system

Commuting Hamiltonians:

$$H_k(x) = \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} \prod_{\substack{m\in I\\ n\notin I}} f_-(x_m - x_n) \prod_{j\in I} \exp\left(-i\hbar\beta\frac{\partial}{\partial_{x_j}}\right) \prod_{\substack{m\in I\\ n\notin I}} f_+(x_m - x_n),$$

where

$$k = 1, \ldots, N, \quad \beta > 0$$

and

$$f_\pm(z) = \left(\frac{\sinh(\mu(z \pm i\beta g)/2)}{\sinh(\mu z/2)}\right)^{1/2}.$$

Physical picture: w/ $\beta = 1/mc$ ($c =$ speed of light)

$$H_{\mathrm{rel}} = mc^2\big(H_1(x) + H_1(-x)\big) \quad \text{(time transl.)}$$
$$P_{\mathrm{rel}} = mc\big(H_1(x) - H_1(-x)\big) \quad \text{(space transl.)}$$
$$B = -m\sum_{j=1}^{N} x_j \quad \text{(Lorentz boost)}$$

yield a representation of the Lie algebra of the Poincaré group.

The hyperbolic relativistic Calogero-Moser system

Length scales:

$$a_+ = 2\pi/\mu \quad \text{(imaginary period / interaction length)}$$
$$a_- = \hbar/mc \quad \text{(shift step size / Compton wave length)}$$

Since $f_\pm(z)$ is ia_+-periodic,

$$\left[H_k, H_l|_{a_+\leftrightarrow a_-}\right] = 0, \quad \forall k, l = 1, \ldots, N,$$

i.e. there are $2N$ commuting Hamiltonians H_k, $H_k|_{a_+\leftrightarrow a_-}$, $k = 1, \ldots, N$.

Differential / non-relativistic limit:

$$\lim_{c\to\infty} \left(H_{\text{rel}} - 2Nmc^2\right) =$$
$$-\frac{\hbar^2}{m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{2g(g-\hbar)}{m}\sum_{1\leq j<k\leq N}\frac{\mu^2}{4\sinh^2(\mu(x_j - x_k)/2)}$$

(The Schrödinger op. for the standard (non-rel.) hyperbolic Calogero-Moser system.)

Motivation and Background

Soliton scattering: conservation of momenta and factorization of each scattering event into a series of pair-scattering events.

The rel. CM sys. introduced by Ruijsenaars & Schneider (1986, classical) and Ruijsenaars (1987, quantum) to reproduce the scattering of N sine-Gordon solitons, w/ eq. of motion

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\phi = \sin\phi,$$

in a model of N point particles.

Classical case settled by Ruijsenaars (1988). We settle the quantum case.

Joint eigenfunctions

Hyperbolic gamma function: $G(a_+, a_-; z)$ is a $a_+ \leftrightarrow a_-$-invariant 'minimal' meromorphic solution of

$$\frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = 2\cosh(\pi z/a_{-\delta}), \quad \delta = +, -.$$

Kernel function:

$$\Psi_N^\sharp(b; x, z) = \mathcal{S}_N^\sharp(b; x, z)\big(W_N(b; x)W_{N-1}(b; z)\big)^{1/2}$$

where

$$\mathcal{S}_N^\sharp(b; x, z) = \prod_{j=1}^{N}\prod_{k=1}^{N-1} \frac{G(x_j - z_k - ib/2)}{G(x_j - z_k + ib/2)}$$

and

$$W_N(b; x) = \prod_{1\le j\ne k\le N} \frac{1}{c(b; x_j - x_k)}, \quad c(b; z) = \frac{G(z + i(a_+ + a_-)/2 - ib)}{G(z + i(a_+ + a_-)/2)}.$$

Lemma

$$H_k^{(N)}(x)\Psi_N^\sharp(x, z) = \Big(H_k^{(N-1)}(-z) + H_{k-1}^{(N-1)}(-z)\Big)\Psi_N^\sharp(x, z)$$

Joint eigenfunctions

Recursive scheme: Let

$$\mathcal{F}_1(x,y) = \exp(i\alpha xy), \quad \alpha = \frac{2\pi i}{a_+ a_-},$$

and define

$$\begin{aligned}
&\mathcal{F}_N(b;x,y) \\
&= \frac{1}{N!}\exp\left(i\alpha y_N \sum_{j=1}^{N} x_j\right)\left(\prod_{j=1}^{N-1}\prod_{\delta=+,-}\frac{1}{c(a_+ + a_- - b;\delta(y_j - y_N))}\right)^{1/2} \\
&\quad\cdot \int_{\mathbb{R}^{N-1}} \Psi_N^{\sharp}(b;x,z)\mathcal{F}_{N-1}(b;z,(y_1 - y_N,\ldots,y_{N-1}-y_N))dz
\end{aligned}$$

(recursively in $N = 2,3,\ldots$)

Theorem (HR 2014)

For $k = 1,\ldots,N$,

$$H_k(x)\mathcal{F}_N(x,y) = S_k\left(e^{2\pi y_1/a_+},\ldots,e^{2\pi y_N/a_+}\right)\mathcal{F}_N(x,y)$$

(w/ S_k the elementary symmetric func. of degree k).

Soliton scattering / factorized asymptotics

Theorem (HR 2018 & 2021)

For $b \in (0, \max(a_+, a_-)]$, we have

$$\mathcal{F}_N(b; x, y) \sim C_N \sum_{\sigma \in S_N} (-1)^{|\sigma|} \prod_{\substack{j<k \\ \sigma^{-1}(j)<\sigma^{-1}(k)}} u(b; x_j - x_k)^{1/2}$$

$$\cdot \prod_{\substack{j<k \\ \sigma^{-1}(j)>\sigma^{-1}(k)}} u(b; x_j - x_k)^{-1/2} \cdot \exp\left(i\alpha\sigma(x) \cdot y\right)$$

as $y_j - y_{j+1} \to \infty$, $j = 1, \ldots, N-1$, w/ the 'scattering' function

$$u(b; z) = -\frac{c(b; z)}{c(b; -z)}, \quad c(b; z) = \frac{G(z + i(a_+ + a_-)/2 - ib)}{G(z + i(a_+ + a_-)/2)}$$

(satisfying $|u(b; z)| = 1$ for $b, z \in \mathbb{R}$).

Rmk: Soliton scattering in the sine-Gordon model recovered for $b = a_\pm/2$.

Idea of proof

Basic ingredients:

- $G(a_+, a_-; z)$ is meromorphic w/

$$\text{poles: } z = -i(a_+ + a_-)/2 - ika_+ - ila_-$$
$$\text{zeros: } z = i(a_+ + a_-)/2 + ika_+ + ila_-$$

  where $k, l = 0, 1, 2, \ldots$

- $G(a_+, a_-; z) = \exp(ig(a_+, a_-; z))$ w/

$$g(a_+, a_-; z) \sim \mp\left(\frac{\pi}{24}\left(\frac{a_+}{a_-} + \frac{a_-}{a_+}\right) + \frac{\alpha}{4}z^2\right), \quad \mathrm{Re}\, z \to \pm\infty.$$

For simplicity, consider

$$\mathcal{F}_2(x, y) = \text{Prefactor} \cdot \exp\left(i\alpha y_2(x_1 + x_2)\right) \cdot \int_{\mathbb{R}} \exp\left(i\alpha z(y_1 - y_2)\right) \prod_{j=1}^{2} \frac{G(x_j - z - ib/2)}{G(x_j - z + ib/2)} dz.$$

Idea of proof

G-asymptotics entails

$$\begin{aligned}\mathrm{Prefactor} &\sim C_1(x)\exp\left(\alpha(a-b/2)(y_1-y_2)\right)\\ \mathrm{Integrand} &\sim C_2(x,z)\exp\left(-\alpha\,\mathrm{Im}\,z(y_1-y_2)\right)\end{aligned}$$

as $y_1 - y_2 \to +\infty$.

Shift contour $\mathbb{R} \to \mathbb{R} + i(a-b/2) + ir$, w/ $0 < r < \min(a_+, a_-)$, picking up residues only at (simple) poles

$$z = x_j + i(a-b/2), \quad j = 1, 2,$$

to get

$$\begin{aligned}\mathcal{F}_2(b;x,y) &= \mathrm{Prefactor} \times \exp\left(i\alpha y_2(x_1+x_2)\right)\\ &\cdot \int_{\mathbb{R}+i(a-b/2)+ir} \exp\left(i\alpha z(y_1-y_2)\right)\prod_{j=1}^{2}\frac{G(x_j-z-ib/2)}{G(x_j-z+ib/2)}dz\\ &+\mathrm{Prefactor}\cdot\exp\left(-\alpha(a-b/2)(y_1-y_2)\right)\Big[u(b;x_1-x_2)^{1/2}\exp\left(i\alpha(x_1y_1+x_2y_2)\right)\\ &\qquad - u(b;x_2-x_1)^{-1/2}\exp\left(i\alpha(x_2y_1+x_1y_2)\right)\Big].\end{aligned}$$

Global meromorphy

Consider the function

$$J_N(b;x,y) = \left(W_N(b;x)W_N(2a-b;y)\right)^{-1/2} \cdot \mathcal{F}_N(b;x,y).$$

Conjecture: We expect that $J_N(b;x,y)$ is meromorphic in (b,x,y) on

$$S(2a) \times \mathbb{C}^N \times \mathbb{C}^N,$$

where

$$S(\epsilon) = \{b \in \mathbb{C} \mid 0 < \mathrm{Re}\, b < \epsilon\}.$$

At the moment, we can prove:

Theorem (HR, in preparation)

$J_N(b;x,y)$ is meromorphic in (b,x,y) on

$$\left\{(b,x,y) \in S(\epsilon_N) \times \mathbb{C}^N \times \mathbb{C}^N \mid \max_{1\le j<k\le N} |\mathrm{Im}\,(y_j - y_k)| < \mathrm{Re}\, b\right\},$$

where $\epsilon_N = 2\max(a_+,a_-)/(3N-2)$.

(We have also obtained the locations of the poles and bounds on their orders.)

References

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- *Part IV. Global meromorphy*
  in preparation.